

Review problems on power series

There are four things you need to know about power series: (a) the definition: power series in x and in $(x - a)$ (b) convergence properties: radius of convergence, endpoints (c) how to differentiate and integrate power series, and (d) power series to know by heart. For (d), the only power series that you need to know by heart for this class are the following:

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \dots \quad |x| < 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \dots\end{aligned}$$

Differentiating and integrating the first of these then gives you power series expansions for $\frac{1}{(1-x)^2}$ and $\ln(1-x)$ (and so also $\ln(1+x)$).

For practice on how to use these properties of power series, you should start by reviewing the homework from week 14 (plenty of examples of radii of convergence), then do the following problems which illustrate how these things can fit together.

Problem 1: (done in class) Find a power series expansion for $\frac{1}{1+x^2}$. What is its radius of convergence? Does it converge at the endpoints? From this, find a power series expansion for $\tan^{-1} x$. What is its radius of convergence? Does it converge at the endpoints?

Problem 2: Consider the power series

$$f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}$$

1. For which values of x does it converge? (find the radius of convergence and determine what happens at the endpoints of the interval of convergence).
2. Find a power series expansion for $f'(x)$. When does this series converge? (be careful about the endpoints).
3. Find a power series expansion for $f''(x)$. When does this series converge? (again, be careful about the endpoints).
4. Recognize this last power series to find expressions for $f'''(x)$, $f'(x)$ and $f(x)$ using ordinary functions. You may need to use an antiderivative of $\ln x$ (Hint 1: integrate by parts; Hint 2: as we've seen in class, the result is $x \ln x - x$).
5. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)2^n}$. (Why do these series converge?)