Consider the function \( f(x, y) = x e^{-(x^2+y^2)} \) and its graph \( S \). (1) Find the first partial derivatives and gradient of \( f \). (2) (a) Find an equation for the tangent plane to \( S \) at the point \((1, 1, e^{-2})\). (b) What is the steepest slope of a line in this plane? (c) Find a vector tangent to the level curve of \( f \) through \((1, 1)\). (3) Find the critical points of \( f \). Then find the local maxima and minima of \( f \). Bonus: are these also global maxima/minima? (4) If \( x = r \cos \theta \) and \( y = r \sin \theta \), find the partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= -x e^{-(x^2+y^2)} - 2xe^{-(x^2+y^2)} = (1-2x) e^{-(x^2+y^2)}, \\
\frac{\partial f}{\partial y} &= -2xy e^{-(x^2+y^2)}; \\
\nabla f(x, y) &= \langle (1-2x) e^{-(x^2+y^2)}, -2xy e^{-(x^2+y^2)} \rangle \\
\nabla f(1, 1) &= \langle -e^{-2}, -2e^{-2} \rangle \\
\text{So an equation for the tangent plane is:} \\
z - e^{-2} &= -e^{-2}(x-1) - 2e^{-2}(y-1)
\end{align*}
\]

(\textbf{b}) We know that the directional derivative of \( f \) at a point \((x, y)\) in the direction of a unit vector \( \overrightarrow{u} \) is:

\[
D_{\overrightarrow{u}} f(x, y) = \overrightarrow{u} \cdot \nabla f(x, y)
\]

This is maximal when \( \overrightarrow{u} \) has the same direction as \( \nabla f(x, y) \); in that direction the slope of the line tangent to the graph of \( f \) is \( \| \nabla f(x, y) \| \).

Here: \[ \| \nabla f(1, 1) \| = \sqrt{(e^{-1})^2 + (-2e^{-1})^2} = \sqrt{5e^{-2}} = \sqrt{5}. e^{-1} \]

(\textbf{c}) This also tells us that level curves of \( f \) are everywhere perpendicular to the gradient of \( f \). Any vector \( \overrightarrow{w} = \langle w_1, w_2 \rangle \) perpendicular to \( \nabla f(1, 1) = \langle -e^{-1}, -2e^{-1} \rangle \) will do, such as \( \langle 2, -1 \rangle \).

\[
\langle 2, -1 \rangle \cdot \langle -e^{-1}, -2e^{-1} \rangle = 0
\]

(3) \((x, y)\) is a critical point of \( f \) \(\Rightarrow\) \( \nabla f(x, y) = \overrightarrow{0} = \langle 0, 0 \rangle \)

(i.e. \( \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0 \)). Here the only critical points are \((1, 0)\) and \((-1, 0)\).
We use the second partial derivatives test to decide if these points are local max/min for $f$:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= -4xe^{-(x^2+y^2)} - 2x(1-2x)e^{-(x^2+y^2)} = (4x^3 - 6x)e^{-(x^2+y^2)} \\
\frac{\partial^2 f}{\partial y\partial x} &= -2y(1-2x)e^{-(x^2+y^2)} \\
\frac{\partial^2 f}{\partial y^2} &= -2x e^{-(x^2+y^2)} + 4xy e^{-(x^2+y^2)} = -2x(1-2y)e^{-(x^2+y^2)}
\end{align*}
\]

Evaluating these 3 functions at $(\frac{1}{\sqrt{2}}, 0)$ then $(-\frac{1}{\sqrt{2}}, 0)$ gives:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2}(\frac{1}{\sqrt{2}}, 0) &= -2\sqrt{2} e^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial y\partial x}(\frac{1}{\sqrt{2}}, 0) &= 0 \\
\frac{\partial^2 f}{\partial y^2}(\frac{1}{\sqrt{2}}, 0) &= -\sqrt{2} e^{-\frac{1}{2}}
\end{align*}
\]

\[
D = \Delta = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} = 4e^{-1} > 0
\]

Therefore $f$ has a local maximum at $(\frac{1}{\sqrt{2}}, 0)$ (value: $f(\frac{1}{\sqrt{2}}, 0) = e^{-\frac{1}{2}}$)

\[
D < 0
\]

Therefore $f$ has a local minimum at $(-\frac{1}{\sqrt{2}}, 0)$ (value $f(-\frac{1}{\sqrt{2}}, 0) = -\frac{e^{-\frac{1}{2}}}{\sqrt{2}}$)

BONUS: These local min/max values are also global min/max values, because $f(x, y) \to 0$ as $x^2+y^2 \to \infty$ (meaning that $(x, y) \to \infty$).

4. Using the Chain Rule:

\[
\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}
\]

\[
\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi}
\]

so:

\[
\begin{align*}
\frac{\partial f}{\partial \phi} &= (1-2x)e^{-(x^2+y^2)} \cos \phi + 2xy e^{-(x^2+y^2)} \sin \phi \\
\frac{\partial f}{\partial \theta} &= -(1-2x)e^{-(x^2+y^2)} \sin \phi + 2xy e^{-(x^2+y^2)} \cos \phi.
\end{align*}
\]