Positive and null recurrent-Branching Process

Pejman Mahboubi

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In last discussion we studied the transience and recurrence of Markov chains.

There are 2 other closely related issues about Markov chains that we address:

- Is there an invariant distribution?
- Does the chain converge to this distribution? (Asymptotic behavior)

For irreducible finite MC, if \( R \) is an aperiodic recurrent class, then there is

- an invariant dist. \( \bar{\pi} \) such that \( \bar{\pi}(x) > 0 \) for all \( x \) [p. 16, (1.9)]
- \( p_n(x, y) \to \bar{\pi}(y) \) as \( n \to \infty \).
- \( \bar{\pi} \) is unique (by irreducibility).
- But for the countable MC, the case is different.
- For example for the Simple Random Walk (d=1) we have
  - \( p_{2n} = \frac{(2n)!}{n!n!2^{2n}}, \quad p_{2n+1} = 0. \) Therefore:
    - \( \sum_{n=1}^{\infty} p_n(0,0) = \infty, \) then it is recurrent (1 recurrent class)
    - \( \lim p_n(0,0) = 0 \) (p. 47), then the invariant dist. does not exist, why? bc, then \( \bar{\pi} = \vec{0}, \) which is not a probability dist.
- Therefore we must have \( \lim_{n \to \infty} p_n(y,x) > 0. \)
- This condition is not possible for the transient chains because it contradicts \( \sum p_n < \infty \)
- But it is possible for recurrent chains to have \( \lim p_n(x,y) > 0. \)
- We say a MC is positive recurrent if \( \lim_{n \to \infty} p_n(x,y) > 0 \)
- Remember that \( \lim_{n \to \infty} p_n(x,y) > 0 \) implies \( \sum p_n(x,y) = \infty. \)
- A MC which is not positive recurrent is called null recurrent.
Positive recurrent MC behave similar to the finite MC’s. For example:

Assume a countable MC \( \{X_n\} \) is aperiodic, and irreducible. If \( X_n \) is positive recurrent with \( \bar{\pi}(x) = \lim_{n \to \infty} p_n(y, x) \), then

- \( \bar{\pi}(x) \) is a probability dist. i.e \( \sum_x \bar{\pi}(x) = 1 \).
- \( \bar{\pi}(x) \) is an invariant dist. i.e if \( P(X_0 = x) = \bar{\pi}(x), \forall x \), then \( P(X_1 = x) = \bar{\pi}(x) \forall x \). OR equivalently

\[
\bar{\pi}(x) = \sum_y \bar{\pi}(y)p(y, x)
\]

(Next page we repeat this fact in a stronger form)

Remember that in the finite case, we discussed the return time \( T \) defined by

\[
T_x = \inf\{n \geq 1 : X_n = x\}
\]

If \( X_n \) is transient then \( P^x(T_x = \infty) > 0 \) (bc by definition \( \rho_{xx} < 1 \))

Therefore if \( X_n \) is transient then \( E^xT_x = \infty \).
If $X_n$ is recurrent, then

$$\begin{cases} 
E^{x} T_x = \frac{1}{\bar{\pi}(x)} < \infty & \text{positive recurrent} \\
\infty & \text{null recurrent}
\end{cases}$$

Consider the irreducible and aperiodic MC. The following 3 are equivalent

- State $x$ is positive recurrent ($\forall y : \lim_{n \to \infty} p_n(y, x) = \bar{\pi}(x) > 0$)
- There is an invariant dist. $\bar{\pi}(x) > \bar{0}$: $\bar{\pi}(x) = \sum_y \bar{\pi}(y)p(y, x)$
- All states are positive recurrent

This theorem shows that being positive recurrent is a class property
Branching Process

- Assume a population starts with $n$ individuals at time 0: $X_0 = n$
- Between time $n$ and $n + 1$, each individual gives birth to a random number of children $Y_1, Y_1, \ldots, Y_n$, and die. Furthermore, $\{Y_k\}_{k=1}^n$ are iid.
- $Y_n$ are the number of children, then they are integer-valued
- let $p_0 = P(Y_i = 0), p_1 = P(Y_i = 1), p_2 = P(Y_i = 2) \ldots$.
- Let $X_n$ denote the population in time $n$. Then

$$X_n = Y_1 + \cdots + Y_{X_{n-1}}$$

- If we know the population at $n$, then the MC holds:

$$P(X_{n+1} = j | X_n = j, X_{n-1} = i_1, \ldots, X_0 = 1) = P(X_{n+1} = j | X_n = j)$$

- indeed we can compute the r-h-s

$$p(k, j) := P(X_{n+1} = j | X_n = j) = P(Y_1 + \cdots + Y_k = j)$$
Therefore, \( p(k,j) \) is the probability of having \( k \) individuals in state \( n+1 \), given we have \( k \) individuals in state \( n \).

We can calculate a few things about the expectations:

- Let \( \mu = EY_i = \sum_{j=1}^{\infty} jp_j \).
- The average number of individuals at time \( n \)
- The conditional average is easy to compute

\[
E(X_n|X_{n-1} = k) = E(Y_1 + \cdots + Y_k) = kEY_1 = k\mu
\]

\[
EX_n = \sum_{k=0}^{\infty} E(X_n|X_{n-1} = k)P(X_{n-1} = k) = \mu \sum_{k=0}^{\infty} kP(X_{n-1} = k) = \mu EX_{n-1}
\]

- By iteration we get \( EX_n = \mu^n EX_0 \)
- Remember that in the Branching process \( \{0\} \) is recurrent.
Consider a few paths of the Branching process.
Two important sets

Define:

\[ A_n := \{ \omega : X_n(\omega) = 0 \} . \]

- When \( X_n = 0 \), then \( X_{n+k} = 0 \) for all \( k \geq 1 \).
- \( A_n \) are increasing: \( A_n \subset A_{n+1} \)

Define:

\[ \mathcal{A} := \{ \omega : \omega \text{ dies out} \} = \{ X_n = 0 \text{ for some } n \} \]

Then we have

\[ \mathcal{A} = \bigcup_{n=1}^{\infty} A_n \quad (1) \]

Next, we find a way to compute \( P(\mathcal{A}) \) in terms of \( P(A_n) \).
Here we state an important fact about the continuity of $P$

**Theorem**

Assume $A_n$ are increasing events: $A_n \subset A_{n+1}$. Then

$$P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} P(A_n)$$

Therefore,

$$P(\mathcal{A}) := P(\text{population dies out}) = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} P(A_n) \quad (2)$$
Theorem ($\mu < 1$)

If $\mu < 1$, then the populations dies out with probability one:

$$P(X_n = 0 \text{ for some } n) = 1$$

Proof.

1. $P(X_n \geq 1) = \sum_{k=1}^{\infty} P(X_n = k) \leq \sum_{k=1}^{\infty} kP(X_n = k) = EX_n = \mu^n EX_0$

2. Note that $\{X_n \geq 1\} = A_n^c$. Therefore

$$P(A_n) = 1 - P(A_n^c) \geq 1 - \mu^n EX_0$$

3. Therefore

$$1 \leq P(A_n) \leq 1 - \mu^n EX_0$$

4. Let $n \to \infty$. Using the squeeze theorem $\lim P(A_n) = 1$. Then (2) finishes the proof.
When $\mu > 1$, then $E X_n = \mu^n \to \infty$. Is it possible that $X_n \to 0$:

$$
\lim_{n \to \infty} P(X_n = 0) > 0?
$$

**Example**

Let $X_n \in \{0, 2^n\}$. $P(X_n = 0) = 1 - \frac{1}{n}$, and $P(X_n = 2^n) = \frac{1}{n}$. Then

- $P(X_n = 0) \to 1$
- $E X_n = 2^n \frac{1}{n} \to \infty$.

Therefore, we have to be careful with the situation:

- The case that $p_0 = 0$ is trivially survives with probability 1
- If $p_0 > 0$, but $p_0 + p_1 = 1$, then $\mu := E X_n < 1$
- Therefore, the only nontrivial case is

$$
p_0 > 0; p_0 + p_1 < 1
$$
Assume $p_0 > 0$, and $p_0 + p_1 < 1$. Furthermore, let

$$a_n(k) := P(A_n | X_0 = k) \text{ and } a(k) := P(\mathcal{A} | X_0 = k)$$

Then

$$a(k) := P^k(\mathcal{A}) = \lim_{n \to \infty} P^k(A_n) = \lim_{n \to \infty} a_n(k) \quad (3)$$

$a_n(k)$ is the probability of extinction, given we started with $k$ individuals

but these $k$ individuals form $k$ independent Branching processes, each starting from one individuals

Therefore, Probability of extinction of a B-P starting with $k$ individuals is equal to the extinction of $k$ independent B-P each starting with one individual, ie,

$$a(k) = [a(1)]^k = [\lim_{n \to \infty} a_n(1)]^k$$

Let $a := a(1)$. 

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Fixed Point Property of \( a \)

- Define the moment generating function of \( X \) by \( \varphi_X(s) = E s^X \).
- Starting from 1, after taking one step we will have \( k \) children with probability \( p_k \):

\[
a = \sum_{k=0}^{\infty} P^1(\mathcal{A}|X_1 = k) p_k = p_0 + \sum_{k=1}^{\infty} P^1(\mathcal{A}|X_1 = k) p_k = p_0 + \sum_{k=1}^{\infty} a(k) p_k = \sum_{k=0}^{\infty} p_k a^k = E a^{X_1}
\]

- The last equality holds because when \( X_0 = 1 \), then \( X_1 = Y_1 \)

\[
a = E a^{X_1} \quad \text{or} \quad a = \varphi_{X_1}(a) \quad (4)
\]

- Therefore, \( a \) is a fixed point of the mgf \( \varphi \)
But equation (4) is not enough for finding $a := P(\text{population dies out})$

This is because $\varphi_{X_1}(\alpha) = \alpha$ might have more than one solution. Let us define

$$\varphi^n(s) := \varphi_{X_n}(s), \quad n \geq 1$$

i.e $\phi^n$ denotes the generating function $\phi_{X_n}$ of $X_n$

where $\{X_n\}$ is the B-P : $\varphi^n(s) := \mathbb{E}s^{X_n}$

Write the definition for $\mathbb{E}s^{X_n}$ to see that

$$a_n = P(A_n) := P(X_n = 0) = \varphi^n(0). \quad (5)$$

If we work a bit harder we find that $\varphi_{X_n}(s)$ is $\varphi_{X_1}$ evaluated at $\varphi^{n-1}(s)$

$$\varphi^n(s) := \varphi_{X_n}(s) = \varphi(\varphi^{X_{n-1}}(s)) = : \varphi(\varphi^{n-1}(s))$$
We claim \( a := P^1(A) \) is the smallest root of \( \varphi(s) = s \):

**Theorem**

\( a := P^1(A) \) is the smallest root of \( \varphi(s) = s \).

We know that \( a = \lim_{n \to \infty} a_n \). We also know that

\[
a_n = \varphi^n(0) = \varphi(\varphi^{n-1}(0)) = \varphi(a_{n-1}).
\]

We prove this theorem by induction. Let \( \hat{a} \) be the smallest root of \( \varphi(s) = s \). We want to show by induction that \( a_n \leq \hat{a} \) for all \( n \). Because then we can take a limit and show that \( a \leq \hat{a} \), which forces \( a = \hat{a} \). \( a_0 = 0 \leq \hat{a} \). Next assume \( a_{n-1} \leq \hat{a} \). Then

\[
a_n = \varphi(a_{n-1}) \leq \varphi(\hat{a}) = \hat{a}
\]

The inequality follows from the fact that \( \phi \) is an increasing function.
here we state some of the properties of $\varphi$. Most of them are easy to check

**Lemma**

*For the positive random variable $X$, let $\varphi(s) := \varphi_X(s)$*

- $\varphi(s)$ is increasing on $[0, \infty)$
- $\varphi(0) = p_0 = P(X = 0)$
- $\varphi'(1) = EX$

- Since $p_0 + p_1 < 1$ by assumption ($\mu \geq 1$), then

$$\varphi''(s) \geq 0$$

- If $X_1, \ldots, X_n$ are iid, then

$$\varphi_{X_1+\ldots+X_n}(s) = \varphi_{X_1}(s) \times \cdots \times \varphi_{X_n}(s)$$
Lemma

Let $\varphi^n(s) := \varphi_{X_n}(s)$. Then

$$\varphi^n(s) = \varphi(\varphi^{n-1}(s))$$

Proof.

$$\varphi^n(s) := E^1 s^{X_n} = \sum_{j=0}^{\infty} P^1(X_n = j) s^j = \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} P^k(X_{n-1} = j) P^1(X_1 = k) =$$

$$\sum_{k=0}^{\infty} p_k \sum_{j=0}^{\infty} P^k(X_{n-1} = j) s^j = \sum_{k=0}^{\infty} p_k [\varphi^{n-1}(s)]^k = \varphi(\varphi^{n-1}(s))$$

$P^k(X_n = j)$ is the probability of starting from $k$ individual, and having $j$ individual at time $n$. \qed