Research statement

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1 Background and motivations: discrete subgroups and lattices in $\mathrm{PU}(n,1)$

The setting of my research is the still rather unexplored area of discrete groups of isometries of complex hyperbolic space, in particular that of lattices in $\mathrm{PU}(n,1)$. Celebrated work of Margulis (for which he was awarded a Fields medal in 1974) roughly classifies lattices and their representations in all so-called symmetric spaces, except real and complex hyperbolic spaces, whose corresponding (orientation-preserving) isometry groups are $\mathrm{PO}(n,1)$ and $\mathrm{PU}(n,1)$. (Symmetric spaces are in one-to-one correspondance with real semisimple Lie groups). In particular, Margulis' results imply that the only symmetric spaces where non-arithmetic lattices can exist are real and complex hyperbolic spaces. Lattices in these hyperbolic isometry groups (or equivalently, tessellations of the corresponding hyperbolic space) were first investigated by Poincaré in the 1890's and have attracted much attention, with regained interest since celebrated work of Thurston (other Fields medalist) in the 1970's in the 3-dimensional real case (so, for $\mathrm{PO}(2,1) \sim \mathrm{GL}(2,\mathbb{C})$). As opposed to the real hyperbolic case, very little is known about discrete groups and lattices in $\mathrm{PU}(n,1)$, beginning with interesting examples, even in the smallest dimensions. We start by working with $\mathrm{PU}(2,1)$ which is the smallest dimension where almost nothing is known. Shortly after Poincaré, Picard constructed the first examples of lattices in $\mathrm{PU}(2,1)$, some of which turned out to be non-arithmetic. The only other known examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$ were constructed by Mostow in 1980 and Deligne and Mostow in 1986. No new complex hyperbolic lattices have been found since then.

Recall that there exist in the real hyperbolic case reflection groups (see the constructions of Vinberg, Makarov in small dimensions), and that there are examples of non-arithmetic lattices in all dimensions, by the construction of Gromov–Piatetskii-Shapiro. In the case of $\mathrm{PU}(n,1)$, these questions remain wide open, even in the smallest dimensions. This is mostly due to the fact that there are no totally geodesic real hypersurfaces in complex hyperbolic geometry, and in particular no natural notion of polyhedra or reflection groups as in the real hyperbolic (or Euclidean) case. This situation makes it difficult to construct not only discrete subgroups of $\mathrm{PU}(n,1)$, but also fundamental polyhedra for such groups. Possible substitutes in $\mathrm{H}^n_{\mathbb{C}}$ are *complex reflections* (or $\mathbb{C}$-reflections) which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $\mathrm{H}^{n-1}_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$), and *real reflections* (or $\mathbb{R}$-reflections) which are antiholomorphic involutions fixing pointwise a Lagrangian subspace (a copy of $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$). Most known examples of discrete groups are generated by complex reflections, or are arithmetic constructions. The use of $\mathbb{R}$-reflections and of the corresponding Lagrangian planes is very recent (it was introduced by Falbel and Zocca in [FZ] in 1999) and is one of the guiding principles of my research.

2 Summary of past results

- **Fundamental domains for finite groups in U(2)** (joint with Falbel, see [FPau]). We proved that all finite subgroups of U(2) are (contained with index 2 in) real reflection groups.

We characterized the corresponding configurations of Lagrangian planes in $\mathbb{C}^2$ and constructed fundamental domains based on these real planes. Part of the motivation was to understand the finite groups occurring in Mostow's lattices ([Mos1]).

- **New fundamental domains for Mostow's lattices** (joint with Deraux and Falbel, see [DFP]). We constructed new fundamental domains for Mostow's lattices ([Mos1]) which are simpler and allow synthetic geometric arguments to prove the main result (that the groups in question are indeed lattices). Mostow's construction (using Dirichlet, or Voronoi, domains) was notoriously difficult and poorly understood, and the proof resorted to massive computer use.
- **Elliptic triangle groups in PU(2,1)** (see [Pau1]). We define such groups to be generated by 2 elliptic isometries whose product is also elliptic (think of triangle groups in the hyperbolic plane). All known (non-arithmetic) lattices in $\mathrm{PU}(2,1)$ are of this type. Before addressing the difficult question of discreteness (and finite covolume), a first question is to understand which configuartions can occur in these groups, in other words what the possible angle pairs are for the 2 generators and their product. We classified these configurations (and the related configurations of Lagrangian planes).This gave surprising results in terms of convexity of a related momentum map, namely that the image is not always convex (even locally), rather it is the union of at most 3, possibly overlapping, convex polygons.

3 Current projects

3.1 Non-arithmetic lattices generated by higher-order complex reflections

This project is mostly joint work with Parker ([ParPau1], [Pau2], [ParPau2]), and the last part is joint with Deraux ([DP]).

Main goal: to obtain new discrete subgroups and lattices in $\mathrm{PU}(2,1)$, more specifically many new non-arithmetic lattices (possibly infinitely many).

Summary of approach: The questions or problems which arise are the following. First, one needs some kind of method to produce good candidates; a possible approach is given by the decription of configurations from the last part of my thesis (see [Pau0] and [Pau1]). Then, given a group with explicit generators in matrix form, it remains to see if it is (contained in) an arithmetic lattice in $\mathrm{PU}(2,1)$. If this isn't the case, discreteness is a delicate problem. There are several methods to decide whether or not the group in question has indeed a good chance of being discrete, for instance systematically studying all words of a given maximal length (following Schwartz, see [Sz]) or using Deraux's algorithm which effectively implements the Dirichlet method (see [De1], [De2] and [DP]). If these first tests are conclusive, one then tries to construct a fundamental domain, using among others the techniques which were developed in the second part of my thesis (see [Pau0] and [DFP]). Aside from proving discreteness (and possibly finite covolume), this will give us more information on the group, such as a presentation and the Euler characteristic/volume of the quotient orbifold in the lattice case. One last thing to check is if the group is really new, i.e. in the lattice case whether or not it is commensurable to one of the lattices on the lists of Deligne–Mostow, Mostow and Thurston (see [DM], [Mos2] and [Th]).

At the moment, my main project is the study of so-called symmetric triangle groups, which we started with Parker in [ParPau1] (following [Par]) and continued in [Pau2] and [ParPau2]. We study more specifically an infinite subfamily of groups which we call sporadic groups. We answer some of the above questions for these groups in [Pau2] (arithmeticity, commensurability with known lattices) and construct fundamental domains for some of them in [ParPau2] (work in progress). We also find (conjectural) Dirichlet domains numerically for many of these sporadic groups in work in progress with

Deraux ([DP]). This gives us the following conjectural picture for small values of p (to be continued for larger values). The notation $\Gamma(p,\tau)$ is for the group generated by R_1 and J with R_1 a complex reflection through angle $2\pi/p$, J a regular elliptic isometry of order 3, and $\operatorname{Tr} R_1 J = \tau$. The special values σ_i of τ are the so-called sporadic values (see [ParPau1], [Pau2] or longer version of research project for more details).

Conjecture 3.1 *$\Gamma(3,\sigma_1)$, $\Gamma(3,\sigma_5)$, $\Gamma(4,\overline{\sigma}_4)$ and $\Gamma(5,\overline{\sigma}_4)$ are lattices, with only the latter cocompact.*

Note that we know from [Pau2] that these groups are non-arithmetic and not commensurable to previously known lattices (lists of [DM], [Mos2] and [Th]). In the most optimistic case one might even be able to find infifnitely many (non-commensurable) non-arithmetic lattices in this family.

3.2 Geodesic triangles in $\mathrm{H}^2_{\mathbb{C}}$

This section concerns a question of "elementary"geometry in $\mathrm{H}^2_{\mathbb{C}}$ which had arisen at the beginning of my thesis, and which we are now trying to answer with Domingo Toledo.

Consider then a geodesic triangle in $\mathrm{H}^2_{\mathbb{C}}$, i.e. three points (and the geodesic arcs joining them). It is known that such a configuration depends on 4 real parameters (see for instance [B], which gives an explicit decription in terms of the 3 side-lengths and a fourth invariant which Brehm calls the "shape invariant"). We wish to give a description of this configuration space in terms of the 3 (Riemannian) angles between the sides at the vertices, denote them α, β and γ, the goal being to choose the fourth invariant well, in such a way that the inequalities between the 4 parameters become as simple as possible. Our idea is that this fourth invariant should be the integral I of the Kähler form over the triangle (more precisely, over any surface having the given triangle as boundary). The extreme cases are: if the triangle is contained in an $\mathbb{R}$-plane, $I = 0$, and if the triangle is contained in a $\mathbb{C}$-plane, $|I| = \pi - (\alpha + \beta + \gamma)$ (with a certain normalization of the curvature). What we want to say is:

Conjecture 3.2 *For all triangles in $\mathrm{H}^2_{\mathbb{C}}$: $0 \leqslant |I| \leqslant \pi - (\alpha + \beta + \gamma)$.*

Note that when the triangle is ideal (all vertices on the boundary) the result is known (see [DT]). This would then extend easily to give necessary and sufficient conditions on the parameters (the above inequality, as well as $\alpha + \beta + \gamma < \pi$). Unfortunately, we aren't yet able to completely prove this inequality (at least, not in a neighborhood of $\mathbb{C}$-planar configurations), but we have experimental evidence that it should always hold.

4 Projects for a near future

4.1 Reflection groups

Apart from purely loxodromic groups (arising for instance by deformation of Fuchsian groups), the only known discrete groups in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$ are of this type (except for arithmetic lattices and their subgroups). Recall that there are two types of reflections in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, $\mathbb{C}$-reflections which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $\mathrm{H}^{n-1}_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$), and $\mathbb{R}$-reflections which are antiholomorphic isometries fixing pointwise a Lagrangian subspace (a copy of $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$). There are many open problems concerning reflection groups in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, among which one can address the following:

- **Finite reflection groups:**
  Finite $\mathbb{C}$-reflection groups have been classified by Shephard and Todd (see also Broué–Malle–Rouquier). In [FPau] we have proven that all finite subgroups of $\mathrm{U}(2)$ are (of index 2 in a group)

generated by $\mathbb{R}$-reflections. Is this true in $\mathrm{U}(n)$? Probably not. The question is then to classify such subgroups of $\mathrm{U}(n)$. The most important aspect of this project is to find some geometry-to-algebra dictionary for this type of groups (analogous to root systems and Coxeter diagrams for real reflection groups). This is the focus of current joint work with Falbel.

- **Lattices generated by reflections in higher dimensions:**

  In real hyperbolic space, it is known that lattices generated by reflections only exist in small dimensions. Vinberg has proven that there are no compact Coxeter polyhedra in $\mathrm{H}^n_{\mathbb{R}}$ for $n \geqslant 30$, and Prokhorov has proven that there are no Coxeter polyhedra of finite volume in $\mathrm{H}^n_{\mathbb{R}}$ for $n \geqslant 996$ (the known examples are for $n \leqslant 8$ in ther first case and $n \leqslant 21$ in the second).

  Is the situation analogous in $\mathrm{PU}(n,1)$? The known examples (Deligne–Mostow, Mostow, Allcock) are in dimension $n \leqslant 13$ (in fact, $n \leqslant 9$ except for one of Allcock's examples). An obvious difficulty is that there is no counterpart of root systems or Coxeter polyhedra.

- **Non-arithmetic lattices generated by reflections:**

  Non-arithmetic lattices in $\mathrm{PU}(n,1)$ are only known for $n = 2$ (lattices due to Picard, Mostow, Deligne–Mostow, between 7 and 9 commensurability classes) and $n = 3$ (one non-cocompact example due to Deligne–Mostow). We hope to obtain new non-arithmetic lattices in $\mathrm{PU}(2,1)$ in the families of symmetric $\mathbb{C}$-reflection triangle groups described above. We can hope to apply our methods in dimension 3 and maybe 4, but after that the direct geometric method (construction of fundamental domains) becomes unrealistic. The truly outstanding question in this area would be to find some complex hyperbolic analog of the Gromov–Piatetskii-Shapiro construction.

4.2 Discrete groups generated by regular elliptic motions

We could also explore the general case of groups generated by two regular elliptic motions (i.e. not $\mathbb{C}$-reflections), where the parameter space is much bigger. The problem of finding discrete groups in this parameter space seems difficult, but the methods from the last part of my thesis allow to determine some "good" one-parameter families to investigate. More precisely, if the three angle pairs of the elliptic motions A, B, and AB are prescribed, then there is a one-parameter family of such groups generated by Lagrangian (or real) reflections, and this allows in principle to systematically search such families.

5 Related questions

- Representation spaces of surface groups in $\mathrm{PU}(2,1)$ and complex hyperbolic quasi-Fuchsian groups. Character varieties of 3-manifolds in $\mathrm{PU}(2,1)$. (Toledo, Goldman, Xia, Parker, Schwartz,...)
- Spherical CR structures on 3-manifolds (Schwartz, Falbel)
- Fake projective planes (Mumford, Klingler, Yeung, Prasad)
- Complex hyperbolic structures on moduli spaces of algebraic objects (Allcock–Carlson–Toledo)
- Negatively curved compact Kähler manifolds not covered by the ball (Mostow–Siu, Deraux)
- Mapping class group dynamics on surface group representations in $\mathrm{PU}(n,1)$ (Goldman, Burger–Iozzi–Wienhard)
- Spectrum of the automorphic Laplace-Beltrami operator on arithmetic ball quotients (Francsics–Lax, Sarnak)

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