MATH 6210: PROBLEM SET #2

extra problem

1. Let $S^1 = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$. For each $n \in \mathbb{Z}$, define a map $\chi_n : S^1 \to \mathbb{C}^\times$ via
$$\chi_n(e^{i\theta}) = e^{in\theta}.$$
   (0) Prove that χ_n is a continuous homorphism from the multiplicative group S^1 to the multiplicative group $\mathbb{C}^\times$.
   (1) Suppose χ is any continuous homorphism from the multiplicative group S^1 to the multiplicative group $\mathbb{C}^\times$. Prove that there exists an n such that $\chi = \chi_n$.
   (2) Suppose χ is a continuous homomorphism from S^1 to $\mathrm{G}L(N, \mathbb{C})$ so that χ admits no invariant subspaces in the following sense: if V is a subspace of $\mathbb{C}^N$ such that
   $$[\chi(x)](v) \in V \text{ for all } x \in S^1 \text{ and } v \in V,$$
   then $V = \{0\}$ or $V = \mathbb{C}^N$. Prove that $N = 1$.

Hence the maps χ_n are precisely the set of continuous homomorphism from S^1 to $\mathrm{G}L(N, \mathbb{C})$ so that χ admits no invariant subspaces.