Rudin, Chapter 4, Problem #3. The space \( L^p(T) \) is separable since the trigonometric polynomials with complex coefficients whose real and imaginary parts are rational form a countable dense subset. (Denseness follows from Theorem 3.14 and Theorem 4.25; countability is clear since \( \{e^{in\theta} \mid n \in \mathbb{Z}\} \) is a countable basis of the trigonometric polynomials.)

Meanwhile the space \( L^\infty(T) \) is not separable. To see this, let \( S \) denote the set of indicator functions of subintervals of the circle (viewed as \([0,1)\)) with irrational endpoints. Then \( S \) is uncountable and the \( L^\infty \) distance between any two elements of \( S \) is 1. Any dense subset of \( L^\infty(T) \) must contain an element in the (disjoint!) collection of balls \( \{B_{1/3}(x) \mid x \in S\} \). So any dense subset must be uncountable. \( \square \)

Rudin, Chapter 4, Problem #4. If \( H \) contains a countable maximal orthonormal set \( B \), then linear combinations \( S \) of the elements of \( B \) with complex coefficients whose real and imaginary parts are rational form a countable dense subset of \( H \). So \( H \) is separable. Conversely suppose \( H \) contains an uncountable orthonormal set \( B \). Any dense subset of \( H \) must contain at least one element in the disjoint collection of balls \( \{B_{1/3}(x) \mid x \in B\} \). So such a dense subset must be uncountable.

Rudin, Chapter 4, Problem #5. Suppose \( L \) is a nonzero continuous linear functional on \( H \). We are to show that \( M^\perp \) is one-dimensional where \( M \) is the kernel of \( L \).

Since \( L \) is nonzero, \( M \neq H \), so (by the Corollary to Theorem 4.11), \( M^\perp \) is nonzero. Fix \( h \) nonzero in \( M^\perp \). By Theorem 4.12, there is \( y \in H \) such that \( L(\cdot) = (\cdot, y) \). Take any \( z \in M^\perp \); we show it’s a multiple of \( h \). Consider

\[
z' = z - \frac{L(z)}{(y, y)} y
\]

Since \( z \) and \( y \) are in \( M^\perp \), so is \( z \). But it’s easy to see that \( L(z') = (z', y) = 0 \). So \( z' \) is also in \( M \). So \( z \) is zero. So \( z \) is a multiple of \( h \), as claimed. \( \square \)

Rudin, Chapter 4, Problem #6. We covered this in class. The main observation is that the natural map \( \psi \) from the Hilbert cube \( Q \) to \( \mathbb{C}^\mathbb{N} := \prod_{n \in \mathbb{N}} \mathbb{C} \) (mapping \( \sum c_nu_n \) to \( (c_1, c_2, \ldots) \)) is a homeomorphism onto its image. Here we take the product topology on \( \mathbb{C}^\mathbb{N} \) which is the coarsest topology so that each component projection to \( \mathbb{C} \) is continuous. (It is obvious that \( \psi \) is a continuous bijection onto its image. The slightly tricky point is that \( \psi^{-1} \) is continuous.) Since \( \psi(Q) \) is a product of the closed (compact) balls of radius \( 1/n \), Tychonoff’s Theorem implies \( \psi(Q) \), hence \( Q \), is compact.

Given a sequence \( \delta_1, \delta_2, \ldots \) of positive real numbers, define a generalized Hilbert cube \( Q \) by

\[
Q = \left\{ \sum_{n} c_n u_n \mid |c_n| \leq \delta_n \right\}.
\]
If $\sum n \delta_n^2 < \infty$, the above argument again shows that $Q$ is compact.

Now suppose $\sum \delta_n^2$ is infinite. Then the map $\psi$ is no longer surjective onto the product of balls of radius $\delta_n$ in $\mathbb{C}^N$ (why?), so the argument breaks down. We will show that $Q$ isn’t compact by finding a sequence with no convergent subsequence. (Since $H$ is a metric space the notions of sequentially compact and compact are equivalent.) Consider the sequence $\delta_1 u_1, \delta_1 u_1 + \delta_2 u_2, \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3, \ldots$. This has no convergent subsequence.

**Rudin, Chapter 4, Problem #7.** Follow the suggestions given in the problem. □

**Rudin, Chapter 4, Problem #8.** If $H_1$ and $H_2$ are two Hilbert spaces, let $\{u_\alpha \mid \alpha \in A_1\}$ and $\{u_\alpha \mid \alpha \in A_2\}$ denote respective maximal orthonormal subsets. Without loss of generality, we can assume there is an injection $\phi$ from $A_1$ to $A_2$ (why?). Define a map $\Phi$ from $H_1$ to $H_2$ sending $\sum c_\alpha u_\alpha \in H_1$ to $\sum c_{\phi(\alpha)} u_\alpha \in H_2$. Then $H_1 \cong \Phi(H_1)$ is a subspace of $H_2$. □

**Rudin, Chapter 4, Problem #11.** The set $\{(n/n + 1)e^{i\alpha} \mid n \in \mathbb{Z}\}$ is closed, nonempty, and contains no element of smallest norm. (Of course it is not convex!) □

**Rudin, Chapter 4, Problem #17.** The mapping $\gamma$ from $[0, 1]$ into $L^2(T)$ taking $a$ to the characteristic function of $[0, a]$ (viewed as a subinterval of the circle) is a continuous injection such that $\gamma(b) - \gamma(a)$ is orthogonal to $\gamma(d) - \gamma(c)$ whenever $0 \leq a \leq b \leq c \leq d \leq 1$. If $H$ is an infinite-dimensional Hilbert space, Problem 4.8 gives a continuous map $\gamma$ from $[0, 1]$ to $H$ with the same properties. □

**Rudin, Chapter 5, Problem #1.** You get diamonds for $p = 1$, rectangles for $p = \infty$, and ellipses for $p = 2$. □

**Rudin, Chapter 5, Problem #2.** Take $x, y$ of norm less than 1. Then

$$||tx + (1-t)y|| \leq t||x|| + (1-t)||y|| \leq 1.$$ □

**Rudin, Chapter 5, Problem #3.** One approach to strict convexity rests on the so called Clarkson Inequalities stating that

$$\left|\frac{f + g}{2}\right|_p^p + \left|\frac{f - g}{2}\right|_p^p \leq \frac{1}{2} \left(||f||_p^p + ||g||_p^p\right)$$

if $2 \leq p < \infty$, and

$$\left|\frac{f + g}{2}\right|_p^q + \left|\frac{f - g}{2}\right|_p^q \leq \left[\frac{1}{2} \left(||f||_p^p + ||g||_p^p\right)\right]^{q/p}$$
if $1 < p < 2$ and $p$ and $q$ are conjugate.

Now that $f \neq g$ of unit norm in $L^p$ for $p < 1 < \infty$. Since

$$\left\| \frac{f - g}{2} \right\|^p_p \text{ and } \left\| \frac{f - g}{2} \right\|^q_p$$

are strictly positive, the above inequalities show that $1/2(f + g)$ has norm strictly less than 1, as desired.

Strict convexity fails in $L^\infty(X)$ assuming $X$ is the disjoint union of two sets $X_1$ and $X_2$ of nonzero measure. Just take $f$ to be the indicator function of $X$ and $g$ to the indicator function of $X_1$. Strict convexity also fails in $L^1(X)$ assuming $X$ contains two sets $X_1$ and $X_2$ of nonzero finite measure. This time take $f$ to be the indicator function of $X_1$ and $g$ to the indicator function of $X_2$.

\[\square\]

\textbf{Rudin, Chapter 5, Problem \#4.} Convexity follows from the definitions. Since continuous functions are bounded, the Dominated Convergence Theorem can be used to show that $M$ is closed. To see that it has no elements of minimal norm, take any $f \in M$. We need to modify $f$ so that its sup norm decreases but so that the integral condition is unchanged. There are a variety of explicit ways to do this. I leave them to you.

\[\square\]

\textbf{Rudin, Chapter 5, Problem \#5.} Convexity of $M$ is clear from the definitions. The fact that $M$ is closed follow from the Dominated Convergence Theorem (which applies since any convergent sequence in $L^1$ has a pointwise convergent subsequence). The minimal possible norm is easily seen to be 1, and any positive function in $M$ attains this minimum.

\[\square\]

\textbf{Rudin, Chapter 5, Problem \#6.} Let $M$ be a subspace of a Hilbert space $H$. Let $\Lambda_M$ be a bounded linear functional on $M$. We are to prove it has a unique norm preserving extension to all of $H$.

To start, $\Lambda_M$ is continuous, so extends uniquely to the closure of $M$. So assume $M$ is closed, hence a Hilbert space. Thus there exists $m \in M$ such that $\Lambda_M(x) = (x, m)$ such that $||\Lambda_M|| = ||m||$ (Theorem 4.12 plus Schwarz’s inequality). There is one obvious norm-preserving extension of $\Lambda_M$ to $H$ mapping any $x \in H$ to $(x, m)$. Suppose $\Lambda$ were another norm-preserving extension. by Theorem 4.12, $\Lambda(x) = (x, h)$ for a unique $h \in H$ with $||h|| = ||m||$. Write the orthogonal decomposition of $h$ with respect to $M$ as $h = h_1 + h_2$. Since $\Lambda|_M = \Lambda_M$ and $h_2 \in M^\perp$,

$$(x, m) = (x, h_1 + h_2) = (x, h_1)$$

for all $x \in M$. By the uniqueness of Theorem 4.12, $h_1 = m$. By the norm preserving property and the Pythagorean Theorem

$$||m|| = ||h|| = ||m + h_2|| = \sqrt{||m||^2 + ||h_2||^2}.$$ 

So $h_2 = 0$ and the only extension of $\Lambda_M$ to $H$ is $\Lambda(x) = (x, m)$ for $x \in H$. 

Rudin, Chapter 5, Problem #7. Let $X$ be a finite set with the counting measure. So $V := L^1(X) \simeq \mathbb{C}^n$. Let $U$ be any nonzero subspace of $V$ of dimension $m < n$. A linear functional $\Lambda$ on $U$ is given by a $1$-by-$m$ matrix. Its norm is the maximum of the absolute value of its entries. We can extend $\Lambda$ to $V$ by adding $n - m$ new entries. If they all have absolute value less than the norm of $\Lambda$, the extension is norm-preserving. So there are infinitely many norm-preserving extensions of $\Lambda$ from $U$ to $V$. 

Rudin, Chapter 5, Problem #8. For (a), note that it follows from the definitions that $X^*$ is a normed vector space. The more subtle fact is that it is complete. So let $\Lambda_1, \Lambda_2, \ldots$ be a Cauchy sequence in $X^*$. Then since each difference $\Lambda_i - \Lambda_j$ is bounded, the sequence $\Lambda_1(x), \Lambda_2(x), \ldots$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $x \in X$. So the pointwise limit $\Lambda(x) := \lim_{n \to \infty} \Lambda_i(x)$ is well-defined. A quick check shows $\Lambda$ is linear. To see it is bounded, note first that since $\{\Lambda_i\}$ is Cauchy, $||\Lambda|| = 1$, $||\Phi(\Lambda)|| = \lim_{n \to \infty} |\Lambda_i(x)| \leq ||\Lambda|| \cdot ||x|| = ||x||$. So, indeed, $\Phi$ is bounded.

Finally, for (c), let $\{x_n\}$ be a sequence in $X$ such that $\{\Lambda(x_n)\}$ is bounded for all $\Lambda \in X^*$. Let $\Phi_n$ be the map from $X^*$ to $\mathbb{C}$ by evaluation at some fixed $x$ in $X$. This is clearly linear. To check that it is bounded, compute for $||\Lambda|| = 1$, $|\Phi(\Lambda)| = |\Lambda(x)| \leq ||\Lambda|| \cdot ||x|| = ||x||$. So, indeed, $\Phi$ is bounded.

Rudin, Chapter 5, Problem #20. For (a), a short argument shows that the set of points $x$ for which $\{f_n(x)\}$ is a $G_\delta$ set. If it consisted of the rationals, then the rationals would be a dense $G_\delta$ in $\mathbb{R}$, something prohibited by the Baire Category Theorem.

For (c) (and hence (b)), enumerate the rationals in $(0,1)$ as $r_1, r_2, \ldots$ and let $f_n$ be the piecewise linear function through $(0,0), (r_1,1), \ldots, (r_n,n), (1,0)$. This is a sequence of functions such that the set of $x$ with $f_n(x) \to \infty$ is exactly the irrationals.

Rudin, Chapter 6, Problem #1. Yes, we need only work with finite partitions. Let $\{F_\alpha\}_{\alpha \in A}$ be any (infinite) partition of $E$. It suffices to find a finite subset $S \subset A$ such that $\sum_{\alpha \in S} \mu(F_\alpha)$ is within $\epsilon$ of $\sum_{\alpha \in A} \mu(F_\alpha)$. Since the latter sum is finite and absolutely convergent, there is a finite partial sum which is within $\epsilon$ of the total sum. This partial sum defines $S$, as required.
Rudin, Chapter 6, Problem #3. Suppose we can show the space of complex measures on $X$ is really a vector space. Then the Riesz Representation Theorem shows that this space (with the total variation norm) is isometrically isomorphic to $X^*$, which is a Banach space by Chapter 5, #8. I leave the vector space verification to you.

Rudin, Chapter 6, Problem #4.

Rudin, Chapter 6, Problem #5. Note that $L^1$ is one-dimensional, but $L^\infty$ is two-dimensional. So these spaces are not dual to each other.