Consider the function \( f(x, y) = (2x + y^2)e^{-x^2} \) and its graph \( S \). (1) (5 points) Find the first partial derivatives and gradient of \( f \). (2) (4 points) (a) Find an equation for the tangent plane to \( S \) at the point \((0, -1, 1)\). (b) What is the directional derivative of \( f \) at \((0, -1)\) in the direction of the vector \( v = <3, 4> \)? (c) What is the maximal directional derivative of \( f \) at \((0, -1)\)? (3) (9 points) Find the critical points of \( f \), then find the local maxima and minima of \( f \). Bonus (2 points): are these also global maxima/minima? (4) (2 points) If \( x = st \) and \( y = s + t \), find the partial derivatives \( \frac{df}{ds} \) and \( \frac{df}{dt} \).

1. \( \frac{\partial f}{\partial x} = 2e^{-x^2} - 2x(2x + y^2)e^{-x^2} = (2-4x^2-2xy)e^{-x^2} \); \( \frac{\partial f}{\partial y} = 2 ye^{-x^2} \). Therefore: \( \nabla f(x, y) = <(2-4x^2-2xy)e^{-x^2}, 2 ye^{-x^2}> \).

2. (a) \( \nabla f(0, -1) = <2, -2> \) so an equation for the tangent plane is: \( z - 1 = 2x - 2(y+1) \). 
   
   (b) Recall that for a unit vector \( \vec{w} \), the directional derivative of \( f \) at \((x, y)\) in the direction of \( \vec{w} \) is: \( D_{\vec{w}}f(x, y) = \vec{w} \cdot \nabla f(x, y) \). We first rescale \( \vec{w} \) to have unit length: \( \vec{w} = \frac{\vec{v}}{||\vec{v}||} = <\frac{3}{5}, \frac{4}{5}> \); then: \( D_{\vec{w}}f(0, -1) = D_{\vec{w}}f(0, -1) = \vec{w} \cdot \nabla f(0, -1) = 2 \cdot \frac{3}{5} - 2 \cdot \frac{4}{5} = -\frac{2}{5} \).

3. We know that the maximal directional derivative is \( ||\nabla f|| \) (in the direction of \( \nabla f \)). Here: \( ||\nabla f|| = \sqrt{4+4} = 2\sqrt{2} \).

3. (3) For the critical points, solve \( \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0 \) for \( x \) and \( y \).

This gives: \( y = 0 \) and \( (2-4x^2-2xy) = 0 \).

Therefore \( f \) has two critical points: \( \left( \frac{1}{\sqrt{2}}, 0 \right)^{P^+} \) and \( \left( -\frac{1}{\sqrt{2}}, 0 \right)^{P^-} \).

We then use the second partial derivative test to determine whether or not \( f \) has a local maximum or minimum at \( P^+ \) and \( P^- \).

We start by computing the second partial derivatives of \( f \):
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= (-8x - 2y) e^{-x^2} - 2x(2 - 4x^2 - 2xy^2) e^{-x^2} = (-12x - 2y^2 + 8x^3 + 4x^2 y^2) e^{-x^2} \\
\frac{\partial^2 f}{\partial x \partial y} &= -4xy e^{-x^2} \\
\frac{\partial^2 f}{\partial y^2} &= 2 e^{-x^2}
\end{align*}
\]

Evaluating these 3 functions at \( p^+ = \left( \frac{1}{\sqrt{2}}, 0 \right) \) and \( p^- = \left( -\frac{1}{\sqrt{2}}, 0 \right) \) gives:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x}(\frac{1}{\sqrt{2}}, 0) &= 4x(2x^2 - 3) e^{-x^2} = -4\sqrt{2} e^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial x}(\frac{1}{\sqrt{2}}, 0) &= 0 \\
\frac{\partial^2 f}{\partial y}(\frac{1}{\sqrt{2}}, 0) &= 2 e^{-\frac{1}{2}}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2}(\frac{1}{\sqrt{2}}, 0) &= -8\sqrt{2} e^{-\frac{1}{2}} < 0 \\
\frac{\partial^2 f}{\partial y^2}(\frac{1}{\sqrt{2}}, 0) &= 8\sqrt{2} e^{-\frac{1}{2}} > 0
\end{align*}
\]

The determinant \( D = \frac{\partial^2 f}{\partial x \partial y} - \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) = -8\sqrt{2} e^{-\frac{1}{2}} < 0 \) for \( \Delta = 8\sqrt{2} e^{-\frac{1}{2}} > 0 \) and \( \frac{\partial^2 f}{\partial x} > 0 \Rightarrow f \) has a local maximum at \( p^+ \) ("Saddle point").

Min. value: \( f(p^-) = -\sqrt{2} e^{-\frac{1}{2}} \).

**Bonuy:** Along vertical lines \( (x = \text{constant}) \), \( f(x, y) \xrightarrow{y \to \pm \infty} \infty \).

In all other directions, \( f(x, y) \xrightarrow{} 0 \) \( (e^{-x^2} \text{ dominates}) \).

Therefore the local min at \( p^- \) is also a global minimum for \( f \).

4) Write the Chain Rule:

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

\[
\begin{align*}
\frac{\partial f}{\partial s} &= (2 - 4x^2 - 2xy^2) e^{-x^2} t + 2ye^{-x^2} \\
\frac{\partial f}{\partial t} &= (2 - 4x^2 - 2xy^2) e^{-x^2} s + 2ye^{-x^2}
\end{align*}
\]