MATH 6210: SOLUTIONS TO PROBLEM SET #3

Rudin, Chapter 4, Problem #3. The space $L^p(T)$ is separable since the trigonometric polynomials with complex coefficients whose real and imaginary parts are rational form a countable dense subset. (Denseness follows from Theorem 3.14 and Theorem 4.25; countability is clear since $\{e^{in\theta} \mid n \in \mathbb{Z}\}$ is a countable basis of the trigonometric polynomials.)

Meanwhile the space $L^\infty(T)$ is not separable. To see this, let S denote the set of indicator functions of subintervals of the circle (viewed as $[0, 1)$) with irrational endpoints. Then S is uncountable and the L^∞ distance between any two elements of S is 1. Any dense subset of $L^\infty(T)$ must contain an element in the (disjoint!) collection of balls $\{B_{1/3}(x) \mid x \in S\}$. So any dense subset must be uncountable. $\square$

Rudin, Chapter 4, Problem #4. If H contains a countable maximal orthonormal set B, then linear combinations S of the elements of B with complex coefficients whose real and imaginary parts are rational form a countable dense subset of H. So H is separable. Conversely suppose H contains an uncountable orthonormal set B. Any dense subset of H must contain at least one element in the disjoint collection of balls $\{B_{1/3}(x) \mid x \in B\}$. So such a dense subset must be uncountable.

Rudin, Chapter 4, Problem #5. Suppose L is a nonzero continuous linear functional on H. We are to show that $M^\perp$ is one-dimensional where M is the kernel of L.

Since L is nonzero, $M \neq H$, so (by the Corollary to Theorem 4.11), M^perp is nonzero. Fix h nonzero in $M^\perp$. By Theorem 4.12, there is $y \in H$ such that $L(\cdot) = (\cdot, y)$. Take any $z \in M^\perp$; we show it's a multiple of h. Consider

$$z' = z - [L(z)/(y, y)]\, y$$

Since z and y are in $M^\perp$, so is z. But it's easy to see that $L(z') = (z', y) = 0$. So z' is also in M. So z is zero. So z is a multiple of h, as claimed. $\square$

Rudin, Chapter 4, Problem #6. We covered this in class. The main observation is that the natural map ψ from the Hilbert cube Q to $\mathbb{C}^{\mathbb{N}} := \prod_{i \in \mathbb{N}} \mathbb{C}$ (mapping $\sum c_n u_n$ to $(c_1, c_2, \ldots)$) is a homeomorphism onto its image. Here we take the product topology on $\mathbb{C}^{\mathbb{N}}$ which is the coarsest topology so that each component projection to $\mathbb{C}$ is continuous. (It is obvious that ψ is a continuous bijection onto its image. The slightly tricky point is that ψ^{-1} is continuous.) Since $\psi(Q)$ is a product of the closed (compact) balls of radius $1/n$, Tychonoff's Theorem implies $\psi(Q)$, hence Q, is compact.

Given a sequence $\delta_1, \delta_2, \ldots$ of positive real numbers, define a generalized Hilbert cube Q by

$$Q = \left\{ \sum_n c_n u_n \mid |c_n| \leq \delta_n \right\} .$$

If $\sum_n \delta_n^2 < \infty$, the above argument again shows that Q is compact.

Now suppose $\sum \delta_n^2$ is infinite. Then the map ψ is no longer surjective onto the product of balls of radius δ_n in $\mathbb{C}^{\mathbb{N}}$ (why?), so the argument breaks down. We will show that Q isn't compact by finding a sequence with no convergent subsequence. (Since H is a metric space the notions of sequentially compact and compact are equivalent.) Consider the sequence $\delta_1 u_1, \delta_1 u_1 + \delta_2 u_2, \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3, \ldots$. This has no convergent subsequence.

Rudin, Chapter 4, Problem #7. Follow the suggestions given in the problem. □

Rudin, Chapter 4, Problem #8. If H_1 and H_2 are two Hilbert spaces, let $\{u_\alpha \mid \alpha \in A_1\}$ and $\{u_\alpha \mid \alpha \in A_2\}$ denote respective maximal orthonormal subsets. Without loss of generality, we can assume there is an injection ϕ from A_1 to A_2 (why?). Define a map Φ from H_1 to H_2 sending $\sum c_\alpha u_\alpha \in H_1$ to $\sum c_{\phi(\alpha)} u_\alpha \in H_2$. Then $H_1 \simeq \Phi(H_1)$ is a subspace of H_2. □

Rudin, Chapter 4, Problem #11. The set $\{(n/n+1))e^{in\theta} \mid n \in \mathbb{Z}\}$ is closed, nonempty, and contains no element of smallest norm. (Of course it is not convex!) □

Rudin, Chapter 4, Problem #17. The mapping γ from $[0,1]$ into $L^2(T)$ taking a to the characteristic function of $[0,a]$ (viewed as a subinterval of the circle) is a continuous injection such that $\gamma(b) - \gamma(a)$ is orthogonal to $\gamma(d) - \gamma(c)$ whenever $0 \le a \le b \le c \le d \le 1$. If H is an infinite-dimensional Hilbert space, Problem 4.8 gives a continuous map γ from $[0,1]$ to H with the same properties. □

Rudin, Chapter 5, Problem #1. You get diamonds for $p = 1$, rectangles for $p = \infty$, and ellipses for $p = 2$. □

Rudin, Chapter 5, Problem #2. Take x, y of norm less than 1. Then

$$||tx + (1-t)y|| \le t||x|| + (1-t)||y|| \le 1.$$

□

Rudin, Chapter 5, Problem #3. One approach to strict convexity rests on the so called Clarkson Inequalities stating that

$$\left|\left|\frac{f+g}{2}\right|\right|_p^p + \left|\left|\frac{f-g}{2}\right|\right|_p^p \le \frac{1}{2}\left(||f||_p^p + ||g||_p^p\right)$$

if $2 \le p < \infty$, and

$$\left|\left|\frac{f+g}{2}\right|\right|_p^q + \left|\left|\frac{f-g}{2}\right|\right|_p^q \le \left[\frac{1}{2}\left(||f||_p^p + ||g||_p^p\right)\right]^{q/p}$$

if $1 < p < 2$ and p and q are conjugate.

Now that $f \neq g$ of unit norm in L^p for $p < 1 < \infty$. Since

$$\left|\left|\frac{f-g}{2}\right|\right|_p^p \text{ and } \left|\left|\frac{f-g}{2}\right|\right|_p^q$$

are strictly positive, the above inequalities show that $1/2(f+g)$ has norm strictly less than 1, as desired.

Strict convexity fails in $L^\infty(X)$ assuming X is the disjoint union of two sets X_1 and X_2 of nonzero measure. Just take f to be the indicator function of X and g to the indicator function of X_1. Strict convexity also fails in $L^1(X)$ assuming X contains two sets X_1 and X_2 of nonzero finite measure. This time take f to be the indicator function of X_1 and g to the indicator function of X_2. □

Rudin, Chapter 5, Problem #4. Convexity follows from the definitions. Since continuous functions are bounded, the Dominated Convergence Theorem can be used to show that M is closed. To see that it has no elements of minimal norm, take any $f \in M$. We need to modify f so that its sup norm decreases but so that the integral condition is unchanged. There are a variety of explicit ways to do this. I leave them to you.

□

Rudin, Chapter 5, Problem #5. Convexity of M is clear from the definitions. The fact that M is closed follow from the Dominated Convergence Theorem (which applies since any convergent sequence in L^1 has a pointwise convergent subsequence). The minimal possible norm is easily seen to be 1, and any positive function in M attains this minimum.

□

Rudin, Chapter 5, Problem #6. Let M be a subspace of a Hilbert space H. Let Λ_M be a bounded linear functional on M. We are to prove it has a unique norm preserving extension to all of H.

To start, Λ_M is continuous, so extends uniquely to the closure of M. So assume M is closed, hence a Hilbert space. Thus there exists $m \in M$ such that $\Lambda_M(x) = (x, m)$ such that $||\Lambda_M|| = ||m||$ (Theorem 4.12 plus Schwarz's inequality). There is one obvious norm-preserving extension of Λ_M to H mapping any $x \in H$ to (x, m). Suppose Λ were another norm-preserving extension. by Theorem 4.12, $\Lambda(x) = (x, h)$ for a unique $h \in H$ with $||h|| = ||m||$. Write the orthogonal decomposition of h with respect to M as $h = h_1 + h_2$. Since $\Lambda_{|M} = \Lambda_M$ and $h_2 \in M^\perp$,

$$(x, m) = (x, h_1 + h_2) = (x, h_1)$$

for all $x \in M$. By the uniqueness of Theorem 4.12, $h_1 = m$. By the norm preserving property and the Pythagorean Theorem

$$||m|| = ||h|| = ||m + h_2|| = \sqrt{||m||^2 + ||h_2||^2}.$$

So $h_2 = 0$ and the only extension of Λ_M to H is $\Lambda(x) = (x, m)$ for $x \in H$.

Rudin, Chapter 5, Problem #7. Let X be a finite set with the counting measure. So $V := L^1(X) \simeq \mathbb{C}^n$. Let U be any nonzero subspace of V of dimension $m < n$. A linear functional Λ on U is given by a 1-by-m matrix. It's norm is the maximum of the absolute value of its entries. We can extend Λ to V by adding $n-m$ new entries. If they all have absolute value less than the norm of Λ, the extension is norm-preserving. So there are infinitely many norm-preserving extensions of Λ from U to V. □

Rudin, Chapter 5, Problem #8. For (a), note that is follows from the definitions that X^* is a normed vector space. The more subtle fact is that it is complete. So let $\Lambda_1, \Lambda_2, \ldots$ be a Cauchy sequence in X^*. Then since each difference $\Lambda_i - \Lambda_j$ is bounded, the sequence $\Lambda_1(x), \Lambda_2(x), \ldots$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $x \in X$. So the pointwise limit $\Lambda(x) := \lim_{n\to\infty} \Lambda_i(x)$ is well-defined. A quick check shows Λ is linear. To see it is bounded, note first that since $\{\Lambda_i\}$ is Cauchy, $\{||\Lambda_i\}$ is bounded by, say, N. Fix $x \in X$ with $||x|| = 1$. Then

$$|\Lambda(x)| = \lim_{n\to\infty} |\Lambda_i(x)| \le N||x|| = N.$$

So, indeed, Λ is bounded.

The last thing to check is that $\Lambda_i \to \Lambda$ *in the norm on* X^*. I leave this to you.

For (b), define Φ from X^* to $\mathbb{C}$ by evaluation at some fixed x in X. This is clearly linear. To check that it is bounded, compute for $||\Lambda|| = 1$,

$$|\Phi(\Lambda)| = |\Lambda(x)| \le ||\Lambda|| \cdot ||x|| = ||x||.$$

So, indeed, Φ is bounded.

Finally, for (c), let $\{x_n\}$ be a sequence in X such that $\{\Lambda(x_n)\}$ is bounded for all $\Lambda \in X^*$. Let Φ_n be the map from X^* to $\mathbb{C}$ defined by evaluating at x_n. So $\{\Lambda(x_n)\}$ bounded for all Λ literally means $\{\Phi_n(\Lambda)\}$ is bounded for all Λ. By Banach-Steinhaus, $\{||\Phi_n||\} = \{||x_n||\}$ is bounded.

□

Rudin, Chapter 5, Problem #20. For (a), a short argument shows that the set of points x for which $\{f_n(x)\}$ is a G_δ set. If it consisted of the rationals, then the rationals would be a dense G_δ in $\mathbb{R}$, something prohibited by the Baire Category Theorem.

For (c) (and hence (b)), enumerate the rationals in $(0,1)$ as $r_1, r_2, ...$ and let f_n be the piecewise linear function through $(0,0), (r_1, 1), \ldots, (r_n, n), (1,0)$. This is a sequence of functions such that the set of x with $f_n(x) \to \infty$ is exactly the irrationals. □

Rudin, Chapter 6, Problem #1. Yes, we need only work with finite partitions. Let $\{F_\alpha\}_{\alpha\in A}$ be any (infinite) partition of E. It suffices to find a finite subset $S \subset A$ such that $\sum_{\alpha\in S} |\mu(F_\alpha)|$ is within ϵ of $\sum_{\alpha\in A} |\mu(F_\alpha)|$. Since the latter sum is finite and absolutely convergent, there is a finite partial sum which is within ϵ of the total sum. This partial sum defines S, as required. □

Rudin, Chapter 6, Problem #3. Suppose we can show the space of complex measures on X is really a vector space. Then the Riesz Representation Theorem shows that this space (with the total variation norm) is isometrically isomorphic to X^*, which is a Banach space by Chapter 5, #8. I leave the vector space verification to you. $\square$

Rudin, Chapter 6, Problem #4.

$\square$

Rudin, Chapter 6, Problem #5. Note that L^1 is one-dimensional, but L^∞ is two-dimensional. So these spaces are not dual to each other. $\square$