University of Utah
Math 1220, Fall 2007

Complements on differential equations

This worksheet contains a summary of what you must know about linear differential equations (first order and second order with constant coefficients) for this class. They are treated in sections 6.6 and 15.1 of our textbook.

1 First order linear equations

These are equations of the form:

$$(*) \quad y' + p(x)y = q(x)$$

The trick to solve this kind of equation is to multiply it by the "integrating factor" $e^{H(x)}$, where $H(x)$ is any antiderivative of $p(x)$. The equation then becomes (writing the missing variable x in y' and y):

$$(*') \quad e^{H(x)}y'(x) + e^{H(x)}p(x)y(x) = e^{H(x)}q(x)$$

and we recognize the left-hand-side as the derivative of $e^{H(x)}y(x)$ (check this). In other words, our equation $(*')$ now reads:

$$\left[e^{H(x)}y(x)\right]' = e^{H(x)}q(x)$$

We know how to solve this in terms of y: integrate the right-hand side, then divide by $e^{H(x)}$. The solutions of $(*)$ are thus:

$$y(x) = e^{-H(x)}\left[\int e^{H(t)}q(t)\,dt + C\right]$$

Exercises: Solve the following differential equations:

1. $y' - y = e^x$
2. $y' + 2xy = x$
3. $xy' + 2y = \frac{\cos x}{x}$
4. $(1 + e^x)y' + y = 1$

2 Equations for physical or electrical oscillations: second order linear equations with constant coefficients

These are equations of the form:

$$(**) \quad y'' + ay' + by = 0$$

The idea to solve these equations is to look for exponential functions as solutions. Consider the function $y(x) = e^{rx}$; then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$. Thus this function is a solution of $(**)$ if and only if:

$$e^{rx}(r^2 + ar + b) = 0$$

and since $e^{rx} \neq 0$ for all x, this is in turn equivalent to:

$$(***) \quad r^2 + ar + b = 0$$

This quadratic equation in r is called the auxiliary equation of $(**)$. If this quadratic polynomial has two real roots r_1 and r_2, then we obtain two independent solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ and we are happy. It turns out that in all cases we obtain the solutions from the (maybe complex) roots of $(***)$. We will quote the result from the general theory (which you will see in a specific course on differential equations):

- If $(***)$ has two distinct real roots r_1 and r_2, then all solutions of $(**)$ are:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad \text{with } C_1, C_2 \in \mathbb{R}$$

- If $(***)$ has a double real root r, then all solutions of $(**)$ are:

$$y(x) = (C_1 + C_2 x)e^{rx} \quad \text{with } C_1, C_2 \in \mathbb{R}$$

- If $(***)$ has two complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, then all solutions of $(**)$ are:

$$y(x) = (C_1 \cos \beta x + C_2 \sin \beta x)e^{\alpha x} \quad \text{with } C_1, C_2 \in \mathbb{R}$$

Exercises: Solve the following differential equations:

1. $y'' + 2y' + 5y = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 2y' - 15y = 0$. Give all solutions and find the solution satisfying $y(0) = 0$ and $y'(0) = 1$.