If $A$ is closed and $x \notin A$, $d(x, A) > 0$.

Suppose not, that is $d(x, A) = 0$. Then $\exists x_n \in A$ s.t. $d(x, x_n) \to 0$, that is $x_n \to x$.

But $A$ is closed so this would mean that $x \in A$.

3.64: (a) $B(x, \epsilon) = \{y \in X/d(x, y) < \epsilon\}$ is closed because it is the preimage of a closed set by a continuous function (namely $C : X \to \mathbb{R}$).
(b) If $X$ is a discrete set (think of $\mathbb{Z}^n$ in $\mathbb{R}^n$), then $y \to d(x, y)$ is finite, set, say $X = \{x_1, x_2\}$ with $d(x_1, x_2) = 1$ then $B(x_1, 1) = \{x_1, x_2\}$ and $B(x_2) = \{x_2\}$.

5.64: (a) $A \cap B \Rightarrow A \in B$ and $A \cap B \Rightarrow (\text{we used } A \text{ for } \text{Int}(A))$.

There are obvious by the characterization of $A$ (resp. $A$) as the largest open set contained in $A$ (resp. smallest closed set containing $A$), which we've seen in class.
(b) $A \cup B \subseteq A \cup B$ and $A \cap B = A \cap B$.
Again this follows from the fact that $A \cup B$ (resp. $A \cap B$) is the largest open set contained in $A \cup B$ (resp. $A \cap B$).
(c) $A \cup B = A \cup B$ and $A \cap B = A \cap B$.
(f) $\overline{A \cup B}$ (resp. $\overline{A \cap B}$) is the largest open set contained in $A \cup B$ (resp. $A \cap B$).

6.64: $N(A \cap \emptyset) = \{x \in X/d(x, A) > 0\}$ is open (preimage of the open set $(-1, 1)$ by the continuous function $x \to d(x, A)$) seen in class. This function is continuous because $1$-lipschitz.

7.64: (a) Open and $A \cap B \neq \emptyset$ $\Rightarrow$ $A \cap B$ $\neq \emptyset$: Let $x \in A \cap B$. This means that $x \in A$ and $\exists x_n \in \overline{B}$ s.t. $x_n \to x$. But $A$ is open so it contains an open ball $B(x, \epsilon)$.

Then for some $N$, $m > N \Rightarrow x_m \in B(x, r) \subseteq A$ by convergence of $x_n$ to $x$. So $A \cap B \neq \emptyset$.

8.64: $\overline{A} \subseteq \overline{B}$, recall that $\overline{A} = \{x \in X/A \neq \emptyset \}$.

Let $x \in A$; then $d(x, A) > 0$.

Then $\exists y \in (X-A)$ s.t. $d(x, y) < 1/n$. But $x - A = \overline{X} \text{ is open and contained in } X - A$.

Also, for fixed $m$, each $x_m \in \overline{A}$ is the limit of some sequence $(x_{m_n})$ with $x_{m_n} \in A$.

So if $B(x, r)$ is any open ball around $x$, it contains some $x_m$ and an $\epsilon > 0$ s.t. $B(x, \epsilon) \subseteq B(x, r)$.

Therefore for large enough $m_n$, $x_m \in \overline{A} \cap B(x, r)$.

9.64: obvious because $\overline{A} \subseteq \overline{B}$.

Example: In $I = [0, 1]$, $A = Q \cap I$. Then $A = \emptyset$, $A = I$ so $\partial A = \emptyset$ and $\overline{A} \cap \partial A = \emptyset$, $\overline{A} = I$. 

\[ \text{Example: In } I = [0, 1], A = Q \cap I. \text{ Then } A = \emptyset, A = I \text{ so } \partial A = \emptyset \text{ and } \overline{A} \cap \partial A = \emptyset, \overline{A} = I. \]
7.58: \( \mathbb{IR} \) is homeomorphic to \((0,1)\):

Take for instance \( \tan : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{IR} \). This is a homeomorphism.

Then open intervals \((a, b)\) are homeomorphic to each other (for instance by a linear transformation).

4.75: (a) Obvious (and seen in class.)

(b) If \((x_n)\) has 2 limit points, say \(x_1\) and \(x_2\), take 2 disjoint open balls \(B_1\) and \(B_2\) centered at \(x_1\) and \(x_2\). Then take a subsequence \((x_{n_k})\) converging to \(x_1\), and another subsequence \((x_{n'_k})\) converging to \(x_2\). Then for all \(k\) \(d(x_{n_k}, x_{n'_k}) > \varepsilon\) so \(\varepsilon\) is not Cauchy, as doesn't converge.

14.76: \( f \) uniformly continuous:

\[ \forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X \left( d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \right) \]

\( (x_n) \) is Cauchy: \( \forall \varepsilon > 0 \exists \delta > 0 \forall m, n \in \mathbb{N} \left( m, n > N \Rightarrow d(x_m, x_n) < \delta \right) \)

Put the 2 together and stir.

Example: If \( f \) is only continuous (\( \mathbb{R} \) not compact), maybe not.

Let \( X = (0, 1] \), \( Y = [1, \infty) \), \( f(x) = \frac{1}{x} \), and \( x_n = \frac{1}{n} \).

Then \((x_n)\) is Cauchy in \( X \) but \( f(x_n) = n \) is not Cauchy in \( Y \).