#2 p. 64 <u>If A is closed and $x \notin A$, $d(x,A)>0$:</u>

Suppose not, that is $d(x,A)=0$. Then $\exists x_n \in A$ s.t. $d(x,x_n)\to 0$, that is $x_n \to x$. But A is closed so this would mean that $x \in A$.

<u>#3 p. 64:</u> (a) $B[x,r]=\{y\in X / d(x,y)\le r\}$ is closed because it is the preimage of a closed set by a continuous function (namely, $C: X \longrightarrow \mathbb{R}$, $y \longmapsto d(x,y)$).

b) If X is a discrete set (think of $\mathbb{Z}^n$ in $\mathbb{R}^n$) or a finite set, say $X=\{x_1,x_2\}$ with $d(x_1,x_2)=1$, then $B[x_1,1]=\{x_1,x_2\}$ and $\overline{B(x_1,1)}=\{x_1\}$.

<u>#5 p. 64:</u> (a) <u>$A\subset B \Rightarrow \mathring{A} \subset \mathring{B}$ and $\overline{A}\subset\overline{B}$:</u> (we used $\mathring{A}$ for $\mathrm{Int}(A)$).
These are obvious by the characterization of $\mathring{A}$ (resp. $\overline{A}$) as the largest open set contained in A (resp. smallest closed set containing A), which we've seen in class.

(b) <u>$\mathring{A}\cup\mathring{B} \subset \mathring{\overline{A\cup B}}$ and $\mathring{A}\cap\mathring{B} = \mathring{\overline{A\cap B}}$:</u> Again this follows from the fact that $\mathring{\overline{A\cup B}}$ (resp. $\mathring{\overline{A\cap B}}$) is the largest open set contained in $A\cup B$ (resp. $A\cap B$).

(c) <u>$\overline{A\cup B}=\overline{A}\cup\overline{B}$ and $\overline{A\cap B}\subset\overline{A}\cap\overline{B}$:</u> follows from (b) by taking complements.

(d) Let $A=\mathbb{Q}$ and $B=\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$ (more generally, any dense A with dense complement)
Then $\mathring{A}=\mathring{B}=\emptyset$ ($\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ don't contain any open (nonempty) interval but $A\cup B=\mathbb{R}=\mathring{\overline{A\cup B}}$.

<u>#6 p. 64:</u> $N(A;r)=\{x\in X / d(x,A)<r\}$ is open (preimage of the open set $(-1,r)$ by the continuous function $x\mapsto d(x,A)$). Seen in class: this function is continuous because 1-Lipschitz.

<u>#7 p. 64:</u> <u>(A open and $A\cap\overline{B}\neq\emptyset) \Rightarrow A\cap B\neq\emptyset$:</u> Let $x\in A\cap\overline{B}$. This means that $x\in A$ and ($\exists x_n\in B$) s.t. $x_n\to x$. But A is open so it contains an open ball $B(x,r)$.
Then for some N, $n\ge N \Rightarrow x_n\in B(x,r)\subset A$ by convergence of x_n to x. So $A\cap B\neq\emptyset$.

<u>#8 p. 64:</u> <u>* $\partial\overline{A}\subset\partial A$:</u> ~~In fact,~~ Recall that: $\partial A=\{x\in X / \forall r>0, B(x,r)\cap A\neq\emptyset$ and $B(x,r)\cap(X-A)\neq\emptyset\}$

Let $x\in\partial\overline{A}$; then $\exists x_n\in\overline{A}$ s.t. $d(x,x_n)<1/n$.
and $\exists y_n\in(X-\overline{A})$ s.t. $d(x,y_n)<\frac{1}{n}$. But $X-\overline{A}=\mathring{\overline{X-A}}$ is (open and) contained in $X-A$, so $y_n\in X-A$.
$\underbrace{}_{=\overline{X-A}}$

Also, for fixed n, each $x_n\in\overline{A}$ is the limit of some sequence $(x_{i,n})_i$ with $x_{i,n}\in A$.
So if $B(x,r)$ is any open ball around x, it contains some x_n and an $\varepsilon>0$ s.t. $B(x_n,\varepsilon)\subset B(x,r)$;
therefore for i large enough $x_{n_i}\in A\cap B(x,r)$.

<u>* $\partial\mathring{A}\subset\partial A$:</u> obvious because $\mathring{A}\subset A$.

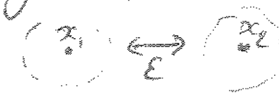
* <u>Example:</u> In $I=[0;1]$, $A=\mathbb{Q}\cap I$. Then: $\mathring{A}=\emptyset$, $\overline{A}=I$ so $\partial\mathring{A}=\emptyset\subsetneq\partial\overline{A}=\{0;1\}\subsetneq\partial A=I$.

#7 p.68: $\mathbb{R}$ is homeomorphic to $(0;1)$:

Take for instance $\tan : \left(-\frac{\pi}{2};\frac{\pi}{2}\right) \to \mathbb{R}$. This is a homeomorphism.
Then open intervals (a,b) are all homeomorphic to each other (for instance by a linear transformation).

#4 p.75: (a) Obvious (and seen in class.)

(b) If (x_n) has 2 limit points, say x_1 and x_2, take 2 disjoint open balls B_i centered at x_i. Then take a subsequence $x_{n_{k,1}}$ converging to x_1 and $x_{n_{k,2}}$ converging to x_2. Then for all k $d(x_{n_{k,1}}, x_{n,k,2}) \geq \varepsilon$ so (x_n) is not Cauchy, so doesn't converge.



#14 p.76: * f uniformly continuous:

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon)$.

* (x_n) is Cauchy: $(\forall \delta > 0)(\exists N \in \mathbb{N})(\forall m,n \in \mathbb{N})(m,n \geq N \Rightarrow d_1(x_m, x_n) < \delta)$.

Put the 2 together and stir.

* Example: If f is only continuous (X not compact), maybe not.

Let $X = (0;1]$, $Y = [1;\infty)$, $f(x) = \frac{1}{x}$ and $x_n = \frac{1}{n}$.

Then (x_n) is Cauchy in X but $f(x_n) = n$ is not Cauchy in Y.