Research project, long version

Julien Paupert

1 Background and motivations: discrete subgroups and lattices in $\mathrm{PU}(2,1)$

The setting of my research is the still rather unexplored area of discrete groups of isometries of complex hyperbolic space, in particular that of lattices in $\mathrm{PU}(2,1)$. This space is one of the two occurrences (with real hyperbolic space) of a rank 1 symmetric space in which Margulis' super-rigidity result does not apply. But, as opposed to the real hyperbolic case, very little is known about these groups, beginning with interesting examples. Recall that there exist in the real case reflection groups (see the constructions of Vinberg, Makarov in small dimensions), and that there are examples of non-arithmetic lattices in all dimensions, by the construction of Gromov–Piatetskii-Shapiro.

In the case of $\mathrm{PU}(n,1)$, all these questions remain wide open, even in the smallest dimensions. This is mostly due to the fact that there are no totally geodesic real hypersurfaces in complex hyperbolic geometry, and in particular no natural notion of polyhedra or reflection groups as in the real hyperbolic (or Euclidean) case. This situation makes it difficult to construct not only discrete subgroups of $\mathrm{PU}(n,1)$, but also fundamental polyhedra for such groups. Possible substitutes in $\mathrm{H}^n_{\mathbb{C}}$ are *complex reflections* (or $\mathbb{C}$-reflections) which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $\mathrm{H}^{n-1}_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$), and *real reflections* (or $\mathbb{R}$-reflections) which are antiholomorphic involutions fixing pointwise a Lagrangian subspace (a copy of $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$). Most known examples are generated by complex reflections, or are arithmetic constructions. The use of $\mathbb{R}$-reflections and of the corresponding Lagrangian planes is very recent (it was introduced by Falbel and Zocca in [FZ] in 1999) and is one of the guiding principles of my research.

2 Current projects

Currently my main goal is to obtain new discrete subgroups and lattices in $\mathrm{PU}(2,1)$. The questions or problems which arise are the following. First, one needs some kind of method to produce good candidates; a possible approach is given by the decription of configurations from the last part of my thesis (see [Pau0] and [Pau1]). Then, given a group with explicit generators in matrix form, it remains to see if it is (contained in) an arithmetic lattice in $\mathrm{PU}(2,1)$. If this isn't the case, discreteness is a delicate problem. There are several methods to decide whether or not the group in question has indeed a good chance of being discrete, for instance systematically studying all words of a given maximal length (following Schwartz, see [Sz]) or using Deraux's algorithm which effectively implements the Dirichlet method (see [De]). If these first tests are conclusive, one then tries to construct a fundamental domain, using among others the techniques which were developed in the second part of my thesis (see [Pau0], [DFP] and subsection "Discretess and fundamental domains II: without parabolics" below). Aside from proving discreteness (and possibly finite covolume), this will give us more information on the group, such as a presentation and the Euler characteristic/volume of the quotient orbifold in the lattice case. One last thing to check is if the group is really new, i.e. in the lattice case whether or

not it is commensurable to one of the lattices on the lists of Deligne–Mostow, Mostow and Thurston (see [DM], [Mos2] and [Th]).

At the moment, my main project is the study of so-called symmetric triangle groups (see definition below), which we started with John Parker in [ParPau1] and continued in [Pau2] and [ParPau2]. We study more specifically an infinite subfamily of groups which we call sporadic groups (again, see definition below). We answer some of the above questions for these groups in [Pau2] (arithmeticity, commensurability with known lattices) and construct fundamental domains for some of them in [ParPau2] (work in progress).

2.1 Discrete groups generated by higher-order complex reflections

This project is mostly joint work with John Parker ([ParPau1], [Pau2], [ParPau2]).

We consider *symmetric* triangle groups in $\mathrm{PU}(2,1)$, which are generated by 3 complex reflections R_1, R_2 and R_3 of order $p \geqslant 3$, where symmetric means that there exists an isometry J of order 3 such that $R_{j+1} = JR_jJ^{-1}$ (with j mod 3). We are particularly interested in groups where certain given words are elliptic, and give necessary conditions for discreteness for such groups. The main motivation is that these groups are candidates for non-arithmetic lattices.

In [Mos1], Mostow constructed the first examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$. These lattices are generated by 3 complex reflections R_1, R_2 and R_3 which are symmetric in the above sense. In Mostow's examples, the generators R_j have order $p = 3$, 4 or 5. Moreover these groups can be characterized among all symmetric triangle groups as those for which the braid relations $R_iR_jR_i = R_jR_iR_j$ hold. Subsequently Deligne–Mostow and Mostow constructed further non-arithmetic lattices as monodromy groups of certain hypergeometric functions in [DM] and [Mos2] (the lattices from [DM] in dimension 2 were known to Picard who did not consider their arithmetic nature). These lattices are (commensurable with) groups generated by symmetric complex reflections R_j with other values of p; see Mostow [Mos2] and Sauter [Sa]. Since then no new non-arithmetic lattices have been constructed.

2.1.1 Sporadic groups

In this section we recall the setting and main results from[ParPau1]. Our starting point was that groups $\Gamma = \langle R_1, J\rangle$ are parametrized up to conjugacy by $\tau = \mathrm{Tr}(R_1J)$. We denote $\Gamma(\psi,\tau)$ the group generated by a complex reflection R_1 through angle ψ and a regular elliptic isometry J of order 3 such that $\mathrm{Tr}(R_1J) = \tau$. We give explicit generators in the following form:

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{1}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\,\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix} \tag{2}$$

These preserve the Hermitian form: $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where:

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} \\ ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau \\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) \end{bmatrix}. \tag{3}$$

This always produces a subgroup Γ of $\mathrm{GL}(3,\mathbb{C})$, but the signature of H_τ depends on the values of ψ and τ. We determined the corresponding configuration space for τ for all fixed values of $\psi = 2\pi/p$

(see sections 2.4 and 2.6 of [ParPau1]). When Γ preserves a Hermitian form of signature $(2,1)$ we will say that Γ is *hyperbolic*.

We found necessary conditions for these groups to be discrete, under the assumption that certain words in the group are elliptic (or parabolic), and these conditions produced a countable family of groups on top of Mostow's groups (which we expect) and their subgroups. More precisely:

Theorem 2.1 *Let R_1 be a complex reflection of order $p \geqslant 3$ and J a regular elliptic isometry of order 3 in* $\mathrm{PU}(2,1)$. *Suppose that $R_1 J$ is elliptic and $R_1 R_2 = R_1 J R_1 J^{-1}$ is elliptic or parabolic. If the group $\Gamma = \langle R_1, J\rangle$ is discrete then one of the following holds:*

- *Γ is one of Mostow's lattices.*
- *Γ is a subgroup of one of Mostow's lattices.*
- *Γ is one of the sporadic groups described below.*

Mostow's lattices correspond to $\tau = e^{i\phi}$ for some angle ϕ; subgroups of Mostow's lattices to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle ϕ, and sporadic groups (this can be taken as a definition) are those for which τ takes one of the 18 values $\{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$ where the σ_i are given in the following list:

$$\begin{array}{ccc}
\sigma_1 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/4) & \sigma_2 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/5) & \sigma_3 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(2\pi/5) \\
\sigma_4 := e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7} & \sigma_5 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/5) & \sigma_6 := e^{2\pi i/9} + e^{-\pi i/9}\, 2\cos(4\pi/5) \\
\sigma_7 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/7) & \sigma_8 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(4\pi/7) & \sigma_9 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(6\pi/7).
\end{array}$$

Therefore, for each value of $p \geqslant 3$, we have a finite number of new groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \overline{\sigma_i})$ which are hyperbolic. We determined exactly which sporadic groups are hyperbolic (see table in section 3.3 of [ParPau1]); notably these exist for all values of p, and more precisely:

Proposition 2.1 *For $p \geqslant 4$ and $\tau = \sigma_1, \sigma_2, \sigma_3, \overline{\sigma_4}, \sigma_5, \sigma_6, \sigma_7, \overline{\sigma_8}$ or σ_9, $\Gamma(2\pi/p, \tau)$ is hyperbolic.*

In the following paragraphs we summarize what we know about these groups, finishing with the part in progress where we construct a fundamental domain for the first non-arithmetic example.

Figure 1 illustrates the case where R_1 has order 3 (so $p = 3$); the boundary triangle is the configuration polygon studied in the last part of my thesis and [Pau1]. The zig-zag-shaped curve actually has two components, one of them (the bottom half) corresponding to Mostow groups and the other to subgroups of Mostow groups. We have also marked as crosses the 16 sporadic points (2 of the 18 from the above theorem are in fact outside of the configuration space). Note that the only groups known to be discrete in this picture are Mostow's lattices from [Mos1], all located on the bottom horizontal segment (and the corresponding subgroups on the top horizontal segment), as well as the only arithmetic sporadic group (see below).

Figure 2 illustrates the various configuration polygons for $p = 2, 3, ...10$, as well as the sporadic points.

2.1.2 Some non-discrete groups

In [ParPau1] we already eliminated a certain number of candidates. The main argument (as already in [Mos1]) is to consider certain triangle subgroups (hyperbolic, Euclidean or spherical) generated by pairs of complex reflections, and to find nondiscrete such subgroups (see Corollary 4.2, Proposition 4.5 and Corollary 6.4 of [ParPau1]):

Figure 1: Mostow groups and sporadic groups in the configuration triangle for $p = 3$

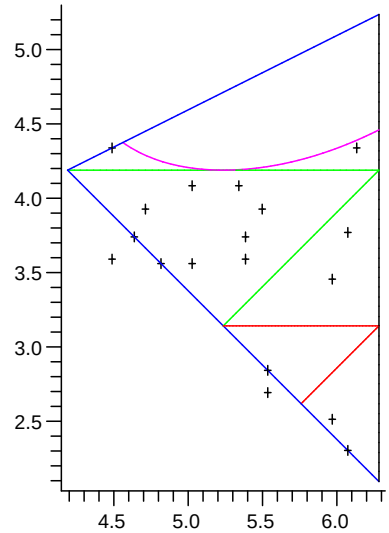
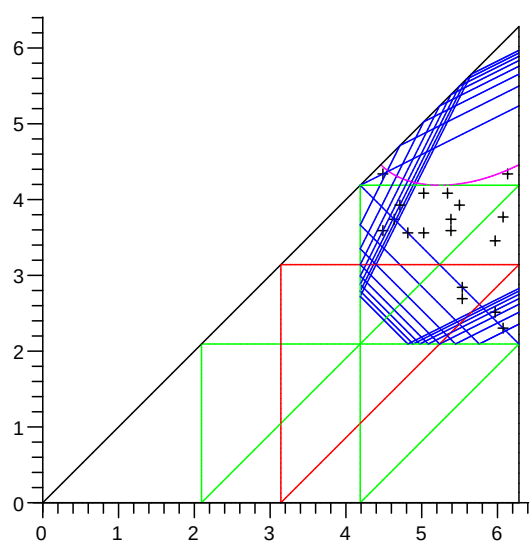


Figure 2: Mostow groups, sporadic groups and configuration polygons for $p = 3, 4, ...10$



Proposition 2.2 *For $p \geqslant 3$ and (τ or $\overline{\tau} = \sigma_3, \sigma_8$ or σ_9), $\Gamma(2\pi/p, \tau)$ is not discrete. Also, for $p \geqslant 3$, $p \neq 5$ and (τ or $\overline{\tau} = \sigma_6$), $\Gamma(2\pi/p, \tau)$ is not discrete.*

I was led to find other non obvious complex reflections in some groups. By construction, some power of $R_1 J$ is a complex reflection (or a complex reflection in a point), except when $\tau = \overline{\sigma_4}$. This allowed me to discard a few more groups (unpublished notes):

Lemma 2.1 *$\Gamma(2\pi/3, \overline{\sigma_1})$, $\Gamma(2\pi/3, \sigma_2)$ and $\Gamma(2\pi/3, \overline{\sigma_2})$ are not discrete.*

This is seen by observing that in the first case $R_1(R_1 J)^4$ is elliptic with infinite order, and in the two other cases $R_1(R_1 J)^5$ is.

2.1.3 (Non)-arithmeticity of sporadic groups

This section and the following one summarize the main results of [Pau2]. These depend in part on determining (or rather estimating) the trace field for the adjoint representation, $\mathbb{Q}[\mathrm{TrAd}\Gamma]$. This field is invariant under commensurability, which is our main tool for the next section, but we also use it to study arithmeticity for these groups.

In [ParPau1] (Propositions 6.5 and 6.6) we proved that only one of the sporadic groups with $p = 3$, namely $\Gamma(2\pi/3, \overline{\sigma_4})$, is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$. In [Pau2] we extended this to higher values of p, and showed that in fact this group is the only such example among all sporadic groups (theorem 4.1 of [Pau2]):

Theorem 2.2 *For $p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$, $\Gamma(2\pi/p, \tau)$ is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$ if and only if $p = 3$ and $\tau = \overline{\sigma_4}$.*

When sudying the arithmetic properties of our groups we conjugate the generators slightly to have coefficients in a nice integer ring (proposition 2.8 of [ParPau1]):

Proposition 2.3 *The maps R_1, R_2 and R_3 may be conjugated within $\mathrm{SU}(2,1)$ and scaled so that their matrix entries lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{\pm i\psi}]$.*

Explicitly, we conjugate the previous matrices by $C = \mathrm{diag}(e^{-i\psi/3},\ 1,\ e^{i\psi/3})$ and rescale by $e^{-i\psi/3}$. Conjugating by C and rescaling by $2\sin(\psi/2)$ also brings H_τ to a Hermitian matrix with entries in the same ring $R = \mathbb{Z}[\tau, \overline{\tau}, e^{\pm i\psi}]$. Therefore, a hyperbolic $\Gamma(\psi, \tau)$ can be realized as a subgroup of $\mathrm{SU}(H, R)$ where H is an $R-$defined Hermitian form of signature $(2,1)$.

We then use the following criterion for arithmeticity (proposition 4.1 of [Pau2]):

Proposition 2.4 *Let E be a purely imaginary quadratic extension of a totally real field F, and H an E-defined Hermitian form of signature (2,1) such that a sporadic group Γ is contained in $\mathrm{SU}(H; \mathcal{O}_E)$. Then Γ is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$ if and only if for all $\varphi \in \mathrm{Gal}(F)$ not inducing the identity on $\mathbb{Q}[\mathrm{TrAd}\Gamma]$, the form ${}^{\varphi}H$ is definite.*

The estimation that we give on the trace field $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ is the following (proposition 3.3 of [Pau2]):

Proposition 2.5 $\mathbb{Q}\left[\cos\psi, |\tau|^2, \mathrm{Re}\tau^3, \mathrm{Re}(e^{-i\psi}\tau^3)\right] \subset \mathbb{Q}[\mathrm{TrAd}\Gamma(\psi, \tau)] \subset \mathbb{Q}\left[\tau, \overline{\tau}, e^{i\psi}\right] \cap \mathbb{R}$.

which follows from Proposition 2.3 and the following formula:

Lemma 2.2 *For $\gamma \in \mathrm{SU}(2,1)$, $\mathrm{TrAd}(\gamma) = |\mathrm{Tr}(\gamma)|^2$.*

2.1.4 Commensurability classes

This estimation of $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ for the sporadic groups Γ is enough in all but a few cases to conclude that they are not commensurable to any previously known lattice (Picard and Mostow lattices, see lists in [DM], [Mos2] and [Th]). See the last section of [Pau2] for more details on the trace fields for these groups. We noted there that all known examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$ belong to between 7 and 9 distinct commensurability classes. As for sporadic groups, we proved the following result (the first statement is very easy, the rest is theorem 5.2 of [Pau2]):

Theorem 2.3 *The sporadic groups $\Gamma(2\pi/p,\tau)$ ($p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$) fall into infinitely many distinct commensurability classes. Moreover, they are not commensurable to any Picard or Mostow lattice, except possibly when:*

- $p = 4$ *or* 6 *(and τ is any sporadic value)*
- $p = 3$ *and* $\tau = \sigma_7$
- $p = 5$ *and* τ *or* $\overline{\tau} = \sigma_1, \sigma_2$
- $p = 7$ *and* $\tau = \overline{\sigma_4}$
- $p = 8$ *and* $\tau = \sigma_1$
- $p = 10$ *and* $\tau = \sigma_1, \sigma_2, \overline{\sigma_2}$
- $p = 12$ *and* $\tau = \sigma_1, \sigma_7$
- $p = 20$ *and* $\tau = \sigma_1, \sigma_2$
- $p = 24$ *and* $\tau = \sigma_1$

2.1.5 Discreteness and fundamental domains I: with parabolics

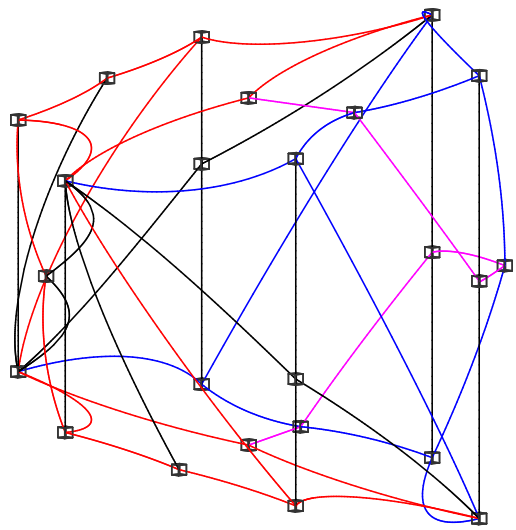
This section concerns work in progress with John Parker ([ParPau2]).

In some cases we have a subgroup generated by two complex reflections whose product is parabolic. More precisely, in $\Gamma(2\pi/3,\sigma_1)$, $\Gamma(2\pi/3,\overline{\sigma_1})$, $\Gamma(2\pi/5,\sigma_3)$ and $\Gamma(2\pi/5,\overline{\sigma_3})$, R_1R_2 is parabolic and in $\Gamma(2\pi/3,\sigma_5)$ $R_2(R_1J)^5$ is parabolic. The latter statement comes again from studying powers of R_1J which are complex reflections (unpublished notes).

Denote q_∞ the fixed point of the parabolic isometry in question, and Γ_∞ the stabilizer of q_∞ in Γ. The idea in these cases is to base our construction of a fundamental domain on a fundamental domain Π_∞ in the boundary $\partial\mathrm{H}^2_{\mathbb{C}}$ (Heisenberg group) for the action of Γ_∞. We will then obtain our fundamental domain on the inside by taking the intersection of a geodesic cone on Π_∞ with the common exterior of certain well-chosen isometric spheres (a Ford domain for some subgroup of Γ). The geodesic cone is taken from q_∞ (vertical geodesic rays in the upper half-space, or Siegel, model).

This idea, for which the naive model is the standard fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper-half plane, was successfully used by Falbel and Parker for the Eisenstein-Picard modular group $\mathrm{SU}(2,1,\mathbb{Z}[e^{2i\pi/3}])$ (see [FPar]). In that case the domain on the boundary Π_∞ was simply a tetrahedron, and the full domain was a 4-simplex (minus a vertex), the intersection of the geodesic cone on the tetrahedron with the exterior of a single isometric sphere (as for $\mathrm{SL}(2,\mathbb{Z})$). In the present case, we are using as fundamental domain on the boundary a hexagonal prism (rather like a drum), and are using the Ford domain for a cyclic subgroup of order 8, i.e. we are intersecting the geodesic cone with the common exterior of 7 isometric spheres (in fact 6, one of them doesn't contribute anything). Note

Figure 3: The core polyhedron, i.e. the union of interesctions of the geodesic cone over Π_∞ with the six relevant isometric spheres. The vertices are arranged along a drum-shaped hexagonal prism, which is the structure of Π_∞. The edges (drawn geometrically, projected to the boundary) come in four flavors: geodesics, intersections of a real plane with a bisector, intersections of a complex line with a bisector, and generic triple bisector intersections.



that a key point is that the fundamental domain on the boundary (the base of the cone) is completely covered by the corresponding isometric spheres (i.e. the intersection is compact in horospheres based at q_∞). This will also guarantee that the group only has one cusp. Figure 3 shows the core part of our domain, which is the union of all intersections of the geodesic cone with each isometric sphere (so that this 3-dimensional picture lives in a union of 6 isometric spheres which are each 3-dimensional hypersurfaces). The combinatorics of this object are a bit complicated, but well understood and so we have side pairings, cycles and a (conjectural) presentation of our new lattice. There are still a few technical steps involved, such as verifying the tiling conditions in Poincaré's polyhedron theorem (a question of time), but we can reasonably say that this construction will work and thus provide our first example of a new non-arithmetic lattice (which would be $\Gamma(2\pi/3, \sigma_1)$).

2.1.6 Discreteness and fundamental domains II: without parabolics

This is a priori the general case (unless all or most of these groups contain surprising parabolic elements as in the case of $\Gamma(2\pi/3, \sigma_5)$ mentioned above). I have tried to adapt the method which we had used in [DFP] for the Mostow lattices, but unfortunately there is so far an obstacle which I will briefly describe.

First of all, consider the point p_{12} which is the intersection point of the mirrors of R_1 and R_2 (in $\mathbb{C}^{2,1}$ or $\mathbb{C}P^2$), and analogously $p_{23} = J(p_{12})$ and $p_{31} = J^2(p_{12})$. (We first find explicit coordinates for these points and determine in which cases they are inside $\mathrm{H}^2_{\mathbb{C}}$). Then we introduce real reflections in the groups, by noting that like in all previously known cases the generators of our group may be decomposed as products of real reflections in the following manner:

$$\begin{cases} J = \sigma_{12}\sigma_{23} \\ R_1 = \sigma_{23}\tau \end{cases} \tag{4}$$

where σ_{ij} acts on $\mathbb{C}^{2,1}$ by the transposition (ij) on coordinates followed by complex conjugation, and τ can be defined by $\tau := \sigma_{23}R_1$ (this requires checking that $\tau^2 = Id$). Geometrically, one then has 3 $\mathbb{R}$-planes L_{12}, L_{23} and L_τ such that $L_{23} \cap L_\tau$ is a geodesic (the "real spine" of bisector $B_1 = B(p_{12}, p_{31})$), $L_{12} \cap L_{23}$ is the isolated fixed point of J, and $L_{12} \cap L_\tau$ is the isolated fixed point of JR_1 (all of these points are in B_1).

Our construction in [DFP] was based on a certain right-angled geodesic hexagon contained in the intersection of the three basic bisectors $B_1 = B(p_{12}, p_{31})$, $B_2 = B(p_{12}, p_{23})$ and $B_3 = B(p_{23}, p_{31})$. (This intersection being generic, i.e. not totally geodesic even the existence of this hexagon was interesting). The six vertices v_{ijk} were obtained in the following way. Denote $\tilde{S}_\eta$ the surface of intersection of the 3 above bisectors (extended to all of $\mathbb{C}P^2$), and $\tilde{\sigma}_i$ the "real spine" (also extended to $\mathbb{C}P^2$) of the bisector B_i. The vertices v_{ijk} are then the intersections of $\tilde{\sigma}_j$ with $\tilde{S}_\eta$. All of this can be reasonably computed, uisng for instance that $\tilde{S}_\eta$ is given by the equations:

$$|\langle x, p_{12}\rangle| = |\langle x, p_{31}\rangle| = |\langle x, p_{23}\rangle|$$

and that $\tilde{\sigma}_1$ for instance is parametrized by $p_{12} - \mu p_{23}$ (with $|\mu| = 1$).

We then observe that we are in the degenerate case where 3 of the vertices of the hexagon are outside of $\mathrm{H}^2_{\mathbb{C}}$, so that the hexagon is in fact a triangle. This provides a tetrahedron in B_1 whose vertices are v_{132}, v_{213}, v_{321} and t_{32}, the last point being the common projection of v_{321} and v_{132} to the mirror of R_1 (this last point is a lemma, saying that v_{321} and v_{132} are in the same complex slice of B_1). This tetrahedron (or its analog) was one of the 3-faces of our polyhedron in [DFP], relying crucially on the fact that $R_1(v_{321}) = v_{132}$ (identifying R_1 as one of the side-pairings). In the present case this is no longer true, and more precisely:

Lemma 2.3 *Consider (a measure $\lambda(A)$ of) the Riemannian angle $\widehat{v_{321}t_{23}v_{132}}$ between geodesic rays. Then: $\lambda(A)$ is an irrational multiple of π for $\tau = \sigma_2, \overline{\sigma_2}, \overline{\sigma_4}, \sigma_5$ and σ_7. For τ or $\overline{\tau} = \sigma_1$, $\lambda(A) = \pi/3$.*

In this last case the angle in question is half of what it would be for the corresponding Mostow group, and this allows us in fact to see that the corresponding sporadic groups are contained with infinite index in a non-discrete Mostow group with $p = 6$.

2.2 Geodesic triangles in $\mathrm{H}^2_{\mathbb{C}}$

This last section concerns a question of "elementary" geometry in $\mathrm{H}^2_{\mathbb{C}}$ which had arisen at the beginning of my thesis, and which we are now trying to answer with Domingo Toledo.

Consider then a geodesic triangle in $\mathrm{H}^2_{\mathbb{C}}$, i.e. three points (and the geodesic arcs joining them). It is known that such a configuration depends on 4 real paremeters (see for instance [B], which gives an explicit decription in terms of the 3 side-lengths and a fourth invariant which Brehm calls the "shape invariant"). We wish to give a description of this configuration space in terms of the 3 (Riemannian) angles between the sides at the vertices, denote them α, β and γ, the goal being to choose the fourth invariant well, in such a way that the inequalities between the 4 parameters become as simple as possible. Our idea is that this fourth invariant should be the integral I of the Kähler form over the triangle (more precisely, over any surface having the given triangle as boundary). The extreme cases are: if the triangle is contained in an $\mathbb{R}$-plane, $I = 0$, and if the triangle is contained in a $\mathbb{C}$-plane, $|I| = \pi - (\alpha + \beta + \gamma)$ (with a certain normalization of the curvature). What we want to say is:

Conjecture 2.1 *For all triangles in $\mathrm{H}^2_{\mathbb{C}}$: $0 \leqslant |I| \leqslant \pi - (\alpha + \beta + \gamma)$.*

Note that when the triangle is ideal (all vertices on the boundary) the result is known (see [DT]). This would then extend easily to give necessary and sufficient conditions on the parameters (the above

inequality, as well as $\alpha + \beta + \gamma < \pi$). Unfortunately, we aren't yet able to completely prove this inequality (at least, not in a neighborhood of $\mathbb{C}$-planar configurations), but we have epxerimental evidence that it should always hold. Some possible approaches:

- geometric: it is known that I is the area of any of the 3 triangles obtained by projecting the original triangle to one of the complex sides (the complex lines containing two vertices). This projection is distance-nonincreasing, but doesn't decrease (all) angles; one then looks for finer arguments using for instance the law of sines in $\mathrm{H}^2_{\mathbb{C}}$. Another idea is to extend the three geodesic sides to the boundary, and to play with the triangles obtained with one vertex at infinity.

- differential geometric: fix a surface (possibly singular) filling the triangle, for instance the union of geodesic arcs from a vertex to the opposite side.The idea is then to compare the Gaussian curvature K of the surface with the ambient sectional curvature σ: $K \leqslant \sigma$ on the surface and $\sigma(x \wedge y) = -(1 + 3\omega^2(x \wedge y))/4$ (ω being the ambient Kähler form). This gives us the desired inequality, except in a neighborhood of $\mathbb{C}$-planar configurations.

- brute force: using explicit formulae for the angles at the vertices. One can normalize to put two of them in a simple form, but not the third (see [B] or the preliminary section in my thesis for explicit normalizations). This is of course a less satisfying approach.

3 Projects for a near future

3.1 Reflection groups

Apart from purely loxodromic groups (arising for instance by deformation of Fuchsian groups), the only known discrete groups in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$ are of this type. Recall that there are two types of reflections in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, $\mathbb{C}$-reflections which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $\mathrm{H}^{n-1}_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$), and $\mathbb{R}$-reflections which are antiholomorphic isometries fixing pointwise a Lagrangian subspace (a copy of $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$). There are many open problems concerning reflection groups in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, among which one can address the following:

- **Finite reflection groups:**
  Finite $\mathbb{C}$-reflection groups have been classified by Shephard and Todd (see also Broué–Malle–Rouquier). In [FPau] we have proven that all finite subgroups of $\mathrm{U}(2)$ are (of index 2 in a group) generated by $\mathbb{R}$-reflections. Is this true in $\mathrm{U}(n)$? Probably not. The question is then to classify such subgroups of $\mathrm{U}(n)$. The most important aspect of this project is to find some geometry-to-algebra dictionary for this type of groups (analogous to root systems and Coxeter diagrams for real reflection groups).

- **Lattices generated by reflections in higher dimensions:**

  In real hyperbolic space, it is known that lattices generated by reflections only exist in small dimensions. Vinberg has proven that there are no compact Coxeter polyhedra in $\mathrm{H}^n_{\mathbb{R}}$ for $n \geqslant 30$, and Prokhorov has proven that there are no Coxeter polyhedra of finite volume in $\mathrm{H}^n_{\mathbb{R}}$ for $n \geqslant 996$ (the known examples are for $n \leqslant 8$ in ther first case and $n \leqslant 21$ in the second).

  Is the situation analogous in $\mathrm{PU}(n, 1)$? The known examples (Deligne–Mostow, Mostow, Allcock) are in dimension $n \leqslant 13$ (in fact, $n \leqslant 9$ except for one of Allcock's examples). An obvious difficulty is that there is no counterpart of root systems or Coxeter polyhedra.

- **Non-arithmetic lattices generated by reflections:**

  Non-arithmetic lattices in $\mathrm{PU}(n,1)$ are only known for $n=2$ (lattices due to Picard, Mostow, Deligne–Mostow, between 7 and 9 commensurability classes) and $n=3$ (one non-cocompact example due to Deligne–Mostow). We hope to obtain new non-arithmetic lattices in $\mathrm{PU}(2,1)$ in the families of symmetric $\mathbb{C}$-reflection triangle groups described above. We can hope to apply our methods in dimension 3 and maybe 4, but after that the direct geometric method (construction of fundamental domains) becomes hopeless.

3.2 Discrete groups generated by regular elliptic motions

We could also explore the general case of groups generated by two regular elliptic motions (ie not $\mathbb{C}$-reflections), where the parameter space is much bigger. The problem of finding discrete groups in this parameter space seems hopeless, but the methods from the last part of my thesis allow to determine some "good" one-parameter families to investigate. More precisely, if the three angle pairs of the elliptic motions A, B, and AB are prescribed, then there is a one-parameter family of such groups generated by Lagrangian (or real) reflections, and this allows in principle to systematically search such families.

4 Related questions

- Representation spaces of surface groups in $\mathrm{PU}(2,1)$ and complex hyperbolic quasi-Fuchsian groups. Character varieties of 3-manifolds in $\mathrm{PU}(2,1)$. (Toledo, Goldman, Xia, Parker, Schwartz,...)
- Spherical CR structures on 3-manifolds (Schwartz, Falbel)
- Fake projective planes (Mumford, Klingler, Yeung, Prasad)
- Complex hyperbolic structures on moduli spaces of algebraic objects (Allcock–Carlson–Toledo)
- Negatively curved compact Kähler manifolds not covered by the ball (Mostow–Siu, Deraux)
- Mapping class group dynamics on surface group representations in $\mathrm{PU}(n,1)$ (Goldman, Burger–Iozzi–Wienhard)
- Spectrum of the automorphic Laplace-Beltrami operator on arithmetic ball quotients (Francsics–Lax, Sarnak)

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