MATH 6210: PROBLEM SET #1
SELECTED SOLUTIONS

A1. The Hausdorff dimension of the Cantor "middle thirds" set C is $s := \ln(2)/\ln(3)$. Moreover $\mathcal{H}^s(C) = 1$.

Solution. Let

$$\begin{aligned} E_0 &= [0,1] \\ E_1 &= [0,1/3] \cup [2/3,1] \\ E_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \\ &\vdots \end{aligned}$$

so that

$$C = E_1 \cap E_2 \cap E_3 \cdots .$$

Thus C is covered by E_j, a set of 2^j intervals of length 3^{-j}. Thus, by definition,

$$\mathcal{H}^s_{3^{-j}}(C) \leq 2^j(3^{-j})^s = 1 \tag{1}$$

with the equality holding by our choice of s. Taking $j \to \infty$, we get

$$\mathcal{H}^s(C) \leq 1.$$

The assertions of (A1) follow if we can prove the reverse inequality. More precisely, if $\mathcal{I}$ is any collection of intervals covering C, we are to show

$$\sum_{I \in \mathcal{I}} |I|^s \geq 1. \tag{2}$$

By compactness of C, we can assume $\mathcal{I}$ is finite. By replacing each open interval with slightly larger closed ones, we can can assume $\mathcal{I}$ consists of closed intervals. Take such an interval I. If I does not contain least two of the closed intervals appearing in the list in the construction of the E_j's above, it belongs to the complement of C and so can be discarded. Otherwise, after intersecting I with C, we can assume I is the smallest interval containing two such intervals J and J'. We write $I = J \coprod K \coprod J'$ with $K \cap C = \emptyset$. By construction $|J|, |J'| < |K|$. So

$$\begin{aligned} |I|^s &= (|J| + |K| + |J'|)^s \\ &\geq [3 \cdot \frac{1}{2}(|J| + |J'|)]^s = 2\left[\frac{1}{2}(|J| + |J'|)\right]^s, \end{aligned}$$

with the last equality following because $3^s = 2$ by definition of s. Using convexity of x^s, we conclude

$$|I|^s \geq 2 \cdot \frac{1}{2}(|J|^s + |J'|^s) = (|J|^s + |J'|^s).$$

So, for the purposes of establishing (2) we can replace I by $J \cup J'$. Continuing in this way, after a finite number of steps we can assume each I is a closed interval of length 3^{-j} appearing in E_j. Since (1) shows (2) holds for the covering of C by E_j, we conclude 2 holds in general. □

A2. Let $(X, \mathcal{M})$ be a measurable space equipped with a positive measure μ such that $\mu(X) < \infty$. Let $f : X \to X$ be a measure preserving transformation of X. That is, suppose that whenever $E \in \mathcal{M}$, then so is $f^{-1}(E)$; and, moreover, $\mu(E) = \mu(f^{-1}(E))$ in this case. Fix $A \in \mathcal{M}$ and assume $\mu(A) > 0$.

(a) Prove that some point of A returns to A. More precisely prove that there exists $x \in A$ and $n \in \mathbb{N}$ such that $f^n(x) \in A$.

(b) Prove that almost every element A returns to A. More precisely prove that

$$\mu\left(\{x \in A \mid f^n(x) \notin A \text{ for all } n\}\right) = 0.$$

(c) Prove that almost every element of A returns to A *infinitely often*. More precisely prove that

$$\mu\left(\{x \in A \mid \text{there exists } N \text{ such that } f^n(x) \notin A \text{ for all } n > N\}\right) = 0.$$

Solution. Suppose no point of A returned to A. Then each of the sets

$$f^{-j}(A) := \{x \in X \mid f^j(x) \in A\}.$$

do not intersect A. So

$$f^{-i}(A) \cap f^{-j}(A) = \emptyset$$

for all $i \neq j$. By hypothesis, $\mu(f^{-j}(A)) = \mu(A)$ for all j. Since $\mu(A) > 0$ and $\mu(X) < \infty$ (and since μ is countably additive), we obtain a contradiction. This gives (a)

For (b), consider the set $A' \subset A$ of point which never return to A. Clearly they never return to A'. By (a), $\mu(A')$ must have measure zero.

For (c), note that

$$\{x \in A \mid \text{there exists } N \text{ such that } f^n(x) \notin A \text{ for all } n > N\}$$

is the union (over N) of $f^{-N}(A')$. This is a countable union of sets of measure zero (by (b)) and hence has measure zero. $\square$