PRACTICE FINAL EXAM

December 7, 2001

There are twelve questions on the practice exam. Calculators are allowed.

1. (10) True or False:
   (a) Every continuous function is differentiable.
   (b) Every differentiable function is continuous.
   (c) Profits are maximized when marginal revenue is zero.
   (d) If revenue is maximized then marginal revenue is zero.
   (e) If f and g are differentiable functions, then $(fg)' = (f')(g')$.

2. (5) Carefully state the Fundamental Theorem of Calculus.

3. (30) A monopoly has a total cost function $C = 500 + 2x^2 + 15x$ for its product, which has a demand function $p = \frac{1}{3}x^2 - 2x + 30$.
   (a) Find the sales quantity that maximizes the monopoly's profits. Make sure to verify that the quantity you find is indeed a maximum.
   (b) Find the consumer surplus at the point at the point found in (a). (Even if you cannot solve (a), you can still receive full credit on (b).)

4. (25) Let

$$f(x) = \ln(1 + x^2 y) + \frac{x}{y}.$$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial^2 f}{\partial x \partial y}$.

5. (30) The income from an oil well is modeled as a continuous stream with an annual flow rate (in millions of dollars) at time t given by

$$f(t) = 12e^{-0.4(t+3)}.$$

   (a) How much total income will the oil well generate if it is operated indefinitely?
   (b) Assume money is worth 10% annually in the short term. Find the value of the next five years of production in present value dollars.

6. (20) The profit from the sales of two products is given by

$$P(x, y) = 20x + 70y - x^2 - y^2.$$

Find all critical values of the profit function. Determine whether each critical point is a maximum, a minimum, or neither.

7. (15) Consider the function

$$f(x) = \frac{x^2 - 4x - 5}{x^3 - 11x^2 + 35x - 25}.$$

Determine if the following limits exist. If they do exist, compute the limit.
   (a) $\lim_{x \to 5} f(x)$
   (b) $\lim_{x \to \infty} f(x)$

8. (15) Compute the following integrals.
   (a) $\int (3x^2 + 4)(x^3 + 4x)^{10} dx$
   (b) $\int \frac{x}{x^2+4} dx$

9. (25) Find the minimum value of $w = x^2+y^2+z^2$ subject to the constraint $2x-4y+z = 21$.
10. (15) Find the tangent line to the curve

$$f(x) = x^3 + 2x + 1$$

at $x = 1$.
11. (15) Suppose the marginal revenue for a product is observed to obey the following equation

$$\overline{MR} = 6(x+1)^2 + 12.$$

Find the revenue as a function of x.
12. (10) Consider the function

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 7x + 3.$$

(a) Compute $f''(x)$.
(b) For what values of x is $f(x)$ concave up?
(c) Does $f(x)$ have any points of inflection? If so, what are they?

Important problems not covered on the practice final:

Applications of maxima and minima (§10.4)
Elasticity of demand; Taxation (§11.5)
Graphing problems (like those on the first two exams)
Implicit Differentiation (§11.3)

Solutions to practice final

1.

(a) F
(b) T
(c) F
(d) T
(e) F

2. Page 934 in the book.

3.

(a) Since it is a monopoly, the company (not the market) sets the price. Hence profits are given by $P(x) = pq - C(x)$, or

$$P(x) = x^3/3 - 2x^2 + 30x - 500 - 2x^2 - 15x = x^3/3 - 4x^2 + 15x - 500.$$

To maximize P, compute P' and set it equal to 0:

$$P'(x) = x^2 - 8x + 15 = (x-5)(x-3) = 0.$$

Thus the critical points are $x = 5$ and $x = 3$. Compute $f''(x)$ to test if each is a maximum, a minimum, or a plateau. Since $f''(x) = 2x - 8$ and $f''(3) < 0$ while $f''(5) > 0$, we conclude that the sale of 3 units maximizes profits.

(b) From (a), we can compute the equilibrium price as

$$p_{\text{eq}} = 3^2/3 - 2(3) + 30 = 27.$$

The consumer surplus is given by

$$\int_0^3 \big((x^2/3 - 2x + 30) - 27\big)\, dx.$$

The indefinite integral is

$$x^3/9 - x^2 + 3x;$$

plugging in the limits gives the final answer of 3.

4.

$$\frac{\partial f}{\partial x} = \frac{2xy}{1+x^2y} + \frac{1}{y}$$
$$\frac{\partial f}{\partial y} = \frac{x^2}{1+x^2y} - \frac{x}{y^2}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x}{(1+x^2y)^2} - \frac{1}{y^2}$$

5. (a) The total value is given by

$$\int_0^\infty 12e^{-0.4(t+3)}\, dt = 9.04 \text{ million dollars.}$$

(b) The present value over the next five years is given by

$$\int_0^5 12e^{-0.4(t+3)}e^{-0.1t}dt = \frac{12}{0.5}e^{-1.2}(1-e^{-2.5}) = 6.64 \text{ million.}$$

6. The only critical point is $(x,y) = (10,35)$. At the critical point, we check that $D = z_{xx}z_{yy} - (z_{xy})^2 = (-2)(-2) - 0 = 4 > 0$. Since $z_{xx} = -2 > 0$, we conclude that the critical point is a relative maximum.

7.

(a) Factor to get

$$f(x) = \frac{(x-5)(x+1)}{(x-5)^2(x-1)} = \frac{(x+1)}{(x-5)(x-1)}.$$

Hence the limit does not exist.

(b) The limit is 0.

8.

(a) $(x^3+4x)^{11}/11 + C$

(b) $\frac{\ln|x^2+4|}{2} + C$

9. Set $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ where $g(x,y) = 2x - 4y + z - 21$. Compute

$$\frac{\partial F}{\partial x} = 2x + 2\lambda$$
$$\frac{\partial F}{\partial y} = 2y - 4\lambda$$
$$\frac{\partial F}{\partial z} = 2z + \lambda$$

Setting these equal to zero, we get

$$x = -\lambda$$
$$y = 2\lambda$$
$$z = -\lambda/2.$$

Plug this into $g(x,y,z) = 0$ to get

$$-2\lambda - 8\lambda - \lambda/2 = 21;$$

so $\lambda = -2$. Thus $(x,y,z) = (2,-4,1)$ is a critical point. One must check values close to this one to conclude it indeed is a minimum.

10. Slope of tangent line is $f'(1) = 5$. So $y = 5x + b$. The line passes through $(1, f(1)) = (1,4)$. So we can plug this point in and solve for b to get $b = -1$. The answer is thus $y = 5x - 1$.

11. The problem says

$$R'(x) = 6(x+1)^2 + 12.$$

Integrating we see

$$R(x) = 2(x+1)^3 + 12x + c.$$

But R is a revenue function, so $R(0) = 0$. This implies $c = -2$. The final answer is thus $R(x) = 2(x+1)^3 + 12x - 2$.

12.
(a) $f''(x) = 2x - 8$.
(b) If $x > 4$, then $f''(x) > 0$, so f is concave up.
(c) Yes; $x = 4$.