Gauss-Bonnet Formula and Gauss-Bonnet Theorem

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This paper will present a proof of the Gauss-Bonnet Formula and an important corollary (The Gauss-Bonnet Theorem). The proofs will follow those given in the book Elements of Differential Geometry by Millman and Parker. They are simpler and shorter than the proofs I've seen from other books. Also, the proofs are for surfaces embedded in 3-space, although the theorems hold, with slightly different restrictions, for surfaces in n-space and more generally on Riemannian 2-Manifolds. This paper will focus only on the proofs of the Gauss-Bonnet Formula/Theorem so all the facts needed will just be listed. So, the easiest way to read this may be to begin with the proof itself and back track.

The material needed in the proof is as follows:

1) Definition of a curve and properties of curves.
2) Definition of a surface, properties of the surface.
3) Some stuff about curvature.
4) Some stuff about vector fields on curves on surfaces.
5) A little topology (simply connected).
6) Lastly Stokes' theorem.

Curves

Definition (Regular Curve) A regular curve is an infinitely differentiable function $\gamma:[a\ ,\ b]\rightarrow R^3$ such that $\gamma'(s)\ \neq\ \mathbf{0}\quad \forall s\in[a\ ,\ b]$.

Definition (Curves Parametrized by Arc Length) A curve parametrized by arc length is a regular curve with the property that the length of the curve equals the distance from the parameter to the initial point. That is, $\|\gamma(s)\|\ =\ s\ -\ a$

The important properties of a curve parametrized by arc length are:

1) The tangent vector has unit length everywhere, and
2) Its second derivative is a vector perpendicular to the tangent vector everywhere. To see this take the derivative of $\|\gamma(s)\|$ and $\|\gamma'(s)\|$ and recall that two vectors are perpendicular if their inner product (our inner product is the dot product) is 0.

Definition (Curvature of a Curve Parametrized by Arc Length) If $\gamma:[a\ ,\ b]\rightarrow R^3$ is a curve parametrized by arc length, then the curvature of γ at s_0 is the real number $k(s_0)\ =\ \|\gamma''(s_0)\|$.

Surfaces

A *surface* M is basically a subset of R^3 where each neighborhood N_P of a point $P\in M$ is like R^2 in a differentiable way. It is defined as the union of the images of a collection of maps called *coordinate patches*. If the images of any two maps coincide there is a further requirement that there exists a *coordinate transformation* between the domains of the maps. This requirement ensures that maps sync up in a smooth way.

Definition (Coordinate Patch) A coordinate patch is a 1-1 C^k function $\boldsymbol{x}:U\rightarrow R^3$ for some $k\geq 1$ where U is an open set of R^2 with coordinates u^1 and u^2 . Also, $\frac{\partial \boldsymbol{x}}{\partial u^1}\times\frac{\partial \boldsymbol{x}}{\partial u^2}\neq\boldsymbol{0}$, and the inverse function $\boldsymbol{x}^{-1}:\boldsymbol{x}(U)\rightarrow U$ is continuous.

Notation: From now on let, $\frac{\partial \boldsymbol{x}}{\partial u^1}=\boldsymbol{x}_1$ and $\frac{\partial \boldsymbol{x}}{\partial u^2}=\boldsymbol{x}_2$ represent the partials of $\boldsymbol{x}$.

Definition (Coordinate Transformation) A coordinate transformation is a bijection $f:U\rightarrow V$ with inverse $g:V\rightarrow U$ such that f and g are C^k differentiable $(k\geq 1)$, where U and V are open subsets of R^2 .

Definition (Surface) A C^k surface M is a subset $M\subset R^3$ such that for each $P\in M$ there is a coordinate patch whose image contains a neighborhood N_P of P in M . Also, if $\boldsymbol{x}:U\rightarrow R^3$ and $\boldsymbol{y}:V\rightarrow R^3$ are coordinate patches whose images coincide ($\boldsymbol{x}(U)\cap\boldsymbol{y}(V)\neq\emptyset$) then it is required that $\boldsymbol{y}^{-1}\boldsymbol{x}:\boldsymbol{x}^{-1}(\boldsymbol{x}(U)\cap\boldsymbol{y}(V))\rightarrow\boldsymbol{y}^{-1}(\boldsymbol{x}(U)\cap\boldsymbol{y}(V))$ be a coordinate transformation.

Now that we have what a surface is, we need some properties of the surface which are important in the proof. They are: the tangent space at a point P on the surface and the metric matrix at P .

Definition (Tangent Vector) A vector $\boldsymbol{X}$ is a tangent vector to a coordinate patch $\boldsymbol{x}:U\rightarrow R^3$ at a point $\boldsymbol{P}=\boldsymbol{x}(a,b)$ if $\boldsymbol{X}=\frac{d\boldsymbol{\gamma}}{dt}(0)$ for a curve $\boldsymbol{\gamma}:I\rightarrow\boldsymbol{x}(U)$ with $0\in I=[-\epsilon,\epsilon]$, $\epsilon>0$ and $\boldsymbol{\gamma}(0)=\boldsymbol{P}$ and $\boldsymbol{\gamma}(I)\subset\boldsymbol{x}(U)$.

Notice that any curve $\boldsymbol{\gamma}$ as above can be written $\boldsymbol{\gamma}(t)=\boldsymbol{x}(\gamma^1(t),\gamma^2(t))$ for some plane curve $(\gamma^1(t),\gamma^2(t))$ in U defined by $\boldsymbol{x}^{-1}(\boldsymbol{\gamma}(t))$.

Definition (Tangent Space) The tangent space of a surface M at $P\in M$ is the set T_PM of all vectors tangent to M at P .

It turns out that the tangent space is a vector space. The first fundamental form is a map which takes two vectors in a tangent space and assigns to them their inner product. If one does this for any two vectors in the tangent plane they get an expression in terms of the inner product of the first partials of the coordinate patch $\boldsymbol{x}$. Namely, an expression in $\langle\boldsymbol{x}_1,\boldsymbol{x}_1\rangle$ $\langle\boldsymbol{x}_1,\boldsymbol{x}_2\rangle$ $\langle\boldsymbol{x}_2,\boldsymbol{x}_1\rangle$ $\langle\boldsymbol{x}_2,\boldsymbol{x}_2\rangle$.

Definition(Metric Matrix) If $\boldsymbol{x}:U\rightarrow R^3$ is a coordinate patch, the metric matrix (g_{ij}) is the matrix

$$\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix}\langle\boldsymbol{x}_1,\boldsymbol{x}_1\rangle & \langle\boldsymbol{x}_1,\boldsymbol{x}_2\rangle\\ \langle\boldsymbol{x}_2,\boldsymbol{x}_1\rangle & \langle\boldsymbol{x}_2,\boldsymbol{x}_2\rangle\end{pmatrix}$$

Definition (Geodesic Curvature) The geodesic curvature, k_g , is defined as $k_g=\langle\boldsymbol{n}\times\boldsymbol{T},\boldsymbol{T}'\rangle$, where $\boldsymbol{T}=\boldsymbol{\gamma}',\boldsymbol{T}'=\boldsymbol{\gamma}''$.

Definition (The Coefficients of The Second Fundamental Form) If $x : U \rightarrow R^3$ is a coordinate patch, then the coefficients of the second fundamental from are the functions $L_{ij} = \langle x_{ij}, n \rangle$ for i,j =1,2.

Definition (Christoffel Symbols) If $x : U \rightarrow R^3$ is a coordinate patch, then the Christoffel symbols are the functions $\Gamma^l_{ij} = \sum_{l=1}^{2} \langle x_{ij}, x_i \rangle g^{lk}$.

Clearly $(n,\ x_1,\ x_2)$ is a basis of R^3 so it makes sense to wright expressions in terms of this basis. The following is a representation of the second partials of the coordinate patch x in the basis $(n,\ x_1,\ x_2)$. It is a nice convenience in the proof of the Gauss-Bonnet Formula.

Lemma (Gauss's Formula) If $x : U \rightarrow R^3$ is a coordinate patch, $x_{ij} = L_{ij} n + \Gamma^1_{ij} x_1 + \Gamma^2_{ij} x_2$.

Definition (Intrinsic Property) An intrinsic property of a surface is a property of the surface which only depends on the entries of the metric matrix (g_{ij}) .

Lemma (The Christoffel Symbols are Intrinsic) Let $x : U \rightarrow R^3$ be a coordinate patch with metric matrix (g_{ij}) . Then

$$\Gamma^l_{ij} = \frac{1}{2} \sum_{k=1}^{2} g^{kl} \left(\frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{kj}}{\partial u^i} \right)$$

Definition (Geodesic Coordinate Patch) If $x : U \rightarrow R^3$ is a coordinate patch such that the metric matrix (g_{ij}) is

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h^2 \end{pmatrix}$$

where

$$h^2 = \langle x_2, x_2 \rangle = \left(\sqrt{\langle x_2, x_2 \rangle} \right)^2 = \|x_2\|^2 \text{ , or simply h } = \|x_2\|$$

then x is called a geodesic coordinate patch.

There is a theorem that says we can always find a geodesic coordinate patch along a curve. What's important is that making the assumption of a geodesic coordinate patch in the Gauss-Bonnet Formula is not a severe restriction.

Definition (Angular Variation) The total angular variation of a vector field T along a curve γ with respect to the vector field X is the integral $\delta_X \alpha = \int_\gamma \alpha' \, ds$. Where α is a function of the parameter s and defined as the angle between a differentiable vector fielda X long γ and T along γ .

Lemma (When the Angular Variation is independent of) X If γ bounds a simply connected region R, and γ is homotopic to a point in R, then $\delta_X \alpha = \int_\gamma \alpha' \, ds$ does not depend on X .

Definition (A Vector Field Parallel Along a Curve) A differentiable vector field X along γ is parallel along γ if X' is perpendicular to the tangent plane.

Lemma (Existance of Parallel Vector Fields Along a Curve) If γ is a regular curve in a surface, then given a vector Y in the tangent plane at a point of γ (say at $\gamma(s_0)$) there exists a a unique vector field P parallel along γ such that $P(s_0) = Y$.

Now for the Gauss-Bonnet Formula.

Theorem (Gauss-Bonnet Formula) Let γ be a piecewise regular curve contained in a simply connected geodesic coordinate patch and bounding a region R in the patch. Let the jump angles between the curves making up γ be $\alpha_1, \ldots, \alpha_n$. Then

$$\iint_R K \, dA + \int_\gamma k_g \, ds + \sum_{k=1}^{n} \alpha_k = 2\pi$$

Proof: First we just make some derivations of the metric and the Christoffel symbols. Since we are in a geodesic coordinate patch, recall that the metric matrix $(g_{ij}) = (\langle x_i, x_j \rangle)$ has the simple form

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h^2 \end{pmatrix}$$

where,

$$h^2 = \langle x_2, x_2 \rangle = \left(\sqrt{\langle x_2, x_2 \rangle}\right)^2 = \|x_2\|^2 = \|x_2\|^2 \text{ , or simply } h = \|x_2\|$$

Then because the Christoffel symbols Γ^k_{ij} can be written in terms of the coefficients of the metric matrix (g_{ij}) we have

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{h_1}{h}, \quad \Gamma^1_{22} = -h h_1, \quad \Gamma^2_{22} = \frac{h_1}{h}, \quad \Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = 0$$

$$where, \quad h_i = \frac{\partial h}{\partial u^i}$$

For instance, computing Γ^2_{12} :

$$\Gamma^2_{12} = \frac{1}{2}\sum_{k=1}^{2} g^{k2}\left(\frac{\partial g_{1k}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^k} + \frac{\partial g_{k2}}{\partial u^1}\right) = \frac{1}{2}\sum_{k=1}^{2} g^{k2}\left(\frac{\partial g_{1k}}{\partial u^2} - 0 + \frac{\partial g_{k2}}{\partial u^1}\right)$$

$$= \frac{1}{2}g^{12}\left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{12}}{\partial u^1}\right) + \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1}\right) = 0 + \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1}\right)$$

$$= \frac{1}{2}\frac{1}{h^2}\left(0 + \frac{\partial g_{22}}{\partial u^1}\right) = \frac{1}{2}\frac{1}{h^2}\left(\frac{\partial (h^2)}{\partial u^1}\right) = \frac{1}{2}\frac{1}{h^2}\left(2h\,\frac{\partial h}{\partial u^1}\right) = \frac{h_1}{h}$$

$$\textit{Since},\ g_{12} = g_{21} = g^{12} = g^{21} = 0,\quad g^{22} = \frac{1}{h^2},\quad g_{22} = h^2$$

The others are obtained just as easily so we skip their proofs. The Gaussian curvature K can also be written in terms of the coefficients of the metric matrix (g_{ij}) . In particular, for a geodesic coordinate patch $K = -\frac{h_{11}}{h}$. That K can be written this way is an important theorem in its own right. It says that K is intrinsic. It is called Gauss's Theorem Egregium and while its computation in terms of the $g_{ij}{}^{'s}$ is reduced in a geodesic coordinate patch it is still a tedious calculation (so I'll omit it).

These derivations will be needed throughout the proof. For now we'll look at the angular variation of a vector field $\boldsymbol{P}$ parallel along γ . There's no loss of generality in assuming γ is parametrized by arc length because the terms $K,\ k_g,\ \alpha_k,$ are independent of parametrization. All this assumption does is limit further computations. In particular, we know that if γ is parametrized by arc length then $\gamma'(s)$ is a unit vector tangent to $\gamma(s)$ for all s. For clarity let $\boldsymbol{T}$ denote $\gamma'(s)$ (i.e. $\gamma'(s) = \boldsymbol{T}(s)$ for all s). We also know that along γ (precisely, along each piece of γ) there exist a unique unit vector field $\boldsymbol{P}$ parallel along γ (i.e. $\boldsymbol{P}'$ is always perpendicular to the tangent plane). Also, because we are in a geodesic coordinate patch the first partial $\boldsymbol{x}_1$ is a unit vector field along γ . That is, $g_{11} = \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle = 1$ which implies $\|\boldsymbol{x}_1\| = 1$ everywhere.

So now we have 3 unit vector fields, each contained in the tangent plane of the surface, along the curve γ . We need to look at the angles between these vectors, so we define the angle β between vectors $\boldsymbol{X},\ \boldsymbol{Y}$ in 3-space in the usual way

$$\cos(\beta) = \frac{\langle \boldsymbol{X}, \boldsymbol{Y}\rangle}{\|\boldsymbol{X}\|\|\boldsymbol{Y}\|}$$

Define the angles between our vector fields at parameter time s as follows:

$$\begin{aligned} \alpha(s) &= \textit{the angle between } \boldsymbol{x}_1,\ \boldsymbol{T} \\ \phi(s) &= \textit{the angle between } \boldsymbol{x}_1,\ \boldsymbol{P}, \qquad \textit{at parameter time } s \\ \theta(s) &= \textit{the angle between } \boldsymbol{P},\ \boldsymbol{T} \end{aligned}$$

Clearly we have the relations $\alpha(s) = \phi(s) + \theta(s),\qquad \alpha'(s) = \phi'(s) + \theta'(s)$.

Then because $\|x_1\| = \|P\| = 1$ the angle ϕ is

$$\cos(\phi) = \frac{\langle x_1, P\rangle}{\|x_1\|\|P\|} = \langle x_1, P\rangle$$

hence, (differentiation with respect to s)

$$(\cos(\phi))' = -\phi'\sin(\phi) = (\langle x_1, P\rangle)' = \langle x_1', P\rangle + \langle x_1, P'\rangle = \langle x_1', P\rangle + 0 = \langle x_1', P\rangle$$

where $\langle x_1, P'\rangle = 0$, since P is a vector field parallel along γ . That is, P' is in the normal space by definition and x_1 is in the tangent space so they are perpendicular which means the dot product is 0. Now recall that γ can be written in the form $\gamma(s)=x(\gamma^1(s), \gamma^2(s))$. Since we are evaluating everything along the curve γ we have by the chain rule

$$x_1' = (x_1(\gamma^1, \gamma^2))' = x_{11}(\gamma^1)' + x_{12}(\gamma^2)'$$

plugging this into the equation for $(\cos(\phi))'$ and using Gauss's Formula for x_{11} and x_{12} (i.e. $x_{11} = L_{11}n + \Gamma^1_{11}x_1 + \Gamma^2_{11}x_2$ and $x_{12} = L_{12}n + \Gamma^1_{12}x_1 + \Gamma^2_{12}x_2$) gives

$$-\phi'\sin(\phi) = \langle x_1', P\rangle = \langle x_{11}(\gamma^1)' + x_{12}(\gamma^2)', \quad P\rangle$$

$$= \langle (L_{11}n + \Gamma^1_{11}x_1 + \Gamma^2_{11}x_2)(\gamma^1)' + (L_{12}n + \Gamma^1_{12}x_1 + \Gamma^2_{12}x_2)(\gamma^2)', \quad P\rangle$$

$$= \langle (\Gamma^1_{11}x_1 + \Gamma^2_{11}x_2)(\gamma^1)' + (\Gamma^1_{12}x_1 + \Gamma^2_{12}x_2)(\gamma^2)', \quad P\rangle$$

$$= \left\langle (0x_1 + 0x_2)(\gamma^1)' + (0x_1 + \left(\frac{h_1}{h}\right)x_2)(\gamma^2)', \quad P\right\rangle$$

$$= \left\langle \left(\frac{h_1}{h}\right)x_2(\gamma^2)', \quad P\right\rangle = \left(\frac{h_1}{h}\right)(\gamma^2)'\langle x_2, P\rangle$$

since $\langle n, P\rangle = 0$ and by the values of the Christoffel coefficients Γ^k_{ij} derived earlier.

Hence

$$-\phi'\sin(\phi) = \left(\frac{h_1}{h}\right)(\gamma^2)'\langle x_2, P\rangle$$

Because $g_{12} = g_{21} = \langle x_1, x_2\rangle = \langle x_2, x_1\rangle = 0$ (i.e. x_1, x_2, are perpendicular), $\left(x_1, \frac{x_2}{\|x_1\|}\right)$ is an orthonormal basis of the tangent plane. From linear algebra we know that if $(u_1, \ldots, u_n)$ is an orthonormal basis of R^n then every vector $u \in R^n$ has the representation

$u = \langle u_1, u \rangle u_1 + \ldots + \langle u_n, u \rangle u_n$. So we can represent $\boldsymbol{P}$ in the basis $\left(\boldsymbol{x}_1, \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|} \right)$:

$$\boldsymbol{P} = \langle \boldsymbol{x}_1, \boldsymbol{P} \rangle \boldsymbol{x}_1 + \left\langle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}, \boldsymbol{P} \right\rangle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}$$

And because P is a unit vector

$$\|\boldsymbol{P}\| = \left\| \langle \boldsymbol{x}_1, \boldsymbol{P} \rangle \boldsymbol{x}_1 + \left\langle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}, \boldsymbol{P} \right\rangle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|} \right\| = \sqrt{\left(\langle \boldsymbol{x}_1, \boldsymbol{P} \rangle^2 + \left\langle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}, \boldsymbol{P} \right\rangle^2 \right)} = 1$$

if and only if

$$\langle \boldsymbol{x}_1, \boldsymbol{P} \rangle = \cos(\phi), \qquad \left\langle \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}, \boldsymbol{P} \right\rangle = \left\langle \frac{\boldsymbol{x}_2}{h}, \boldsymbol{P} \right\rangle = \sin(\phi)$$

Since the left equation already stands, the right equation has to equal $\sin(\phi)$. Plugging the right equation into $-\phi' \sin(\phi) = \left(\frac{h_1}{h} \right) (\gamma^2)' \langle \boldsymbol{x}_2, \boldsymbol{P} \rangle$ gives $\phi' = -h_1 (\gamma^2)'$. Now when we compute the angular variation of ϕ we get $\delta\phi = \int_\gamma \phi' ds = \int_\gamma -h_1 (\gamma^2)' ds = \int_\gamma -h_1 du^1$.

We now consider the angle θ . $\cos(\theta) = \langle \boldsymbol{T}, \boldsymbol{P} \rangle$ implies

$$(\cos(\theta))' = -\theta' \sin(\theta) = (\langle \boldsymbol{T}, \boldsymbol{P} \rangle)' = \langle \boldsymbol{T}', \boldsymbol{P} \rangle + \langle \boldsymbol{T}, \boldsymbol{P}' \rangle = \langle \boldsymbol{T}', \boldsymbol{P} \rangle + 0 = \langle \boldsymbol{T}', \boldsymbol{P} \rangle$$

Hence, $\theta' \sin(\theta) = \langle \boldsymbol{T}', \boldsymbol{P} \rangle$.

Using this result and some familiar rules from vector calculus we can compute the geodesic curvature in terms of θ .

The rules being:

1) $\boldsymbol{X} \times \boldsymbol{Y} = (\|\boldsymbol{X}\| \|\boldsymbol{Y}\| \sin(\theta)) \boldsymbol{N}$ where θ is the angle between $\boldsymbol{X}$ and $\boldsymbol{Y}$. And $\boldsymbol{N}$ is the normal to $\boldsymbol{X}$ and $\boldsymbol{Y}$.
2) $\langle \boldsymbol{X}, \boldsymbol{Y} \times \boldsymbol{Z} \rangle = \langle \boldsymbol{Y}, \boldsymbol{Z} \times \boldsymbol{X} \rangle$.
3) $\boldsymbol{X} \times (\boldsymbol{Y} \times \boldsymbol{Z}) = \boldsymbol{Y} \langle \boldsymbol{X}, \boldsymbol{Z} \rangle - \boldsymbol{Z} \langle \boldsymbol{X}, \boldsymbol{Y} \rangle$.

Then,

$$k_g = \langle \boldsymbol{n} \times \boldsymbol{T}, \boldsymbol{T}' \rangle = \langle \boldsymbol{T}', \boldsymbol{n} \times \boldsymbol{T} \rangle = \langle \boldsymbol{n}, \boldsymbol{T} \times \boldsymbol{T}' \rangle = \left\langle \frac{\boldsymbol{P} \times \boldsymbol{T}}{\sin(\theta)}, \boldsymbol{T} \times \boldsymbol{T}' \right\rangle$$

$$= \frac{1}{\sin(\theta)} \langle \boldsymbol{P}, \boldsymbol{T} \times (\boldsymbol{T} \times \boldsymbol{T}') \rangle = \frac{1}{\sin(\theta)} \langle \boldsymbol{P}, -\boldsymbol{T}' \rangle = \theta'$$

So the geodesic curvature is simply, $k_g = \theta'$ which means $\int_\gamma k_g ds = \int_\gamma \theta' ds$. From the relation $\alpha'(s) = \phi'(s) + \theta'(s)$ we get

$$\int_\gamma \alpha' ds = \int_\gamma \theta' ds + \int_\gamma \phi' ds = \int_\gamma k_g ds + \int_\gamma -h_1 du^1$$

Furthermore, $\int_\gamma \alpha' ds + \sum_{k=1}^{n} \alpha_n'$ is the total angular variation of T along γ with respect to x_1 (taking the angles between the curves into account). Of course, since γ bounds the region R , a simply connected region, and γ is homotopic to a point in R , we know that $\int_\gamma \alpha' ds + \sum_{k=1}^{n} \alpha_n'$ is independent of x_1 . Because we are traveling around a "circle" (around γ) it is geometrical clear that the angle α between T and x_1 will vary by 2π (e.g. imagine traveling around the unit circle in the plane and measuring the angle between the tangent vector and the x-axis. It is clear that the angle sweeps out 2π). Hence, the total angular variation, $\delta\alpha$, is

$$\delta\alpha = \int_\gamma \alpha' ds + \sum_{k=1}^{n} \alpha_n' = 2\pi$$

Putting this into what we derived earlier gives

$$\int_\gamma \alpha' ds = -\sum_{k=1}^{n} \alpha_n' + 2\pi = \int_\gamma k_g ds + \int_\gamma -h_1 du^1$$

Now, recall that Stokes' Theorem says that

$$\int_{\gamma=\delta R} P du^1 + Q du^2 = \iint_R \left(\frac{\partial Q}{\partial u^1} - \frac{\partial P}{\partial u^2} \right) du^1 du^2$$

With $P = 0$ and $Q = h_1$ we get

$$-\int_{\gamma=\delta R} k_1 du^2 = -\iint_R \left(\frac{\partial k_1}{\partial u^1} \right) du^1 du^2 = -\iint_R (h_{11}) du^1 du^2 = -\iint_R (h_{11}) \left(\frac{h}{h} \right) du^1 du^2$$

$$= \iint_R \left(\frac{-h_{11}}{h} \right) (h)\, du^1 du^2 = \iint_R (K)(h\, du^1 du^2) = \iint_R K\, dA$$

The last equality follows from the definition of dA (i.e. $dA = \sqrt{(g_{11} g_{22} - (g_{12})^2)} u^1 u^2$ and in a geodesic coordinate patch this reduces to $dA = \sqrt{((1) g_{22} - 0)} u^1 u^2 = \sqrt{(h^2)} u^1 u^2 = h u^1 u^2$) Putting this with the previous stuff gives

$$-\sum_{k=1}^{n} \alpha_n' + 2\pi = \int_\gamma k_g ds + \int_\gamma -h_1 du^1 = \int_\gamma k_g ds + \iint_R K\, dA$$

proving the formula.

Corollary (Gauss-Bonnet Theorem) If M is a compact surface then $\iint_M K\, dA = 2\pi\chi(M)$.

Proof: We know that M can be triangulated, so define a triangulation on M with V, E, F, the number of vertices, edges, and faces, respectively. Furthermore, we require that the curves making up the triangulation are geodesics so that the total geodesic curvature will be zero. On each face we can find a simply connected geodesic coordinate patch. Define these regions as, $R_i,\ 1 \leqslant i \leqslant F$ with boundaries $\gamma_i,\ 1 \leqslant i \leqslant F$ and interior angles β_{ij} . The interior angles are related to the jump angles by $\alpha_{ij} = \beta_{ij} - \pi$ and from the Gauss-Bonnet Formula we get,

$$\iint_M K\, dA = \sum_{i=1}^{F} \iint_{R_i} K\, dA = \sum_{i=1}^{F}\left(2\pi - 0 - \sum_j (\pi - \beta_{ij})\right) = 2\pi F - \sum_{i=1}^{F}\sum_j \pi + \sum_{i=1}^{F}\sum_j \beta_{ij}$$

Now, $\sum_j \pi = \pi V = \pi E$ and because each angle belongs to two of the triangles we have that

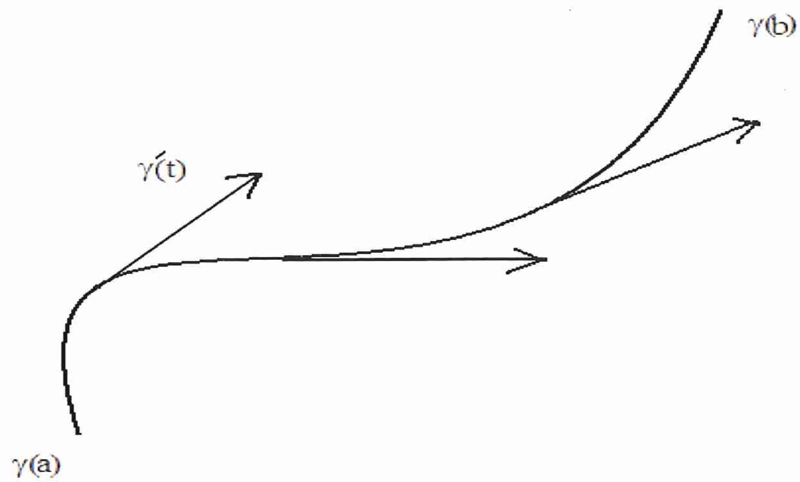
$\sum_{i=1}^{F}\sum_j \pi = 2\pi E$. Also, each vertex is surrounded by interior angles so the sum of interior angles at a vertex is 2π , so the sum of all the interior angles is $2\pi V = \sum_{i=1}^{F}\sum_j \pi$. Then,

$$\iint_M K\, dA = 2\pi F - \sum_{i=1}^{F}\sum_j \pi + \sum_{i=1}^{F}\sum_j \beta_{ij} = 2\pi F - 2\pi E + 2\pi V = 2\pi\chi(M) .$$

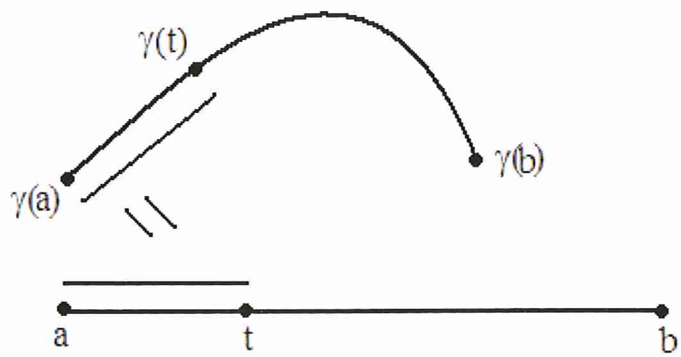
Class Presentation

Curvature of Curves

*A <u>regular curve</u> is an infinitely differentiable function γ:[a,b]→ R^n such that $\gamma'(t) \neq \mathbf{0}$ for all t in I. This insures the derivative of γ exists for all t. $\gamma'(t)$ turns out to be the tangent vector to the curve.

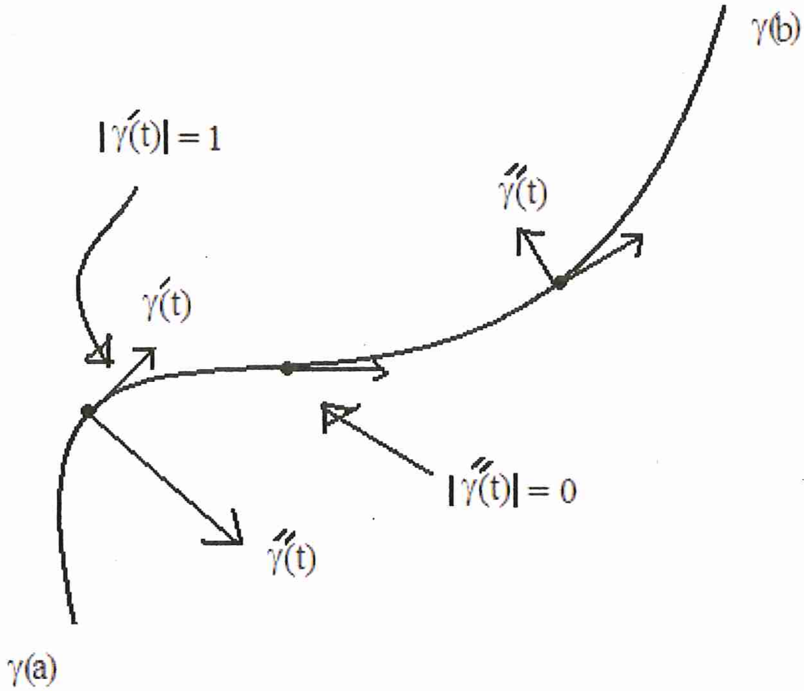


*A <u>curve parametrized by arc length</u> is a curve γ:[a,b]→ R^n with the property that the length of the curve equals the distance from the parameter to the initial point, $\|\gamma(t)\| = t - a$.



*Every regular curve can be changed into a curve parametrized by arc length. That is, for some regular curve α not parametrized by arc length there exists a regular curve β parametrized by arc length with the same image as α.

The important property of these curves is $\|\gamma'(t)\| = 1$ for all t in [a,b], which implies $\gamma'(t)$ is orthogonal to $\gamma''(t)$. That is, $\|\gamma'(t)\|^2 = \langle\gamma'(t),\gamma'(t)\rangle = 1$, implies, $0 = (1)' = (\langle\gamma'(t),\gamma'(t)\rangle)' = \langle\gamma'(t),\gamma'(t)\rangle + \langle\gamma'(t),\gamma'(t)\rangle = 2\langle\gamma''(t),\gamma'(t)\rangle$.



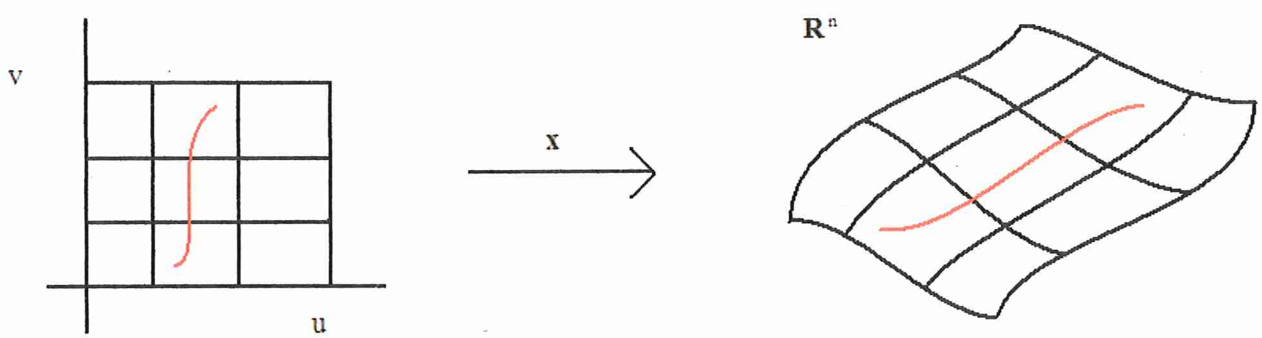
Notice that $\|\gamma'(t)\| = 1$ everywhere along γ, and that the more the curve $\gamma(t)$ curves the larger the vector $\gamma''(t)$ is. The magnitude of $\gamma''(t)$ is the <u>curvature</u> of $\gamma(t)$ at t.

*So again: if $\gamma(t)$ is a regular curve parametrized by arc length then the curvature k of $\gamma(t)$ at t is defined as $k = \|\gamma''(t)\|$.

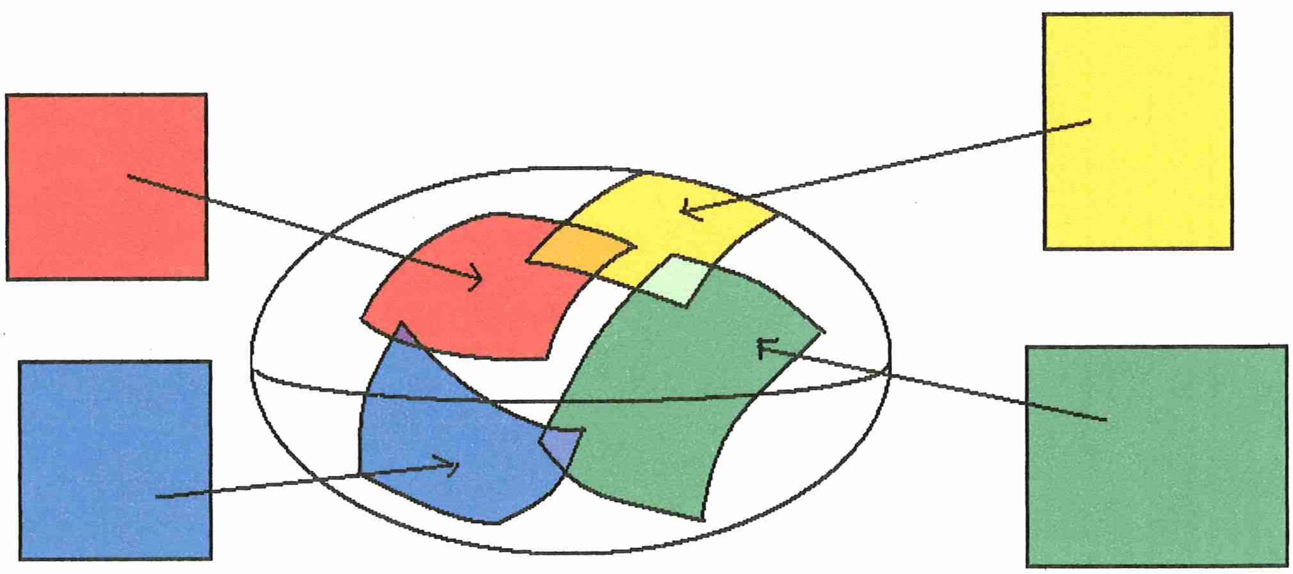
To define the curvature K of a surface (called the gaussian curvature) at a point p on the surface the curvatures of the curves inside the surface at the point p have to be considered.

Surfaces

A surface M is a collection of maps called coordinate patches that cover M, where a coordinate patch is a infinitely differentiable bijective map $\mathbf{x}:U \to R^n$, where U is an open set in R^2 with coordinates u and v with the added requirement that the cross product of the partials $\mathbf{x}_u$ and $\mathbf{x}_v$ of $\mathbf{x}_u \wedge \mathbf{x}_v \neq \mathbf{0}$.



Coordinate Patch

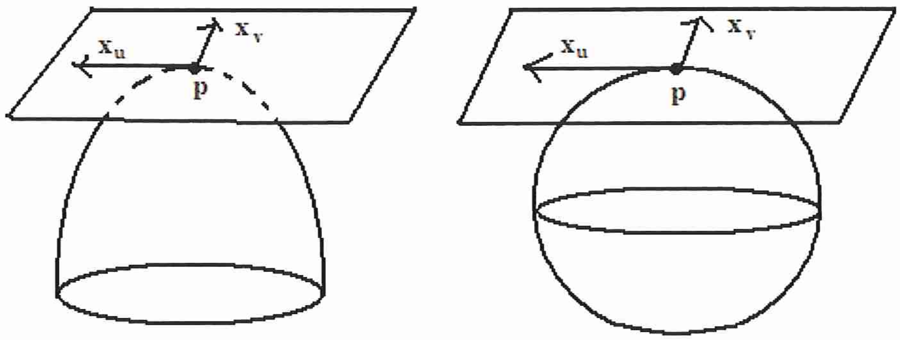


Surface showing 4 coordinate patches

Notice that $\mathbf{x}_u \wedge \mathbf{x}_v \neq \mathbf{0}$ just says $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent so that the tangent plane can be defined everywhere on a surface.

Curvature of Surfaces

*The tangent plane at a point p on a surface M (denoted TpM) is the plane spanned by the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ evaluated at (a,b), where, $p = \mathbf{x}(a,b)$.



Some tangent planes

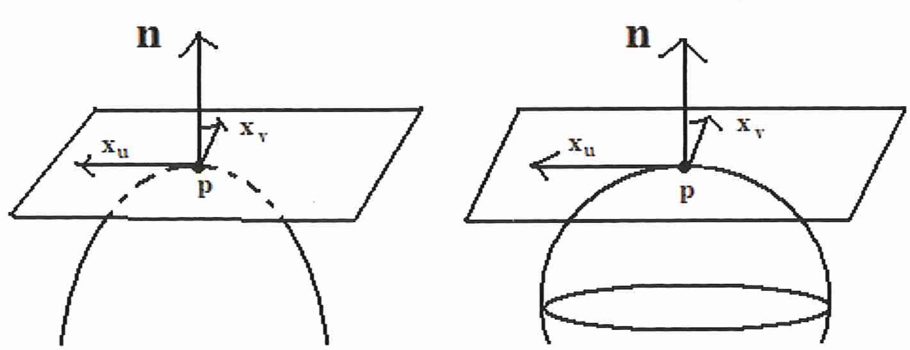
Properties of the tangent plane:

1) The tangent plane is a vector space with basis ($\mathbf{x}_u$, $\mathbf{x}_v$). That is, if v is any vector in the tangent space then it has the form $v = a\mathbf{x}_u + b\mathbf{x}_v$ for some constants a,b. And again, $\mathbf{x}_u$ and $\mathbf{x}_v$ linearly independent.

2) If $\gamma:[a,b]\rightarrow R^n$ is any regular curve through a point p then its tangent vector at p is in TpM.

3) If v is a vector in TpM then there exists a regular curve $\gamma:[a,b]\rightarrow R^n$ through p such that $\gamma' = v$ at p.

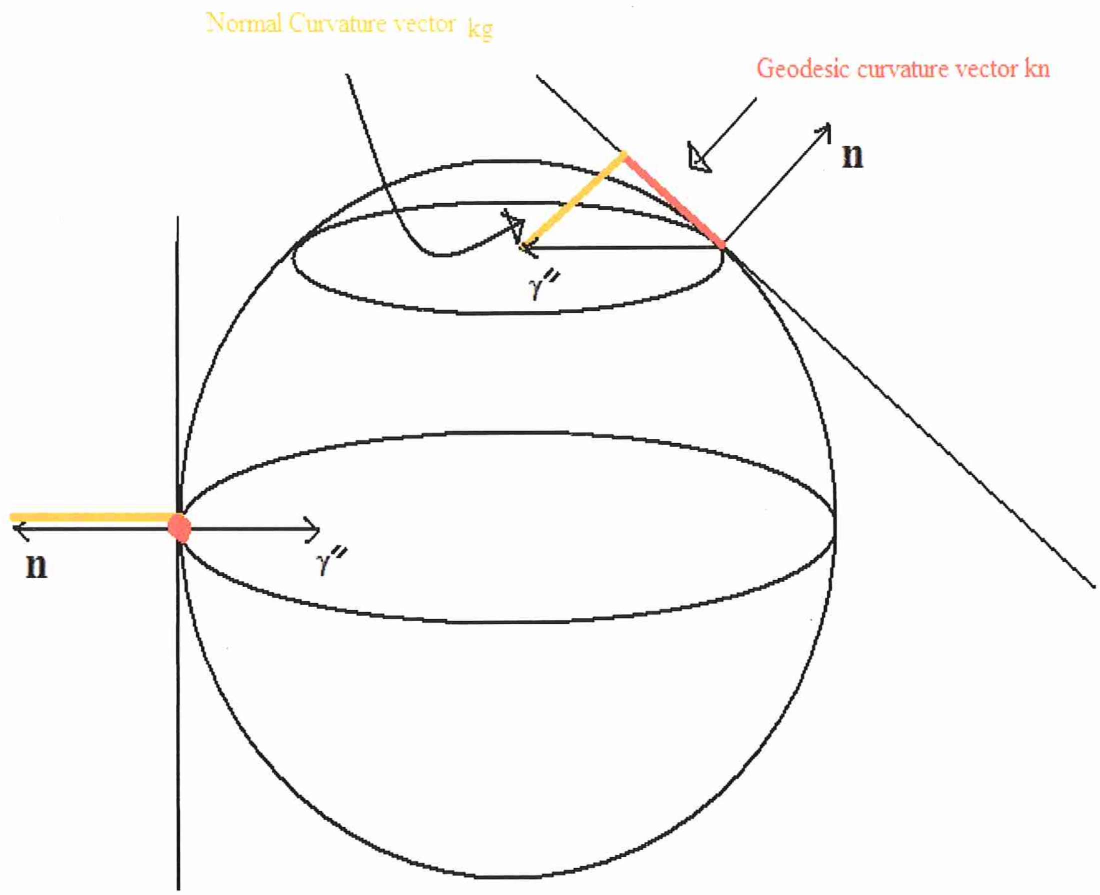
*The normal space is the set of vectors orthogonal to the tangent space. This is the set {r**n**: r is in R} where **n** is a unit vector defined as, $\mathbf{n} = \mathbf{x}_u \wedge \mathbf{x}_v / \| \mathbf{x}_u \wedge \mathbf{x}_v \|$.



Some normals

* If $\gamma:[a,b] \rightarrow R^n$ is a regular curve parametrized by arc length then the geodesic curvature of γ (denoted k_g) at a point p on M is the magnitude of the projection of the curvature vector γ''

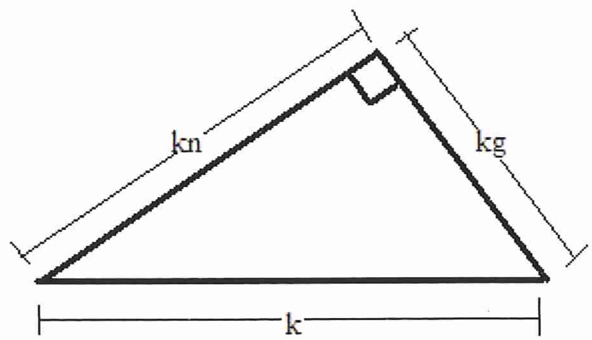
* If $\gamma:[a,b] \rightarrow R^n$ is a regular curve parametrized by arc length then the normal curvature of γ (denoted k_n) at a point p on M is the magnitude of the component of the curvature vector γ'' in the direction of **n**.



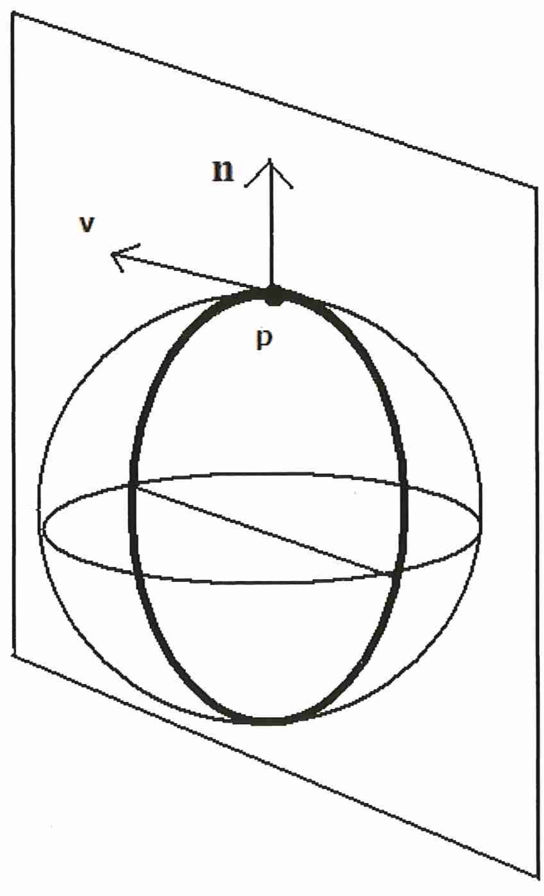
Normal and Geodesic curvatures on the sphere

Notice that the geodesic curvature is 0 at every point along the equator. This is true for any great circle on the sphere. The great circles are lines with the geodesic curvature equal to zero everywhere. They are the only lines with this property and are the analogy of straight lines in the plane. A curve in some surface M with geodesic curvature everywhere 0 is called a geodesic. The Gauss-Bonnet theorem will show that polygons formed by geodesics have analogies to plane geometry. Also, when the Gauss-Bonnet theorem is applied to the plane the same results hold (i.e. the sum of the interior angles of a triangle in the plane is π).

*The normal, geodesic, and γ'' curvature vectors form a right triangle with the γ'' vector as the hypotenuse so that $k^2 = (kg)^2 + (kn)^2$.



Consider a surface M and a tangent plane TpM at a point p in M. One can form curves in the surface M through p by intersecting the surface with planes. In particular consider the planes formed by the span of the normal vector to the surface, **n**, with a vector **v** in the tangent plane TpM (called a normal section).

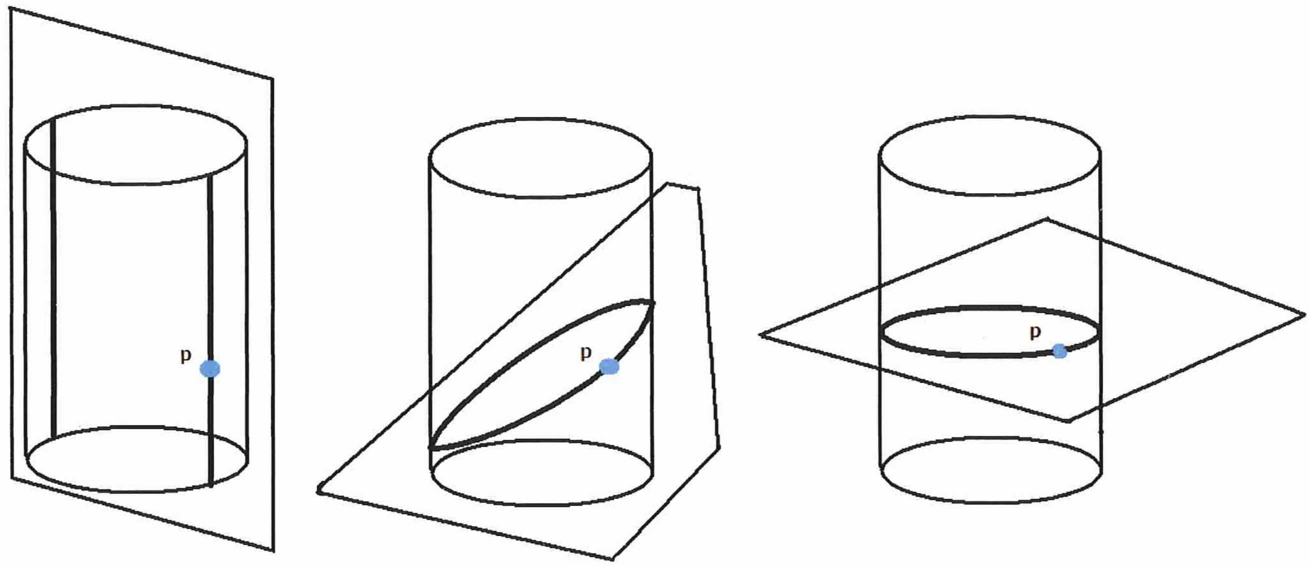


A normal section of the sphere at the north pole: A plane (formed by the tangent vector **v** and **n**) intersecting the sphere and forming a curve (bold line).

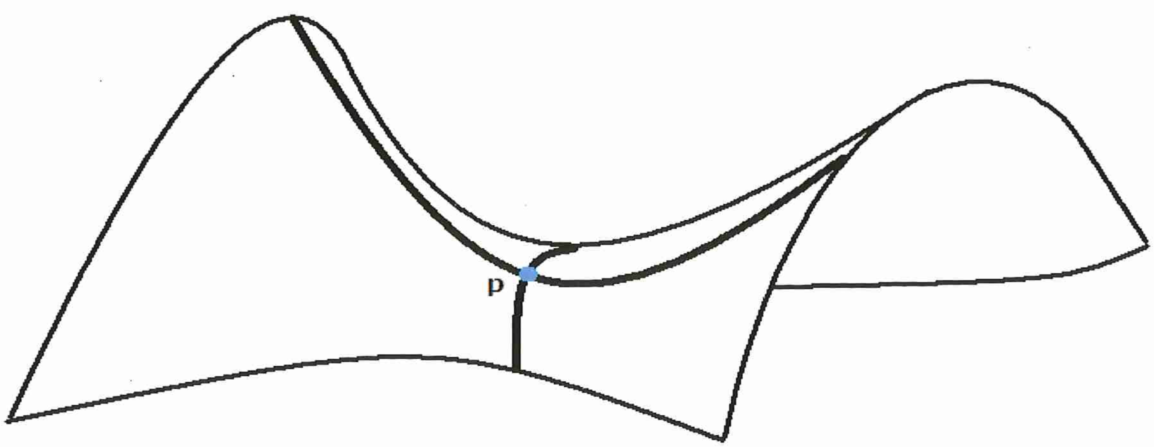
Doing this for every vector v in the tangent plane yields a family of curves on M through p. One can then find the normal curvatures of each of the curves.

*It turns out that this family of curves has a maximum normal curvature (k1) and a minimum normal curvature (k2). k1 and k2 are called the principal curvatures.

*The Gaussian curvature K at a point p on a surface M is defined to be the product of the principle curvatures k1 and k2.



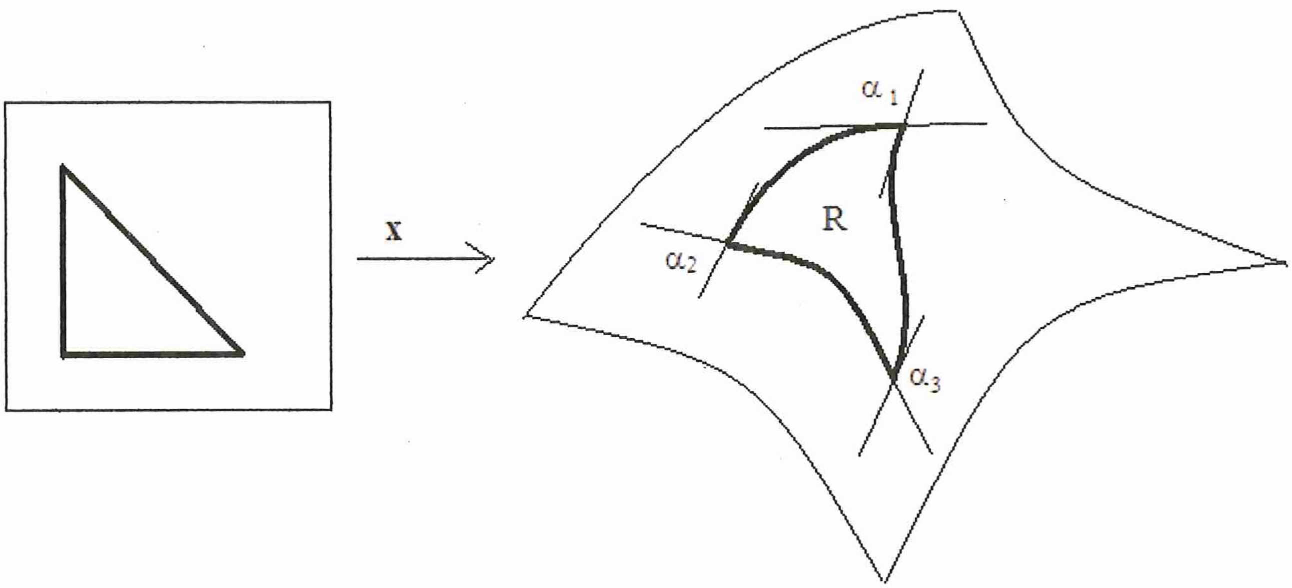
Normal sections of an annulus of radius 1 at p. The left and right images are normal sections associated with k2 and k1 respectively and clearly k2 = 0 and k1 = 1. So the gaussian curvature is K = k1*k2 = 0*1 = 0. The Gauss-Bonnet theorem will show that the geometry is of the annulus is just like that of the plane.



A surface with k2 < 0 < K1, so that K<0.

Gauss-Bonnet Formula

* Let γ be a piecewise regular curve contained within a simply connected coordinate patch and bounding a region R in the patch. Let the jump angles at the endpoints be $\alpha_1, \ldots, \alpha_3$. Then, $\iint_R K\,dA + \int_\gamma k_g\,ds + \Sigma\,\alpha_i = 2\pi$.



A triangular like curve bounding R with jump angles shown.

Note: $\iint_R K\,dA$ is the surface integral over the area element dA, so that $\iint_R dA$ is the area of the region R. So if K is a constant $\iint_R K\,dA$ is a multiple of the area of the region. Also, $\int_\gamma k_g\,ds$ is the ordinary integral of the function kg. So if kg is 0 everywhere along the curve, $\int_\gamma k_g\,ds = 0$.

Example: If $K > 0$ then considering only geodesic triangles with interior angles $\beta_1, \beta_2, \beta_3$, since the jump angles are $\alpha_i = \pi - \beta_i$.

$$\Sigma\,\alpha_i = \Sigma(\pi - \beta_i) = 2\pi - \iint_R K\,dA \text{ or } \Sigma\,\beta_i > \pi.$$

That is, the sum of the interior angles of a geodesic triangle is strictly greater than π.

If $K = 0$, then considering only geodesic triangles $\Sigma\, \alpha_i = 2\pi$ or the sum of the interior angles equals π. So this a agrees with plane geometry and since the annulus has zero curvature it has the same type of geometry as the plane.

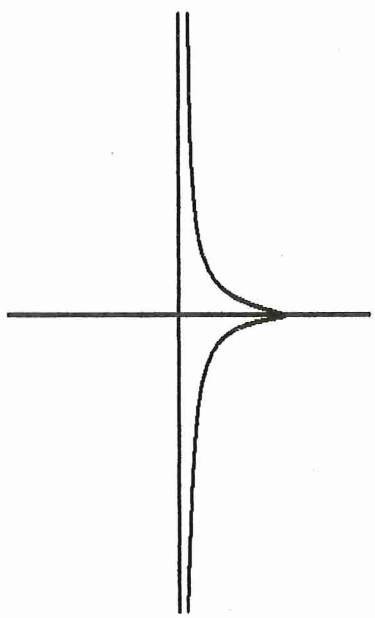
Similarly, if $K < 0$, The sum of the interior angels of a geodesic triangle is strickly less than π.

Example: Let Δ be a triangle on S^2, with boundary γ and sides geodesics and interior angles $\beta_1, \beta_2, \beta_3$. Then, since K=1 everywhere, and $k_g = 0$ everywhere,

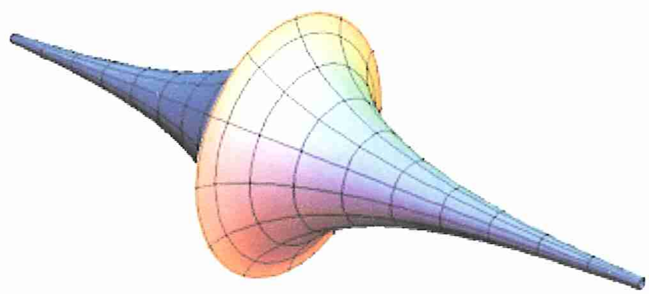
$$2\pi = \iint_M K\, dA + \int_\gamma k_g\, ds + \Sigma\, \alpha_i = \text{Area}(\Delta) + 0 + \Sigma\, (\pi - \beta_i) = \text{Area}(\Delta) + 3\pi - (\Sigma\, \beta_i)$$

Which means, $\text{Area}(\Delta) = (\beta_1 + \beta_2 + \beta_3) - \pi$.

Example: The pseudosphere: The pseudosphere is a 2 dimensional surface of revolution in R^3 generated by a curve called a tractrix.



Tractrix



Pseudosphere

Evidently, this surface has constant negative gaussian curvature $K = -1$. Also, it is isometric to the poincare disk H^2. So, the geodesic on the pseudosphere correspond with the geodesics in the disk. The gauss-bonnet theorem then gives, $\text{Area}(\Delta) = \pi - (\beta_1 + \beta_2 + \beta_3)$. This is the opposite of the formula for the area of the triangle on the sphere.

Gauss-Bonnet Theorem

* If M is an orientable compact surface then $\iint_M K\,dA = 2\pi\chi(M)$.

This means that for any orientable compact surface M, $\iint_M K\,dA$ is a topological invarient. This is weird because K is a very geometric concept. That, $(2\pi)^{-1}\iint_M K\,dA$ is integer valued seems weird enough by itself.

Example: For $M = S^2$ we know $K = 1$ everywhere and $\chi(M) = 1$. Then,

$$\iint_{S^2} K\,dA = \iint_{S^2} 1\,dA = \text{Area}(S^2) = 4\pi$$

Which agrees what we already know.

Example: If $K > 0$ everywhere on a surface M then the theorem implies $\chi(M) > 0$. We know that the only surface with positive euler characteristic is a surface homeomorphic to a sphere. So a surface M with positive gaussian curvature must be homeomorphic to a sphere.