This book is dedicated in affectionate memory of my Mother and Father

K. J. FALCONER
Lecturer in Mathematics, University of Bristol
Sometime Fellow of Corpus Christi College, Cambridge

The geometry of fractal sets

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
London New York New Rochelle
Melbourne Sydney

Measures etc.

$\mathcal{H}^s$	s-dimensional Hausdorff measure or outer measure.
$\mathcal{L}^n$	n-dimensional Lebesgue measure.
$\mathcal{M}^s$	s-dimensional comparable net measure.
$\mathcal{H}^s_\delta, \mathcal{M}^s_\delta$	δ-outer measures used in constructing $\mathcal{H}^s$ and $\mathcal{M}^s$.
$\mathcal{L}(\Gamma)$	length of the curve Γ.
$\dim E$	Hausdorff dimension of E.
ϕ_t, C_t, I_t	t-potential, capacity, energy.

Densities

$D^s(E, x)$	density of E at x.
$\underline{D}^s(E, x)$, $\bar{D}^s(E, x)$,	lower, upper densities.
$\bar{D}^s_c(E, x)$	upper convex density.
$\underline{D}^s(E, x, \theta, \phi)$	lower angular density, etc.

1
Measure and dimension

1.1 Basic measure theory

This section contains a condensed account of the basic measure theory we require. More complete treatments may be found in Kingman & Taylor (1966) or Rogers (1970).

Let X be any set. (We shall shortly take X to be n-dimensional Euclidean space $\mathbb{R}^n$.) A non-empty collection $\mathcal{S}$ of subsets of X is termed a *sigma-field* (or *σ-field*) if $\mathcal{S}$ is closed under complementation and under countable union (so if $E \in \mathcal{S}$, then $X \backslash E \in \mathcal{S}$ and if $E_1, E_2, \ldots \in \mathcal{S}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{S}$). A little elementary set theory shows that a σ-field is also closed under countable intersection and under set difference and, further, that X and the null set $\varnothing$ are in $\mathcal{S}$.

The *lower* and *upper limits* of a sequence of sets $\{E_j\}$ are defined as

$$\varliminf_{j \to \infty} E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j$$

and

$$\varlimsup_{j \to \infty} E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.$$

Thus $\varliminf E_j$ consists of those points lying in all but finitely many E_j, and $\varlimsup E_j$ consists of those points in infinitely many E_j. From the form of these definitions it is clear that if E_j lies in the σ-field $\mathcal{S}$ for each j, then $\varliminf E_j$, $\varlimsup E_j \in \mathcal{S}$. If $\varliminf E_j = \varlimsup E_j$, then we write $\lim E_j$ for the common value; this always happens if $\{E_j\}$ is either an increasing or a decreasing sequence of sets.

Let $\mathcal{C}$ be any collection of subsets of X. Then the σ-field *generated by* $\mathcal{C}$, written $\mathcal{S}(\mathcal{C})$, is the intersection of all σ-fields containing $\mathcal{C}$. A straightforward check shows that $\mathcal{S}(\mathcal{C})$ is itself a σ-field which may be thought of as the 'smallest' σ-field containing $\mathcal{C}$.

A *measure* μ is a function defined on some σ-field $\mathcal{S}$ of subsets of X and taking values in the range $[0, \infty]$ such that

$$\mu(\varnothing) = 0 \tag{1.1}$$

and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) \tag{1.2}$$

for every countable sequence of disjoint sets $\{E_j\}$ in $\mathscr{S}$.

It follows from (1.2) that μ is an increasing set function, that is, if $E \subset E'$ and $E, E' \in \mathscr{S}$, then

$$\mu(E) \le \mu(E').$$

Theorem 1.1 (continuity of measures)
Let μ be a measure on a σ-field $\mathscr{S}$ of subsets of X.
(a) *If $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of sets in $\mathscr{S}$, then*

$$\mu(\lim_{j\to\infty} E_j) = \lim_{j\to\infty} \mu(E_j).$$

(b) *If $F_1 \supset F_2 \supset \ldots$ is a decreasing sequence of sets in $\mathscr{S}$ and $\mu(F_1) < \infty$, then*

$$\mu(\lim_{j\to\infty} F_j) = \lim_{j\to\infty} \mu(F_j).$$

(c) *For any sequence of sets $\{F_j\}$ in $\mathscr{S}$,*

$$\mu(\underline{\lim}_{j\to\infty} F_j) \le \underline{\lim}_{j\to\infty} \mu(F_j).$$

Proof. (a) We may express $\bigcup_{j=1}^{\infty} E_j$ as the disjoint union $E_1 \cup \bigcup_{j=2}^{\infty}(E_j \backslash E_{j-1})$. Thus by (1.2),

$$\begin{aligned}\mu(\lim_{j\to\infty} E_j) &= \mu\left(\bigcup_{j=1}^{\infty} E_j\right)\\ &= \mu(E_1) + \sum_2^{\infty} \mu(E_j \backslash E_{j-1})\\ &= \lim_{k\to\infty}\left[\mu(E_1) + \sum_2^{k} \mu(E_j \backslash E_{j-1})\right]\\ &= \lim_{k\to\infty} \mu\left(E_1 \cup \bigcup_2^{k}(E_j \backslash E_{j-1})\right)\\ &= \lim_{k\to\infty} \mu(E_k).\end{aligned}$$

(b) If $E_j = F_1 \backslash F_j$, then $\{E_j\}$ is as in (a). Since $\bigcap_j F_j = F_1 \backslash \bigcup_j E_j$,

$$\begin{aligned}\mu(\lim_{j\to\infty} F_j) &= \mu\left(\bigcap_{j=1}^{\infty} F_j\right)\\ &= \mu(F_1) - \mu\left(\bigcup_j E_j\right)\\ &= \mu(F_1) - \lim_{j\to\infty} \mu(E_j)\end{aligned}$$

$$\begin{aligned}&= \lim_{j\to\infty} (\mu(F_1) - \mu(E_j))\\ &= \lim_{j\to\infty} \mu(F_j).\end{aligned}$$

(c) Now let $E_k = \bigcap_{j=k}^{\infty} F_j$. Then $\{E_k\}$ is an increasing sequence of sets in $\mathscr{S}$, so by (a),

$$\mu(\underline{\lim}_{j\to\infty} F_j) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k) \le \underline{\lim}_{j\to\infty} \mu(F_j). \qquad \square$$

Next we introduce outer measures which are essentially measures with property (1.2) weakened to subadditivity. Formally, an *outer measure* ν on a set X is a function defined on *all* subsets of X taking values in $[0, \infty]$ such that

$$\nu(\varnothing) = 0, \tag{1.3}$$

$$\nu(A) \le \nu(A') \quad \text{if } A \subset A' \tag{1.4}$$

and

$$\nu\left(\bigcup_1^{\infty} A_j\right) \le \sum_1^{\infty} \nu(A_j) \quad \text{for any subsets } \{A_j\} \text{ of } X. \tag{1.5}$$

Outer measures are useful since there is always a σ-field of subsets on which they behave as measures; for reasonably defined outer measures this σ-field can be quite large.

A subset E of X is called *ν-measurable* or *measurable with respect to the outer measure ν* if it decomposes every subset of X additively, that is, if

$$\nu(A) = \nu(A \cap E) + \nu(A \backslash E) \tag{1.6}$$

for all 'test sets' $A \subset X$. Note that to show that a set E is ν-measurable, it is enough to check that

$$\nu(A) \ge \nu(A \cap E) + \nu(A \backslash E), \tag{1.7}$$

since the opposite inequality is included in (1.5). It is trivial to verify that if $\nu(E) = 0$, then E is ν-measurable.

Theorem 1.2
Let ν be an outer measure. The collection $\mathscr{M}$ of ν-measurable sets forms a σ-field, and the restriction of ν to $\mathscr{M}$ is a measure.

Proof. Clearly, $\varnothing \in \mathscr{M}$, so $\mathscr{M}$ is non-empty. Also, by the symmetry of (1.6), $A \in \mathscr{M}$ if and only if $X \backslash A \in \mathscr{M}$. Hence $\mathscr{M}$ is closed under taking complements.

To prove that $\mathscr{M}$ is closed under countable union, suppose that $E_1, E_2, \ldots \in \mathscr{M}$ and let A be any set. Then applying (1.6) to $E_1, E_2, \ldots$ in turn with appropriate test sets,

$$\begin{aligned}\nu(A) &= \nu(A\cap E_1) + \nu(A\backslash E_1)\\ &= \nu(A\cap E_1) + \nu((A\backslash E_1)\cap E_2) + \nu(A\backslash E_1\backslash E_2)\\ &= \cdots\\ &= \sum_{j=1}^{k} \nu\left(\left(A\backslash\bigcup_{i=1}^{j-1} E_i\right)\cap E_j\right) + \nu\left(A\backslash\bigcup_{j=1}^{k} E_j\right).\end{aligned}$$

Hence

$$\nu(A) \geq \sum_{j=1}^{k} \nu\left(\left(A\backslash\bigcup_{i=1}^{j-1} E_i\right)\cap E_j\right) + \nu\left(A\backslash\bigcup_{j=1}^{\infty} E_j\right)$$

for all k and so

$$\nu(A) \geq \sum_{j=1}^{\infty} \nu\left(\left(A\backslash\bigcup_{i=1}^{j-1} E_i\right)\cap E_j\right) + \nu\left(A\backslash\bigcup_{j=1}^{\infty} E_j\right). \tag{1.8}$$

On the other hand,

$$A\cap\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty}\left(\left(A\backslash\bigcup_{i=1}^{j-1} E_i\right)\cap E_j\right),$$

so, using (1.5),

$$\begin{aligned}\nu(A) &\leq \nu\left(A\cap\bigcup_{j=1}^{\infty} E_j\right) + \nu\left(A\backslash\bigcup_{j=1}^{\infty} E_j\right)\\ &\leq \sum_{j=1}^{\infty} \nu\left(\left(A\backslash\bigcup_{i=1}^{j-1} E_i\right)\cap E_j\right) + \nu\left(A\backslash\bigcup_{j=1}^{\infty} E_j\right) \leq \nu(A),\end{aligned}$$

by (1.8). It follows that $\bigcup_{j=1}^{\infty} E_j \in \mathscr{M}$, so $\mathscr{M}$ is a σ-field.

Now let $E_1, E_2, \ldots$ be *disjoint* sets of $\mathscr{M}$. Taking $A = \bigcup_{j=1}^{\infty} E_j$ in (1.8),

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) \geq \sum_{j=1}^{\infty} \nu(E_j)$$

and combining this with (1.5) we see that ν is a measure on $\mathscr{M}$. □

We say that the outer measure ν is *regular* if for every set A there is a ν-measurable set E containing A with $\nu(A) = \nu(E)$.

Lemma 1.3

If ν is a regular outer measure and $\{A_j\}$ is any increasing sequence of sets,

$$\lim_{j\to\infty} \nu(A_j) = \nu(\lim_{j\to\infty} A_j).$$

Proof. Choose a ν-measurable E_j with $E_j \supset A_j$ and $\nu(E_j) = \nu(A_j)$ for each j. Then, using (1.4) and Theorem 1.1(c),

$$\nu(\lim A_j) = \nu(\underline{\lim}\, A_j) \leq \nu(\underline{\lim}\, E_j) \leq \underline{\lim}\, \nu(E_j) = \lim \nu(A_j).$$

The opposite inequality follows from (1.4). □

Now let (X, d) be a metric space. (For our purposes X will usually be n-dimensional Euclidean space, $\mathbb{R}^n$, with d the usual distance function.) The sets belonging to the σ-field generated by the closed subsets of X are called the *Borel sets* of the space. The Borel sets include the open sets (as complements of the closed sets), the *F_σ-sets* (that is, countable unions of closed sets), the *G_δ-sets* (countable intersections of open sets), etc.

An outer measure ν on X is termed a *metric outer measure* if

$$\nu(E\cup F) = \nu(E) + \nu(F) \tag{1.9}$$

whenever E and F are *positively separated*, that is, whenever

$$d(E, F) = \inf\{d(x, y): x\in E, y\in F\} > 0.$$

We show that if ν is a metric outer measure, then the collection of ν-measurable sets includes the Borel sets. The proof is based on the following version of 'Carathéodory's lemma'.

Lemma 1.4

Let ν be a metric outer measure on (X, d). Let $\{A_j\}_1^\infty$ be an increasing sequence of subsets of X with $A = \lim_{j\to\infty} A_j$, and suppose that $d(A_j, A\backslash A_{j+1}) > 0$ for each j. Then $\nu(A) = \lim_{j\to\infty} \nu(A_j)$.

Proof. It is enough to prove that

$$\nu(A) \leq \lim_{j\to\infty} \nu(A_j), \tag{1.10}$$

since the opposite inequality follows from (1.4). Let $B_1 = A_1$ and $B_j = A_j\backslash A_{j-1}$ for $j \geq 2$. If $j + 2 \leq i$, then $B_j \subset A_j$ and $B_i \subset A\backslash A_{i-1} \subset A\backslash A_{j+1}$, so B_i and B_j are positively separated. Thus, applying (1.9) $(m-1)$ times,

$$\nu\left(\bigcup_{k=1}^{m} B_{2k-1}\right) = \sum_{k=1}^{m} \nu(B_{2k-1}),$$

$$\nu\left(\bigcup_{k=1}^{m} B_{2k}\right) = \sum_{k=1}^{m} \nu(B_{2k}).$$

We may assume that both these series converge – if not we would have $\lim_{j\to\infty} \nu(A_j) = \infty$, since $\bigcup_{k=1}^{m} B_{2k-1}$ and $\bigcup_{k=1}^{m} B_{2k}$ are both contained in A_{2m}. Hence

$$\begin{aligned}\nu(A) = \nu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \nu\left(A_j \cup \bigcup_{k=j+1}^{\infty} B_k\right)\\ &\leq \nu(A_j) + \sum_{k=j+1}^{\infty} \nu(B_k)\end{aligned}$$

$$\leq \lim_{i\to\infty} \nu(A_i) + \sum_{k=j+1}^{\infty} \nu(B_k).$$

Since the sum tends to 0 as $j\to\infty$, (1.10) follows. □

Theorem 1.5

If ν is a metric outer measure on (X,d), then all Borel subsets of X are ν-measurable.

Proof. Since the ν-measurable sets form a σ-field, and the Borel sets form the smallest σ-field containing the closed subsets of X, it is enough to show that (1.7) holds when E is closed and A is arbitrary.

Let A_j be the set of points in $A\backslash E$ at a distance at least $1/j$ from E. Then $d(A\cap E, A_j)\geq 1/j$, so

$$\nu(A\cap E) + \nu(A_j) = \nu((A\cap E)\cup A_j) \leq \nu(A) \tag{1.11}$$

for each j, as ν is a metric outer measure. The sequence of sets $\{A_j\}$ is increasing and, since E is closed, $A\backslash E = \bigcup_{j=1}^{\infty} A_j$. Hence, provided that $d(A_j, A\backslash E\backslash A_{j+1}) > 0$ for all j, Lemma 1.4 gives $\nu(A\backslash E)\leq \lim_{j\to\infty}\nu(A_j)$ and (1.7) follows from (1.11). But if $x\in A\backslash E\backslash A_{j+1}$ there exists $z\in E$ with $d(x,z) < 1/(j+1)$, so if $y\in A_j$ then $d(x,y)\geq d(y,z) - d(x,z) > 1/j - 1/(j+1) > 0$. Thus $d(A_j, A\backslash E\backslash A_{j+1}) > 0$, as required. □

There is another important class of sets which, unlike the Borel sets, are defined explicitly in terms of unions and intersections of closed sets. If (X,d) is a metric space, the *Souslin sets* are the sets of the form

$$E = \bigcup_{i_1 i_2 \ldots} \bigcap_{k=1}^{\infty} E_{i_1 i_2 \ldots i_k},$$

where $E_{i_1 i_2 \ldots i_k}$ is a closed set for each finite sequence $\{i_1, i_2, \ldots, i_k\}$ of positive integers. Note that, although E is built up from a countable collection of closed sets, the union is over continuum-many infinite sequences of integers. (Each closed set appears in the expression in many places.)

It may be shown that every Borel set is a Souslin set and that, if the underlying metric spaces are complete, then any continuous image of a Souslin set is Souslin. Further, if ν is an outer measure on a metric space (X,d), then the Souslin sets are ν-measurable provided that the closed sets are ν-measurable. It follows from Theorem 1.5 that if ν is a metric outer measure on (X,d), then the Souslin sets are ν-measurable. We shall only make passing reference to Souslin sets. Measure-theoretic aspects are described in greater detail by Rogers (1970), and the connoisseur might also consult Rogers *et al.* (1980).

1.2 Hausdorff measure

For the remainder of this book we work in Euclidean n-space, $\mathbb{R}^n$, although it should be emphasized that much of what is said is valid in a general metric space setting.

If U is a non-empty subset of $\mathbb{R}^n$ we define the *diameter* of U as $|U| = \sup\{|x-y| : x, y\in U\}$. If $E\subset \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of E.

Let E be a subset of $\mathbb{R}^n$ and let s be a non-negative number. For $\delta > 0$ define

$$\mathscr{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s, \tag{1.12}$$

where the infimum is over all (countable) δ-covers $\{U_i\}$ of E. A trivial check establishes that $\mathscr{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$.

To get the *Hausdorff s-dimensional outer measure* of E we let $\delta\to 0$. Thus

$$\mathscr{H}^s(E) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(E) = \sup_{\delta>0} \mathscr{H}^s_\delta(E). \tag{1.13}$$

The limit exists, but may be infinite, since $\mathscr{H}^s_\delta$ increases as δ decreases. $\mathscr{H}^s$ is easily seen to be an outer measure, but it is also a *metric* outer measure. For if δ is less than the distance between positively separated sets E and F, no set in a δ-cover of $E\cup F$ can intersect both E and F, so that

$$\mathscr{H}^s_\delta(E\cup F) = \mathscr{H}^s_\delta(E) + \mathscr{H}^s_\delta(F),$$

leading to a similar equality for $\mathscr{H}^s$. The restriction of $\mathscr{H}^s$ to the σ-field of $\mathscr{H}^s$-measurable sets, which by Theorem 1.5 includes the Borel sets (and, indeed, the Souslin sets) is called *Hausdorff s-dimensional measure*.

Note that an equivalent definition of Hausdorff measure is obtained if the infimum in (1.12) is taken over δ-covers of E by convex sets rather than by arbitrary sets since any set lies in a convex set of the same diameter. Similarly, it is sometimes convenient to consider δ-covers of open, or alternatively of closed, sets. In each case, although a different value of $\mathscr{H}^s_\delta$ may be obtained for $\delta > 0$, the value of the limit $\mathscr{H}^s$ is the same, see Davies (1956). (If however, the infimum is taken over δ-covers by balls, a different measure is obtained; Besicovitch (1928*a*, Chapter 3) compares such 'spherical Hausdorff measures' with Hausdorff measures.)

For any E it is clear that $\mathscr{H}^s(E)$ is non-increasing as s increases from 0 to ∞. Furthermore, if $s < t$, then

$$\mathscr{H}^s_\delta(E) \geq \delta^{s-t}\mathscr{H}^t_\delta(E),$$

which implies that if $\mathscr{H}^t(E)$ is positive, then $\mathscr{H}^s(E)$ is infinite. Thus there is a unique value, $\dim E$, called the *Hausdorff dimension* of E, such that

$$\mathscr{H}^s(E) = \infty \text{ if } 0\leq s < \dim E, \quad \mathscr{H}^s(E) = 0 \text{ if } \dim E < s < \infty. \tag{1.14}$$

If C is a cube of unit side in $\mathbb{R}^n$, then by dividing C into k^n subcubes of side $1/k$ in the obvious way, we see that if $\delta \geq k^{-1}n^{\frac{1}{2}}$ then $\mathscr{H}^n_\delta(C) \leq k^n(k^{-1}n^{\frac{1}{2}})^n \leq n^{\frac{1}{2}n}$, so that $\mathscr{H}^n(C) < \infty$. Thus if $s > n$, then $\mathscr{H}^s(C) = 0$ and $\mathscr{H}^s(\mathbb{R}^n) = 0$, since $\mathbb{R}^n$ is expressible as a countable union of such cubes. It follows that $0 \leq \dim E \leq n$ for any $E \subset \mathbb{R}^n$. It is also clear that if $E \subset E'$ then $\dim E \leq \dim E'$.

An $\mathscr{H}^s$-measurable set $E \subset \mathbb{R}^n$ for which $0 < \mathscr{H}^s(E) < \infty$ is termed an *s-set*; a 1-set is sometimes called a *linearly measurable set*. Clearly, the Hausdorff dimension of an s-set equals s, but it is important to realize that an s-set is something much more specific than a measurable set of Hausdorff dimension s. Indeed, Besicovitch (1942) shows that any set can be expressed as a disjoint union of continuum-many sets of the same dimension. Most of this book is devoted to studying the geometric properties of s-sets.

The definition of Hausdorff measure may be generalized by replacing $|U_i|^s$ in (1.12) by $h(|U_i|)$, where h is some positive function, increasing and continuous on the right. Many of our results have direct analogues for these more general measures, though sometimes at the expense of algebraic simplicity. The Hausdorff 'dimension' of a set E may then be identified more precisely as a partition of the functions which measure E as zero or infinity (see Rogers (1970)). Some progress is even possible if $|U_i|^s$ is replaced by $h(U_i)$, where h is simply a function of the set U_i (see Davies (1969) and Davies & Samuels (1974)).

We next prove that $\mathscr{H}^s$ is a regular measure, together with the useful consequence that we may approximate to s-sets from below by closed subsets. This proof is given by Besicovitch (1938) who also demonstrates (1954) the necessity of the finiteness condition in Theorem 1.6(b).

Theorem 1.6

(a) *If E is any subset of $\mathbb{R}^n$ there is a G_δ-set G containing E with $\mathscr{H}^s(G) = \mathscr{H}^s(E)$. In particular, $\mathscr{H}^s$ is a regular outer measure.*

(b) *Any $\mathscr{H}^s$-measurable set of finite $\mathscr{H}^s$-measure contains an F_σ-set of equal measure, and so contains a closed set differing from it by arbitrarily small measure.*

Proof. (a) If $\mathscr{H}^s(E) = \infty$, then $\mathbb{R}^n$ is an open set of equal measure, so suppose that $\mathscr{H}^s(E) < \infty$. For each $i = 1, 2 \ldots$ choose an *open* $2/i$-cover of E, $\{U_{ij}\}_j$, such that

$$\sum_{j=1}^{\infty} |U_{ij}|^s < \mathscr{H}^s_{1/i}(E) + 1/i.$$

Then $E \subset G$, where $G = \bigcap_{i=1}^\infty \bigcup_{j=1}^\infty U_{ij}$ is a G_δ-set. Since $\{U_{ij}\}_j$ is a $2/i$-cover of G, $\mathscr{H}^s_{2/i}(G) \leq \mathscr{H}^s_{1/i}(E) + 1/i$, and it follows on letting $i \to \infty$ that $\mathscr{H}^s(E) = \mathscr{H}^s(G)$. Since G_δ-sets are $\mathscr{H}^s$-measurable, $\mathscr{H}^s$ is a regular outer measure.

(b) Let E be $\mathscr{H}^s$-measurable with $\mathscr{H}^s(E) < \infty$. Using (a) we may find open sets $O_1, O_2, \ldots$ containing E, with $\mathscr{H}^s(\bigcap_{i=1}^\infty O_i \backslash E) = \mathscr{H}^s(\bigcap_{i=1}^\infty O_i) - \mathscr{H}^s(E) = 0$. Any open subset of $\mathbb{R}^n$ is an F_σ-set, so suppose $O_i = \bigcup_{j=1}^\infty F_{ij}$ for each i, where $\{F_{ij}\}_j$ is an increasing sequence of closed sets. Then by continuity of $\mathscr{H}^s$,

$$\lim_{j\to\infty} \mathscr{H}^s(E \cap F_{ij}) = \mathscr{H}^s(E \cap O_i) = \mathscr{H}^s(E).$$

Hence, given $\varepsilon > 0$, we may find j_i such that

$$\mathscr{H}^s(E \backslash F_{ij_i}) < 2^{-i}\varepsilon \qquad (i = 1, 2, \ldots).$$

If F is the closed set $\bigcap_{i=1}^\infty F_{ij_i}$, then

$$\mathscr{H}^s(F) \geq \mathscr{H}^s(E \cap F) \geq \mathscr{H}^s(E) - \sum_{i=1}^{\infty} \mathscr{H}^s(E \backslash F_{ij_i}) > \mathscr{H}^s(E) - \varepsilon.$$

Since $F \subset \bigcap_{i=1}^\infty O_i$, then $\mathscr{H}^s(F \backslash E) \leq \mathscr{H}^s(\bigcap_{i=1}^\infty O_i \backslash E) = 0$. By (a) $F \backslash E$ is contained in some G_δ-set G with $\mathscr{H}^s(G) = 0$. Thus $F \backslash G$ is an F_σ-set contained in E with

$$\mathscr{H}^s(F \backslash G) \geq \mathscr{H}^s(F) - \mathscr{H}^s(G) > \mathscr{H}^s(E) - \varepsilon.$$

Taking a countable union of such F_σ-sets over $\varepsilon = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ gives an F_σ-set contained in E and of equal measure to E. □

The next lemma states that any attempt to estimate the Hausdorff measure of a set using a cover of sufficiently small sets gives an answer not much smaller than the actual Hausdorff measure.

Lemma 1.7

Let E be $\mathscr{H}^s$-measurable with $\mathscr{H}^s(E) < \infty$, and let ε be positive. Then there exists $\rho > 0$, dependent only on E and ε, such that for any collection of Borel sets $\{U_i\}_{i=1}^\infty$ with $0 < |U_i| \leq \rho$ we have

$$\mathscr{H}^s(E \cap \bigcup_i U_i) < \sum_i |U_i|^s + \varepsilon.$$

Proof. From the definition of $\mathscr{H}^s$ as the limit of $\mathscr{H}^s_\delta$ as $\delta \to 0$, we may choose ρ such that

$$\mathscr{H}^s(E) < \sum |W_i|^s + \tfrac{1}{2}\varepsilon \tag{1.15}$$

for any ρ-cover $\{W_i\}$ of E. Given Borel sets $\{U_i\}$ with $0 < |U_i| \leq \rho$, we may find a ρ-cover $\{V_i\}$ of $E \backslash \bigcup_i U_i$ such that

$$\mathscr{H}^s\left(E \backslash \bigcup_i U_i\right) + \tfrac{1}{2}\varepsilon > \sum |V_i|^s.$$

Since $\{U_i\} \cup \{V_i\}$ is then a ρ-cover of E,

$$\mathscr{H}^s(E) < \sum |U_i|^s + \sum |V_i|^s + \tfrac{1}{2}\varepsilon,$$

by (1.15). Hence

$$\mathscr{H}^s\Big(E\cap\bigcup_i U_i\Big)=\mathscr{H}^s(E)-\mathscr{H}^s\Big(E\backslash\bigcup_i U_i\Big)$$
$$<\sum|U_i|^s+\sum|V_i|^s+\tfrac{1}{2}\varepsilon-\sum|V_i|^s+\tfrac{1}{2}\varepsilon$$
$$=\sum|U_i|^s+\varepsilon. \qquad \square$$

Finally in this section, we prove a simple lemma on the measure of sets related by a 'uniformly Lipschitz' mapping

Lemma 1.8
Let $\psi:E\to F$ be a surjective mapping such that

$$|\psi(x)-\psi(y)|\le|x-y| \qquad (x,y\in E)$$

for a constant c. Then $\mathscr{H}^s(F)\le c^s\mathscr{H}^s(E)$.

Proof. For each i, $|\psi(U_i\cap E)|\le c|U_i|$. Thus if $\{U_i\}$ is a δ-cover of E, then $\{\psi(U_i\cap E)\}$ is a $c\delta$-cover of F. Also $\sum_i|\psi(U_i\cap E)|^s\le c^s\sum_i|U_i|^s$ so that $\mathscr{H}^s_{c\delta}(F)\le c^s\mathscr{H}^s_\delta(E)$, and the result follows on letting $\delta\to 0$. $\square$

1.3 Covering results

The Vitali covering theorem is one of the most useful tools of geometric measure theory. Given a 'sufficiently large' collection of sets that cover some set E, the Vitali theorem selects a *disjoint* subcollection that covers almost all of E.

We include the following lemma at this point because it illustrates the basic principle embodied in the proof of Vitali's result, but in a simplified and finite setting. A collection of sets is termed *semidisjoint* if no member of the collection is contained in any different member.

Lemma 1.9
Let $\mathscr{C}$ be a collection of balls contained in a bounded subset of $\mathbb{R}^n$. Then we may find a finite or countably infinite disjoint subcollection $\{B_i\}$ such that

$$\bigcup_{B\in\mathscr{C}}B\subset\bigcup_i B_i', \tag{1.16}$$

where B_i' is the ball concentric with B_i and of three times the radius. Further, we may take the collection $\{B_i'\}$ to be semidisjoint.

Proof. We select the $\{B_i\}$ inductively. Let B_1 be the largest ball in $\mathscr{C}$ (or one of the largest if there is more than one of equal diameter). If $B_1,\ldots,B_m$ have been chosen, let B_{m+1} be one of the largest balls in $\mathscr{C}$ that does not intersect $\bigcup_1^m B_i$. The process terminates if no such ball remains. Certainly, these balls are disjoint; we claim that they also have the required covering property. If $B\in\mathscr{C}$, then either $B=B_i$ for some i, or B intersects some B_i with $|B_i|\ge|B|$. If this was not the case B would have been selected in preference to the first ball B_m for which $|B_m|<|B|$. (Note that, by summing volumes, $\sum|B_i|^2<\infty$ so that $|B_i|\to 0$ as $i\to\infty$ if infinitely many balls are selected.) In either case, $B\subset B_i'$, giving (1.16). To get the $\{B_i'\}$ semidisjoint, simply remove B_i from the subcollection if $B_i'\subset B_j'$ for any $j\ne i$ noting that B_i' can only be contained in finitely many B_j'. $\square$

A collection of sets $\mathscr{V}$ is called a *Vitali class* for E if for each $x\in E$ and $\delta>0$ there exists $U\in\mathscr{V}$ with $x\in U$ and $0<|U|\le\delta$.

Theorem 1.10 (Vitali covering theorem)
(a) *Let E be an $\mathscr{H}^s$-measurable subset of $\mathbb{R}^n$ and let $\mathscr{V}$ be a Vitali class of closed sets for E. Then we may select a (finite or countable) disjoint sequence $\{U_i\}$ from $\mathscr{V}$ such that either $\sum|U_i|^s=\infty$ or $\mathscr{H}^s(E\backslash\bigcup_i U_i)=0$.*
(b) *If $\mathscr{H}^s(E)<\infty$, then, given $\varepsilon>0$, we may also require that*

$$\mathscr{H}^s(E)\le\sum_i|U_i|^s+\varepsilon.$$

Proof. Fix $\rho>0$; we may assume that $|U|\le\rho$ for all $U\in\mathscr{V}$. We choose the $\{U_i\}$ inductively. Let U_1 be any member of $\mathscr{V}$. Suppose that $U_1,\ldots,U_m$ have been chosen, and let d_m be the supremum of $|U|$ taken over those U in $\mathscr{V}$ which do not intersect $U_1,\ldots,U_m$. If $d_m=0$, then $E\subset\bigcup_1^m U_i$ so that (a) follows and the process terminates. Otherwise let U_{m+1} be a set in $\mathscr{V}$ disjoint from $\bigcup_1^m U_i$ such that

$$|U_{m+1}|\ge\tfrac{1}{2}d_m.$$

Suppose that the process continues indefinitely and that $\sum|U_i|^s<\infty$. For each i let B_i be a ball with centre in U_i and with radius $3|U_i|$. We claim that for every k

$$E\backslash\bigcup_1^k U_i\subset\bigcup_{k+1}^\infty B_i. \tag{1.17}$$

For if $x\in E\backslash\bigcup_1^k U_i$ there exists $U\in\mathscr{V}$ not intersecting $U_1,\ldots,U_k$ with $x\in U$. Since $|U_i|\to 0$, $|U|>2|U_m|$ for some m. By virtue of the method of selection of $\{U_i\}$, U must intersect U_i for some i with $k<i<m$ for which $|U|\le 2|U_i|$. By elementary geometry $U\subset B_i$, so (1.17) follows. Thus if $\delta>0$,

$$\mathscr{H}^s_\delta\Big(E\backslash\bigcup_1^\infty U_i\Big)\le\mathscr{H}^s_\delta\Big(E\backslash\bigcup_1^k U_i\Big)\le\sum_{k+1}^\infty|B_i|^s=6^s\sum_{k+1}^\infty|U_i|^s,$$

provided k is large enough to ensure that $|B_i|\le\delta$ for $i>k$. Hence $\mathscr{H}^s_\delta(E\backslash\bigcup_1^\infty U_i)=0$ for all $\delta>0$, so $\mathscr{H}^s(E\backslash\bigcup_1^\infty U_i)=0$, which proves (a).

To get (b), we may suppose that ρ chosen at the beginning of the proof is the number corresponding to ε and E given by Lemma 1.7. If $\sum|U_i|^s=\infty$, then (b) is obvious. Otherwise, by (a) and Lemma 1.7,

$$\begin{aligned}\mathcal{H}^s(E) &= \mathcal{H}^s(E\setminus\bigcup_i U_i) + \mathcal{H}^s(E\cap\bigcup_i U_i)\\ &= 0 + \mathcal{H}^s(E\cap\bigcup_i U_i)\\ &< \sum|U_i|^s + \varepsilon. \qquad \square\end{aligned}$$

Covering theorems are studied extensively in their own right, and are of particular importance in harmonic analysis, as well as in geometric measure theory. Results for very general classes of sets and measures are described in the two books by de Guzmán (1975, 1981) which also contain further references. One approach to covering principles is due to Besicovitch (1945*a*, 1946, 1947); the first of these papers includes applications to densities such as described in Section 2.2 of this book.

1.4 Lebesgue measure

We obtain n-dimensional Lebesgue measure as an extension of the usual definition of the volume in $\mathbb{R}^n$ (we take 'volume' to mean length in $\mathbb{R}^1$ and area in $\mathbb{R}^2$).

Let C be a coordinate block in $\mathbb{R}^n$ of the form

$$C = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

where $a_i < b_i$ for each i. Define the volume of C as

$$V(C) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$$

in the obvious way. If $E \subset \mathbb{R}^n$ let

$$\mathcal{L}^n(E) = \inf \sum_i V(C_i), \tag{1.18}$$

where the infimum is taken over all coverings of E by a sequence $\{C_i\}$ of blocks. It is easy to see that $\mathcal{L}^n$ is an outer measure on $\mathbb{R}^n$, known as *Lebesgue n-dimensional outer measure*. Further, $\mathcal{L}^n(E)$ coincides with the volume of E if E is any block; this follows by approximating the sum in (1.18) by a finite sum and then by subdividing E by the planes containing the faces of the C_i. Since any block C_i may be decomposed into small subblocks leaving the sum in (1.18) unaltered, it is enough to take the infimum over δ-covers of E for any $\delta > 0$. Thus $\mathcal{L}^n$ is a metric outer measure on $\mathbb{R}^n$. The restriction of $\mathcal{L}^n$ to the $\mathcal{L}^n$-measurable sets or *Lebesgue-measurable* sets, which, by Theorem 1.5, include the Borel sets, is called *Lebesgue n-dimensional measure* or *volume*.

Clearly, the definitions of $\mathcal{L}^1$ and $\mathcal{H}^1$ on $\mathbb{R}^1$ coincide. As might be expected, the outer measures $\mathcal{L}^n$ and $\mathcal{H}^n$ on $\mathbb{R}^n$ are related if $n > 1$, in fact

they differ only by a constant multiple. To show this we require the following well-known geometric result, the 'isodiametric inequality', which says that the set of maximal volume of a given diameter is a sphere. Proofs, using symmetrization or other methods, may be found in any text on convexity, e.g. Eggleston (1958), see also Exercise 1.6.

Theorem 1.11

The n-dimensional volume of a closed convex set of diameter d is, at most, $\pi^{\frac{1}{2}n}(\frac{1}{2}d)^n/(\frac{1}{2}n)!$, *the volume of a ball of diameter d.*

Theorem 1.12

If $E \subset \mathbb{R}^n$, *then* $\mathcal{L}^n(E) = c_n\mathcal{H}^n(E)$, *where* $c_n = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2}n)!$. *In particular,* $c_1 = 1$ *and* $c_2 = \pi/4$.

Proof. Given $\varepsilon > 0$ we may cover E by a collection of closed convex sets $\{U_i\}$ such that $\sum|U_i|^n < \mathcal{H}^n(E) + \varepsilon$. By Theorem 1.11 $\mathcal{L}^n(U_i) \le c_n|U_i|^n$, so $\mathcal{L}^n(E) \le \sum\mathcal{L}^n(U_i) < c_n\mathcal{H}^n(E) + c_n\varepsilon$, giving $\mathcal{L}^n(E) \le c_n\mathcal{H}^n(E)$.

Conversely, let $\{C_i\}$ be a collection of coordinate blocks covering E with

$$\sum_i V(C_i) < \mathcal{L}^n(E) + \varepsilon. \tag{1.19}$$

We may suppose these blocks to be open by expanding them slightly whilst retaining this inequality. For each i the closed balls contained in C_i of radius, at most, δ form a Vitali class for C_i. By the Vitali covering theorem, Theorem 1.10(a), there exist disjoint balls $\{B_{ij}\}_j$ in C_i of diameter, at most, δ, with $\mathcal{H}^n(C_i\setminus\bigcup_{j=1}^\infty B_{ij}) = 0$ and so with $\mathcal{H}^n_\delta(C_i\setminus\bigcup_{j=1}^\infty B_{ij}) = 0$. Since $\mathcal{L}^n$ is a Borel measure, $\sum_{j=1}^\infty \mathcal{L}^n(B_{ij}) = \mathcal{L}^n(\bigcup_{j=1}^\infty B_{ij}) \le \mathcal{L}^n(C_i)$. Thus

$$\begin{aligned}\mathcal{H}^n_\delta(E) \le \sum_{i=1}^\infty \mathcal{H}^n_\delta(C_i) &\le \sum_{i=1}^\infty\sum_{j=1}^\infty \mathcal{H}^n_\delta(B_{ij}) + \sum_{i=1}^\infty \mathcal{H}^n_\delta\Big(C_i\setminus\bigcup_{j=1}^\infty B_{ij}\Big)\\ &\le \sum_{i=1}^\infty\sum_{j=1}^\infty |B_{ij}|^n = \sum_{i=1}^\infty\sum_{j=1}^\infty c_n^{-1}\mathcal{L}^n(B_{ij})\\ &\le c_n^{-1}\sum_{i=1}^\infty \mathcal{L}^n(C_i) < c_n^{-1}\mathcal{L}^n(E) + c_n^{-1}\varepsilon,\end{aligned}$$

by (1.19). Thus $c_n\mathcal{H}^n_\delta(E) \le \mathcal{L}^n(E) + \varepsilon$ for all ε and δ, giving $c_n\mathcal{H}^n_\delta(E) \le \mathcal{L}^n(E)$. $\square$

One of the classical results in the theory of Lebesgue measure is the Lebesgue density theorem. Much of our later work stems from attempts to formulate such a theorem for Hausdorff measures. The reader may care to furnish a proof as an exercise in the use of the Vitali covering theorem. Alternatively, the theorem is a simple consequence of Theorem 2.2.