Key

University of Utah
Math 5210, Spring 2010

Midterm Exam # 1

Problem 0: (3 points) (a) Write the definitions of compact and sequentially compact.
(b) Give without proof an example of a non-converging Cauchy sequence in a metric space, and an example of an injective sequence of real numbers with exactly 2 limit points.

(a) X is compact if every open cover of X has a finite subcover.
X is sequentially compact if every sequence in X has a subsequence which converges in X.

(b) $\left(\frac{1}{n}\right)_{n\geq 1}$ is a Cauchy sequence in $X = (0,1]$ which doesn't converge in X.

$x_n = (-1)^n + \frac{1}{n}$ defines an injective sequence (no repeating terms) whose limit points are -1 and 1. $\left(x_{2n} = 1 + \frac{1}{2n} \xrightarrow[n\to\infty]{} 1 \text{ and } x_{2n+1} = -1 + \frac{1}{2n+1} \xrightarrow[n\to\infty]{} -1\right)$.

Problem 1: (3 points) Prove that $A = \{(x,y,z) \in \mathbb{R}^3 | x^3yz^2 + 2xyz + 5x^2y^8z^{11} < 17\}$ is an open subset of $\mathbb{R}^3$.

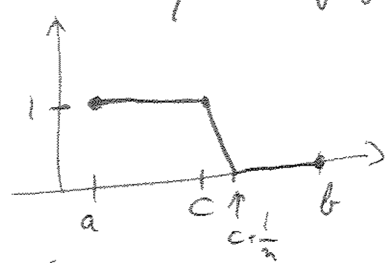
Let $P(x,y,z) = x^3yz^2 + 2xyz + 5x^2y^8z^{11}$.
Then $P : \mathbb{R}^3 \to \mathbb{R}$ is continuous (and even C^∞, it is a polynomial).
Therefore $A = P^{-1}((-\infty; 17))$ is open, as the preimage of an open set by a continuous function.

Problem 2: (8 points) Is the metric space $(C([a,b]), \|\cdot\|_\infty)$ compact? (prove your answer)
What about the unit ball in this space?

* $C([a,b])$ is not compact because it is not bounded.
(For instance, if f_n is the constant function ~~equal~~ equal to n on $[a,b]$,
then $\|f_m - f_n\|_\infty = |m-n| \geq 1$ so (f_n) has no Cauchy subsequence, hence
no converging subsequence.).

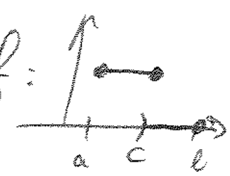
* Let $B = \{ f \in C([a,b]) \mid \|f\|_\infty \leq 1 \}$ be the closed unit ball in $(C([a,b]), \|\,\|_\infty)$.
Consider a sequence of functions in B such as the following (seen in class):

f_n :



<u>Claim: (f_n) has no converging subsequence.</u>
There are some subtleties here.

(f_n) converges <u>pointwise</u> to f :



(i.e. $\forall x \in [a,b]$,
$f_n(x) \longrightarrow 1$ if $a \leq x \leq c$
$f_n(x) \longrightarrow 0$ if $c < x \leq b$).

but certainly not for $\|.\|_\infty$
(indeed, we saw in class that $\|f_n - f_m\|_\infty = 1 - \frac{m}{n}$ so (f_n) is not Cauchy for $\|.\|_\infty$).

The easiest way to conclude is to use the pointwise limit f, by observing that:
$\|.\|_\infty$ convergence $\Rightarrow$ pointwise convergence (i.e. if $\|f_n - f\|_\infty \to 0$ then $\forall x \in [a,b]$ $f_n(x) \to f(x)$).
So if (f_n) had a converging subsequence,
the limit would have to be f (because f is the pointwise limit),
but we know that $\|f_n - f\|_\infty \not\to 0$, so no converging subsequence.

Therefore: $\boxed{B \text{ is not compact}}$

Problem 3: (6 points) Consider the subsets $\mathbb{N}$ and $M = \{n + \frac{1}{2n} | n \in \mathbb{N}\}$ of $\mathbb{R}$. (a) Prove that M is closed. (b) What is the distance $d(M, \mathbb{N})$? (prove your answer).

(a) M is closed because it is discrete (any 2 distinct points in M are at least $1/2$ apart — in fact almost 1).

(b) Consider the points $x_n = n$ in $\mathbb{N}$ and $y_n = n + \frac{1}{2n}$ in M.

Then $d(x_n, y_n) = \frac{1}{2n} \xrightarrow[n\to\infty]{} 0$, so $d(M, \mathbb{N}) = 0$.

Note: This (easy) problem gives an example of 2 closed disjoint sets which are at distance 0 from each other (try this if one of them is compact...)

Problem 4: (Bonus, 5 points) Find all accumulation points of the set $S: \{\frac{1}{m} + \frac{1}{n} | m, n \in \mathbb{N}\}$. (Recall that an *accumulation point* of a subset $S \subset X$ is a point $x \in X$ such that every neighborhood of x contains a point of S distinct from x.)

* 0 is an accumulation point of S by taking a double limit: $\frac{1}{m} + \frac{1}{n} \xrightarrow[m,n\to\infty]{} 0$

* Any $\frac{1}{m}$ is also an accumulation point of S: fix m and look at the sequence $\frac{1}{m} + \frac{1}{n} \xrightarrow[n\to\infty]{} \frac{1}{m}$.

* Exercise: S has no other accumulation points.