MATH 6210: SOLUTIONS TO PROBLEM SET #2

Rudin, Chapter 2, Problem #5. Show that the Cantor set has measure 0 but is uncountable.

Let E_k denote the kth step in the construction of E. So $E_0 = [0,1]$, $E_1 = [0,1/3] \cup [2/3,1]$, and so on; and $E = \bigcap_i E_i$. Add up size of intervals removed to form E_i to get

$$(1/3) + 2(1/3)^2 + 2^2(1/3)^3 + \cdots + 2^{i-1}(1/3)^i.$$

So E is the complement of a set of size

$$(1/3) + 2(1/3)^2 + 2^2(1/3)^3 + \cdots = 1,$$

and hence has measure zero.

Finally, the elements of E are in 1-1 correspondence with infinite base three decimals, so E is uncountable. $\square$

Rudin, Chapter 4, Problem #7. Construct a totally disconnected compact set $K \subset \mathbb{R}$ such that $\mu(K) > 0$.

Repeat the constuction of the Cantor set from the interval $[0,1]$ recalled in the previous problem, but define K_i inductively by removing (open) middle segments from $K_{i=1}$ of length ε^i (for $\varepsilon < 1$). Then a simple argument (along the lines in the previous argument) shows that the resulting intersection K has measure

$$\frac{1-3\varepsilon}{1-2\varepsilon}.$$

K is totally disconnected and compact for the same reasons that the Cantor set is. $\square$

Rudin, Chapter 4, Problem #6. Fix $0 < \varepsilon' < 1$. Find an open dense subset of $[0,1]$ of measure ε'.

Consider the complement of the set constructed in Problem #7. It is dense and can be made to have measure ε'. $\square$

Rudin, Chapter 4, Problem #8. Construct a Borel set $E \subset \mathbb{R}$ such that

$$0 < \mu(I \cap E) < \mu(I)$$

for every nonempty interval I.

Suppose we can find $E \subset [0,1]$ satisfying the requirement for each $I \subset [0,1]$. Then repeating E along the real line gives a solution to the problem.

To find such $E \subset [0,1]$, start with the generalized Cantor set, say E_1, of measure ε constructed in Problem #7. Then in each of the (countable!) number of "holes" in E_1 with appropriately scaled generalized Cantor sets so that the result has meaure

$\varepsilon+\varepsilon/2$. Call the result E_2. Next fill the countable number of holes in E_2 with scaled generalized Cantor sets so that the result now has measure $\varepsilon+\varepsilon/2+\varepsilon/4$. Continue. The union of all E_i will have the required properties and will have measure 2ε.

As we noted, if we repeat E along the real line, it has the required properties of the problem. (By shrinking its measure as we repeat, we can insure the result has *finite* total measure!)

□

Rudin, Chapter 4, Problem #9. Construct a sequence of continuous functions f_n on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$

but so that $f_n(x)$ converges for no $x \in [0, 1]$.

Recall a tent function centered at c of height a and width b is the piecewise linear function

$$f(x) = \begin{cases} 0 & \text{if } x \notin [c-b/2, c+b/2] \\ a(1-|2(x-c)/b|) & \text{if } x \in [-b/2, b/2]. \end{cases}$$

Let g_n be the tent function centered at $1 + 1/2 + \cdots + 1/n$ (taken modulo 1) with height 1 and width $1/\sqrt{n}$, and let $f_n = \chi_{[0,1]} g_n$. This sequence is a "pulse" of overlapping tent functions which wrap around the unit interval. The width of each tent keeps shrinking (so the integral tends to zero), but if we fix $x \in [0, 1]$, x takes values 0 and arbitrarily close to 1 infinitely often.

□

Rudin, Chapter 3, Problem #3. If ϕ is a continuous function on (a, b) such that

$$\phi\left(\frac{x+y}{2}\right) \le \frac{\phi(x)}{2} + \frac{\phi(y)}{2},$$

then ϕ is convex.

The best proof (like that of Theorem 3.2) is by drawing a picture. I omit the details. As Rudin points out, you need to be a little careful since the result fails if ϕ is not assumed to be continuous (for example, by taking $a = 0, b = 2$, $\phi(x) = 0$ for $x < 1$, and $\phi(x) = 3x + 1$ for $x \ge 1$). □

Rudin, Chapter 3, Problem #7. Find necessary and sufficient conditions for the inclusion of $L^p \subset L^q$ to hold.

The solution I give is based on "Another note on the inclusion of $L^p(\mu) \subset L^q(\mu)$" by A. Villani, *The American Mathematical Monthly*, Vol. 92 (1985), No. 7, 485–487.

Proposition 1 Let μ be a positive measure in $\mathcal{M}$. Let $\mathcal{M}_0$ denote the sets of nonzero measure. The the following are equivalent

(a) $L^p(\mu) \subset L^q(\mu)$ for some $p < q$ in $(0, \infty]$;

(b) $\inf_{E\in\mathcal{M}_0} \mu(E) > 0$; and

(c) $L^p(\mu) \subset L^q(\mu)$ for all $p < q$ in $(0, \infty]$;

Proof. (a) implies (b). If $L^p(\mu) \subset L^q(\mu)$, then clearly $L^{pt} \subset L^{qt}$ for all $t > 0$. So assume we can assume $p \geq 1$, so L^p and L^q are normed. The key is *the hypothesized set-theoretic inclusion $L^p \subset L^q$ is a continuous linear map.* The reason is that, as we saw in the proof of the completeness of L^q, any convergent sequence in L^p converges (in L^q) to the pointwise limit of some subsequence (almost everywhere). So the image of L^p inside L^q is closed in the L^q norm. So the Closed Graph Theorem then implies the inclusion is continuous. Hence there is a constant k such that for every $f \in L^p$,

$$||f||_p \leq k||f||_q.$$

Take $f = \chi_E$ for a measurable set E to get

$$\mu(E)^{1/p} \leq k\mu(E)^{1/q},$$

and so

$$\mu(E) \geq k^{\frac{-1}{1/q-1/p}}.$$

So the infinium in (b) must be strictly positive.

(b) implies (c). Given $f \in L^p$, let E_n denote the set of x such that $|f(x) < n$. Since $f \in L^p$, $\mu(E_n)$ must tend to zero as n becomes large. By (b), this means there is some N such that $\mu(E_n) = 0$. So f is actually in L^∞. So f is in L^q too.

(c) trivially implies (a). □

There is also a dual assertion (whose proof I leave to you):

Proposition 2 Let μ be a positive measure in $\mathcal{M}$. Let $\mathcal{M}_\infty$ denote the sets of finite measure. The the following are equivalent

(a) $L^p(\mu) \subset L^q(\mu)$ for some $p > q$ in $(0, \infty)$;
(b) $\sup_{E \in \mathcal{M}_\infty} \mu(E) < \infty$; and
(c) $L^p(\mu) \subset L^q(\mu)$ for all $p > q$ in $(0, \infty)$;

Note the if $\mu(X)$ is finite, Proposition 2 applies. If μ is discrete, Proposition 1 applies. Neither proposition applies to Lebesque measure on $\mathbb{R}$. □

Rudin, Chapter 3, Problem #11. Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Then

$$\int_\Omega f d\mu \cdot \int_\Omega f d\mu \ \geq \ 1$$

Since f and g are positive, we can consider their square roots $\sqrt{f}$ and $\sqrt{g}$. By the H older inequality, we have

$$||\sqrt{f}||_2||\sqrt{g}||_2 \geq ||\sqrt{fg}||_1.$$

The left-hand side is of course

$$\sqrt{\int_\Omega f d\mu \cdot \int_\Omega g d\mu}.$$

By hypothesis $\sqrt{fg} \geq 1$, and so the right-hand side is at least

$$1 \cdot \mu(\Omega) = 1.$$

Squaring both sides gives the desired result.

$\square$

Circle Problem. Let $S^1 = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$. For each $n \in \mathbb{Z}$, define a map $\chi_n : S^1 \to \mathbb{C}^\times$ via

$$\chi_n(e^{i\theta}) = e^{in\theta}.$$

(0) Prove that χ_n is a continuous homorphism from the multiplicative group S^1 to the multiplicative group $\mathbb{C}^\times$.

(1) Suppose χ is any continuous homorphism from the multiplicative group S^1 to the multiplicative group $\mathbb{C}^\times$. Prove that there exists an n such that $\chi = \chi_n$.

(2) Suppose χ is a continuous homomorphism from S^1 to $\mathrm{G}L(N, \mathbb{C})$ so that χ admits no invariant subspaces in the following sense: if V is a subspace of $\mathbb{C}^N$ such that

$$[\chi(x)](v) \in V \text{ for all } x \in S^1 \text{ and } v \in V,$$

then $V = \{0\}$ or $V = \mathbb{C}^N$. Prove that $N = 1$.

Hence the maps χ_n are precisely the set of continuous homomorphism from S^1 to $\mathrm{G}L(N, \mathbb{C})$ so that χ admits no invariant subspaces.

Solution. (0) is trivial. For (1), let H_m denote the subgroup of S^1 consisting of mth roots of unity. Since χ is a homomorphism, it is clear that for each m there exists and integer $0 \leq n_m \leq m - 1$ (possibly depending on m) so that $\chi(z) = z^{n_m}$ for all $z \in H_m$. Let z_m denote a choice of generator for H_m. For primes $p \neq q$, we know we can take $z_{pq} = z_p z_q$. So we compute in the group H_{pq},

$$z_p^{n_{pq}} z_q^{n_{pq}} = z_{pq}^{n_{pq}} = \chi(z_{pq}) = \chi(z_p z_q) = \chi(z_p)\chi(z_q) = z_p^{n_p} z_q^{n_q}.$$

In other words

$$z_p^{n_{pq} - n_p} = z_q^{n_q - n_{pq}}.$$

Since H_p and H_q intersect only in $\{1\}$,

$$n_{pq} = n_p \text{ modulo } p$$

and

$$n_{pq} = n_q \text{ modulo } q.$$

Thus $n_p = n_q \bmod p$ (as well as mod q). So indeed for all primes p, n_p is a constant n (independent of p). Since all prime roots of unity are dense in S^1, the claim follows.

For (2), write A_θ for $\chi(e^{i\theta})$; so A_θ is an $N \times N$ matrix. Since A_θ cannot be the zero matrix, it has a nonzero eigenvalue. Pick one and call it λ_θ. Consider the matrix $B_\theta = A_\theta - \lambda_\theta \mathrm{Id}$. Since χ is a homomorphism and S^1 is abelian, it's clear that B_θ commutes with all A_ϕ. Hence the kernel K of B is an invariant subspace of $\mathbb{C}^N$. By hypothesis, this means that K is either $\{0\}$ or all or $\mathbb{C}^N$. The former case is impossible since any eigenvector corresponding to λ_θ is in the kernel. Thus K must be all of $\mathbb{C}^N$. So $B_\theta = 0$. In other words A_θ is the (nonzero) constant multiple λ_θ of the identity. Thus *any* subspace of $\mathbb{C}^N$ is invariant. This is a contradiction unless $N = 1$. $\square$