Midterm Exam # 1

Problem 0: (3 points) (a) Write the definitions of compact and sequentially compact. (b) Give without proof an example of a non-converging Cauchy sequence in a metric space, and an example of an injective sequence of real numbers with exactly 2 limit points.

(a) $X$ is compact if every open cover of $X$ has a finite subcover.

X is sequentially compact if every sequence in $X$ has a subsequence which converges in $X$.

(b) $(\frac{1}{n})_{n \geq 1}$ is a Cauchy sequence in $\mathbb{R}$ which doesn't converge in $X$.

$x_n = (\frac{1}{n})^{1/n}$ defines an injective sequence (no repeating terms) whose limit points are $-1$ and $1$. $(x_n = 1 + \frac{1}{2n} \to 1$ and $x_{n+1} = -1 + \frac{1}{2n} \to -1$).

Problem 1: (3 points) Prove that $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^3yz^2 + 2xy + 5x^2y^8z^{11} < 17\}$ is an open subset of $\mathbb{R}^3$.

Let $P(x, y, z) = x^3yz^2 + 2xy + 5x^2y^8z^{11}$.

Then $P : \mathbb{R}^3 \to \mathbb{R}$ is continuous (and even $C^\infty$, it is a polynomial).

Therefore $A = P^{-1}((-\infty, 17))$ is open, as the preimage of an open set by a continuous function.
Problem 2: (8 points) Is the metric space \( C([a, b]), \|\cdot\|_\infty \) compact? (prove your answer)

What about the unit ball in this space?

\* \( C([a, b]) \) is not compact because it is not bounded.
(For instance, if \( f_m \) is the constant function equal to \( m \) on \([a, b]\),
then \( \|f_m - f_n\|_\infty = |m - n| \geq 1 \) so \( (f_m) \) has no Cauchy subsequence, hence
no converging subsequence.)

\* Let \( B = \{ f \in C([a, b]) \mid \|f\|_\infty \leq 1 \} \) be the closed unit ball in \( (C([a, b]), \|\cdot\|_\infty) \).
Consider a sequence of functions in \( B \) such as the following (seen in class):

\[ f_n : \begin{array}{c}
  & \downarrow \\
  a & \overset{c}{\longrightarrow} & b
\end{array} \]

\[ f : \begin{array}{c}
  & \downarrow \\
  a & \overset{c}{\longrightarrow} & b
\end{array} \]

Claim: \( (f_n) \) has no converging subsequence.

There are some subtleties here.

\( (f_n) \) converges pointwise to
\[ f : \begin{array}{c}
  & \downarrow \\
  a & \overset{c}{\longrightarrow} & b
\end{array} \]

(\text{i.e.} \( \forall x \in [a, b], \begin{array}{l}
  \begin{array}{c}
  \downarrow \\
  a & \overset{c}{\longrightarrow} & b
  \end{array} \\
  f_n(x) \rightarrow 1 \text{ if } a \leq x \leq c \\
  f_n(x) \rightarrow 0 \text{ if } c < x \leq b
  \end{array} \))

but certainly not for \( \|\cdot\|_\infty \)

(\text{indeed, we saw in class that} \( \|f_n - f\|_\infty = 1 - \frac{m}{n} \) \text{ so} \( (f_n) \) is not Cauchy for \( \|\|_\infty \)).

The easiest way to conclude is to use the pointwise limit \( f \), by observing that:

\( \|\cdot\|_\infty \) convergence \( \Rightarrow \) pointwise convergence (i.e. if \( \|f_n - f\|_\infty \rightarrow 0 \), then
So if \( (f_n) \) had a converging subsequence,
the limit would have to be \( f \) (because \( f \) is the pointwise limit),
but we know that \( \|f_n - f\|_\infty \rightarrow 0 \), so no converging subsequence.

Therefore: \( B \) is not compact.
Problem 3: (6 points) Consider the subsets \(N\) and \(M = \{ n + \frac{1}{2n} | n \in \mathbb{N} \}\) of \(\mathbb{R}\). (a) Prove that \(M\) is closed. (b) What is the distance \(d(M, N)\)? (prove your answer).

(a) \(M\) is closed because it is discrete (any 2 distinct points in \(M\) are at least \(\frac{1}{2}\) apart — in fact almost 1).

(b) Consider the points \(x_n = n\) in \(N\) and \(y_n = n + \frac{1}{2n}\) in \(M\).

Then \(d(x_n, y_n) = \frac{1}{2n}\) \(\to 0\), so \(d(M, N) = 0\).

Note: This (easy) problem gives an example of 2 closed disjoint sets which are at distance 0 from each other (try this if one of them is compact.)

Problem 4: (Bonus, 5 points) Find all accumulation points of the set \(\{ \frac{1}{m} + \frac{1}{n} | m, n \in \mathbb{N} \}\).
(Recall that an accumulation point of a subset \(S \subset X\) is a point \(x \in X\) such that every neighborhood of \(x\) contains a point of \(S\) distinct from \(x\).)

* \(0\) is an accumulation point of \(S\) by taking a double limit: \(\frac{1}{m} + \frac{1}{n} \to 0\).

* Any \(1\) is also an accumulation point of \(S\): fix \(m\) and look at the sequence \(\frac{1}{m} + \frac{1}{n} \to 1\).

* Exercise: \(S\) has no other accumulation points.