Rudin, Chapter 2, Problem #5. Show that the Cantor set has measure 0 but is uncountable.

Let $E_k$ denote the $k$th step in the construction of $E$. So $E_0 = [0,1]$, $E_1 = [0,1/3] \cup [2/3,1]$, and so on; and $E = \bigcap_k E_k$. Add up size of intervals removed to form $E_i$ to get

$$(1/3) + 2(1/3)^2 + 2^2(1/3)^3 + \cdots + 2^{i-1}(1/3)^i.$$ 

So $E$ is the complement of a set of size

$$(1/3) + 2(1/3)^2 + 2^2(1/3)^3 + \cdots = 1,$$

and hence has measure zero.

Finally, the elements of $E$ are in 1-1 correspondence with infinite base three decimals, so $E$ is uncountable. \hfill \Box

Rudin, Chapter 4, Problem #7. Construct a totally disconnected compact set $K \subset \mathbb{R}$ such that $\mu(K) > 0$.

Repeat the construction of the Cantor set from the interval $[0,1]$ recalled in the previous problem, but define $K_i$ inductively by removing (open) middle segments from $K_{i-1}$ of length $\varepsilon^i$ (for $\varepsilon < 1$). Then a simple argument (along the lines in the previous argument) shows that the resulting intersection $K$ has measure

$$\frac{1 - 3\varepsilon}{1 - 2\varepsilon}.$$ 

$K$ is totally disconnected and compact for the same reasons that the Cantor set is. \hfill \Box

Rudin, Chapter 4, Problem #6. Fix $0 < \varepsilon' < 1$. Find an open dense subset of $[0,1]$ of measure $\varepsilon'$.

Consider the complement of the set constructed in Problem #7. It is dense and can be made to have measure $\varepsilon'$. \hfill \Box

Rudin, Chapter 4, Problem #8. Construct a Borel set $E \subset \mathbb{R}$ such that

$$0 < \mu(I \cap E) < \mu(I)$$

for every nonempty interval $I$.

Suppose we can find $E \subset [0,1]$ satisfying the requirement for each $I \subset [0,1]$. Then repeating $E$ along the real line gives a solution to the problem.

To find such $E \subset [0,1]$, start with the generalized Cantor set, say $E_1$, of measure $\varepsilon$ constructed in Problem #7. Then in each of the (countable!) number of “holes” in $E_1$ with appropriately scaled generalized Cantor sets so that the result has measure
$\varepsilon + \varepsilon/2$. Call the result $E_2$. Next fill the countable number of holes in $E_2$ with scaled generalized Cantor sets so that the result now has measure $\varepsilon + \varepsilon/2 + \varepsilon/4$. Continue. The union of all $E_i$ will have the required properties and will have measure $2\varepsilon$.

As we noted, if we repeat $E$ along the real line, it has the required properties of the problem. (By shrinking its measure as we repeat, we can insure the result has finite total measure!)

\[ \square \]

\textbf{Rudin, Chapter 4, Problem \#9.} Construct a sequence of continuous functions $f_n$ on $[0,1]$ such that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0,$$

but so that $f_n(x)$ converges for no $x \in [0,1]$.

Recall a tent function centered at $c$ of height $a$ and width $b$ is the piecewise linear function

$$f(x) = \begin{cases} 0 & \text{if } x \notin [c - b/2, c + b/2] \\ a(1 - |2(x - c)/b|) & \text{if } x \in [-b/2, b/2]. \end{cases}$$

Let $g_n$ be the tent function centered at $1 + 1/2 + \cdots + 1/n$ (taken modulo 1) with height 1 and width $1/\sqrt{n}$, and let $f_n = \chi_{[0,1]}g_n$. This sequence is a “pulse” of overlapping tent functions which wrap around the unit interval. The width of each tent keeps shrinking (so the integral tends to zero), but if we fix $x \in [0,1]$, $x$ takes values 0 and arbitrarily close to 1 infinitely often.

\[ \square \]

\textbf{Rudin, Chapter 3, Problem \#3.} If $\phi$ is a continuous function on $(a, b)$ such that

$$\phi \left( \frac{x + y}{2} \right) \leq \frac{\phi(x)}{2} + \frac{\phi(y)}{2},$$

then $\phi$ is convex.

The best proof (like that of Theorem 3.2) is by drawing a picture. I omit the details. As Rudin points out, you need to be a little careful since the result fails if $\phi$ is not assumed to be continuous (for example, by taking $a = 0, b = 2, \phi(x) = 0$ for $x < 1$, and $\phi(x) = 3x + 1$ for $x \geq 1$).

\[ \square \]

\textbf{Rudin, Chapter 3, Problem \#7.} Find necessary and sufficient conditions for the inclusion of $L^p \subset L^q$ to hold.


\textbf{Proposition 1} Let $\mu$ be a positive measure in $\mathcal{M}$. Let $\mathcal{M}_0$ denote the sets of nonzero measure. The the following are equivalent

(a) $L^p(\mu) \subset L^q(\mu)$ for some $p < q$ in $(0, \infty]$;
(b) $\inf_{E \in \mathcal{M}_0} \mu(E) > 0$; and
(c) \( L^p(\mu) \subseteq L^q(\mu) \) for all \( p < q \) in \((0, \infty)\);

**Proof.** (a) implies (b). If \( L^p(\mu) \subseteq L^q(\mu) \), then clearly \( L^t(\mu) \subseteq L^t(\mu) \) for all \( t > 0 \). So assume we can assume \( p \geq 1 \), so \( L^p \) and \( L^q \) are normed. The key is the hypothesized set-theoretic inclusion \( L^p \subseteq L^q \) is a continuous linear map. The reason is that, as we saw in the proof of the completeness of \( L^q \), any convergent sequence in \( L^p \) converges (in \( L^q \)) to the pointwise limit of some subsequence (almost everywhere). So the image of \( L^p \) inside \( L^q \) is closed in the \( L^q \) norm. So the Closed Graph Theorem then implies the inclusion is continuous. Hence there is a constant \( k \) such that for every \( f \in L^p \),

\[
\|f\|_p \leq k|\|f\|_q|.
\]

Take \( f = \chi_E \) for a measurable set \( E \) to get

\[
\mu(E)^{1/p} \leq k\mu(E)^{1/q},
\]

and so

\[
\mu(E) \geq k^{-1} \mu(E)^{1/(p+q)}.
\]

So the infimum in (b) must be strictly positive.

(b) implies (c). Given \( f \in L^p \), let \( E_n \) denote the set of \( x \) such that \( |f(x)| < n \). Since \( f \in L^p \), \( \mu(E_n) \) must tend to zero as \( n \) becomes large. By (b), this means there is some \( N \) such that \( \mu(E_n) = 0 \). So \( f \) is actually in \( L^\infty \). So \( f \) is in \( L^q \) too.

(c) trivially implies (a).

There is also a dual assertion (whose proof I leave to you):

**Proposition 2** Let \( \mu \) be a positive measure in \( \mathcal{M} \). Let \( \mathcal{M}_\infty \) denote the sets of finite measure. The the following are equivalent

(a) \( L^p(\mu) \subseteq L^q(\mu) \) for some \( p > q \) in \((0, \infty)\);

(b) \( \sup_{E \in \mathcal{M}_\infty} \mu(E) < \infty \); and

(c) \( L^p(\mu) \subseteq L^q(\mu) \) for all \( p > q \) in \((0, \infty)\);

Note the if \( \mu(X) \) is finite, Proposition 2 applies. If \( \mu \) is discrete, Proposition 1 applies. Neither proposition applies to Lebesque measure on \( \mathbb{R} \).

**Rudin, Chapter 3, Problem #11.** Suppose \( \mu(\Omega) = 1 \), and suppose \( f \) and \( g \) are positive measurable functions on \( \Omega \) such that \( fg \geq 1 \). Then

\[
\int_\Omega f d\mu \cdot \int_\Omega f d\mu \geq 1
\]

Since \( f \) and \( g \) are positive, we can consider their square roots \( \sqrt{f} \) and \( \sqrt{g} \). By the Hölder inequality, we have

\[
||\sqrt{f}||_{2}||\sqrt{g}||_{2} \geq ||\sqrt{fg}||_{1}.
\]

The left-hand side is of course

\[
\sqrt{\int_\Omega f d\mu \cdot \int_\Omega g d\mu}.
\]

By hypothesis \( \sqrt{fg} \geq 1 \), and so the right-hand side is at least

\[
1 \cdot \mu(\Omega) = 1.
\]

Squaring both sides gives the desired result.
**Circle Problem.** Let $S^1 = \{ e^{i\theta} \mid 0 \leq \theta < 2\pi \}$. For each $n \in \mathbb{Z}$, define a map $\chi_n : S^1 \to \mathbb{C}^\times$ via
\[ \chi_n(e^{i\theta}) = e^{i n \theta}. \]

(0) Prove that $\chi_n$ is a continuous homomorphism from the multiplicative group $S^1$ to the multiplicative group $\mathbb{C}^\times$.

(1) Suppose $\chi$ is any continuous homomorphism from the multiplicative group $S^1$ to the multiplicative group $\mathbb{C}^\times$. Prove that there exists an $n$ such that $\chi = \chi_n$.

(2) Suppose $\chi$ is a continuous homomorphism from $S^1$ to $GL(N, \mathbb{C})$ so that $\chi$ admits no invariant subspaces in the following sense: if $V$ is a subspace of $\mathbb{C}^N$ such that
\[ \chi(x)(v) \in V \text{ for all } x \in S^1 \text{ and } v \in V, \]
then $V = \{0\}$ or $V = \mathbb{C}^N$. Prove that $N = 1$.

Hence the maps $\chi_n$ are precisely the set of continuous homomorphism from $S^1$ to $GL(N, \mathbb{C})$ so that $\chi$ admits no invariant subspaces.

**Solution.** (0) is trivial. For (1), let $H_m$ denote the subgroup of $S^1$ consisting of $m$th roots of unity. Since $\chi$ is a homomorphism, it is clear that for each $m$ there exists an integer $0 \leq n_m \leq m - 1$ (possibly depending on $m$) so that $\chi(z) = z^{n_m}$ for all $z \in H_m$. Let $z_m$ denote a choice of generator for $H_m$. For primes $p \neq q$, we know we can take $z_{pq} = z_p z_q$. So we compute in the group $H_{pq}$,
\[ z_p^{n_p} z_q^{n_q} = z_{pq}^{n_{pq}} = \chi(z_{pq}) = \chi(z_p z_q) = \chi(z_p) \chi(z_q) = z_p^{n_p} z_q^{n_q}. \]

In other words
\[ z_p^{n_p - n_q} = z_q^{n_q - n_{pq}}. \]

Since $H_p$ and $H_q$ intersect only in \{1\},
\[ n_{pq} = n_p \text{ modulo } p \]
and
\[ n_{pq} = n_q \text{ modulo } q. \]

Thus $n_p = n_q \mod p$ (as well as mod $q$). So indeed for all primes $p$, $n_p$ is a constant $n$ (independent of $p$). Since all prime roots of unity are dense in $S^1$, the claim follows.

For (2), write $A_\theta$ for $\chi(e^{i\theta})$; so $A_\theta$ is an $N \times N$ matrix. Since $A_\theta$ cannot be the zero matrix, it has a nonzero eigenvalue. Pick one and call it $\lambda_\theta$. Consider the matrix $B_\theta = A_\theta - \lambda_\theta \text{Id}$. Since $\chi$ is a homomorphism and $S^1$ is abelian, it’s clear that $B_\theta$ commutes with all $A_\phi$. Hence the kernel $K$ of $B$ is an invariant subspace of $\mathbb{C}^N$. By hypothesis, this means that $K$ is either $\{0\}$ or all of $\mathbb{C}^N$. The former case is impossible since any eigenvector corresponding to $\lambda_\theta$ is in the kernel. Thus $K$ must be all of $\mathbb{C}^N$. So $B_\theta = 0$. In other words $A_\theta$ is the (nonzero) constant multiple $\lambda_\theta$ of the identity. Thus any subspace of $\mathbb{C}^N$ is invariant. This is a contradiction unless $N = 1$. \qed