Midterm Exam # 1
Time: 50 minutes

No calculators, notes, books. Show all work.

Part 1: A point \( P(t) \) is moving in 3-space with position vector \( \mathbf{r}(t) = (t \cos 2\pi t, t \sin 2\pi t, t) \).
1) Find the velocity and acceleration vectors and the speed of the point at any time \( t \), then at time \( t = 0 \). (8 points)
2) Find parametric equations for the tangent line to the curve at \( t = 0 \). (2 points)
3) Write the formulae for the tangential and normal components of acceleration (\( a_T \) and \( a_N \)) and the curvature \( \kappa \) in terms of the vectors \( \mathbf{r}'(t) \) and \( \mathbf{r}''(t) \). Find \( a_T, a_N, \) and \( \kappa \) at \( t = 0 \). (6 points)

Bonus question (2 points): What does the curve look like? (sketch and/or describe it)

1) \( \mathbf{v}(t) = \mathbf{r}'(t) = \left( \cos 2\pi t - 2\pi \sin 2\pi t, \sin 2\pi t + 2\pi t \cos 2\pi t, 1 \right) \)
\( \mathbf{v}(0) = (1, 0, 1) \)
\( \mathbf{a}(t) = \mathbf{r}''(t) = \left( -2\pi \sin 2\pi t - 4\pi^2 \cos 2\pi t, 2\pi \cos 2\pi t + 4\pi \sin 2\pi t - 4\pi^2 \sin 2\pi t, 0 \right) \)
\( \mathbf{a}(0) = (0, 4\pi, 0) \)

Speed: \( v(t) = \| \mathbf{v}(t) \| = \sqrt{(\cos 2\pi t - 2\pi \sin 2\pi t)^2 + (\sin 2\pi t + 2\pi t \cos 2\pi t)^2 + 1} \)
\( = \sqrt{\cos^2 2\pi t + \sin^2 2\pi t + 4\pi^2 t^2 + 4\pi^2 t^2 \sin^2 2\pi t + 4\pi^2 t^2 \cos^2 2\pi t - 4\pi^2 t^2 \sin^2 2\pi t - 4\pi^2 t^2 \cos^2 2\pi t + 1} \)
\( = \sqrt{2 + 4\pi^2 t^2} \)
\( v(0) = \sqrt{2} \)

2) At \( t = 0 \): \( \mathbf{r}'(0) = (0, 0, 0) \), \( \mathbf{v}'(0) = (1, 0, 1) \), so the tangent line has parametric equations: \( x(t) = t, \ y(t) = 0, \ z(t) = t \).

3) We know that: \( a_T = \frac{\mathbf{r}' \cdot \mathbf{n}}{\| \mathbf{n} \|} \) \( a_N = \frac{\| \mathbf{r}' \times \mathbf{n} \|}{\| \mathbf{n} \|} \) \( \kappa = \frac{a_N}{v^2} = \frac{\| \mathbf{r}' \times \mathbf{n} \|}{\| \mathbf{n} \|^3} \)

Evaluating these at \( t = 0 \): \( a_T = 0 \) \( a_N = \frac{\| (-4\pi, 0, 4\pi) \|}{\sqrt{2}} = \sqrt{32\pi^2} = 4\pi \)
and \( \kappa = \frac{4\pi}{\sqrt{2} \sqrt{2}} = 2\pi \).
Part 2: (4 points) Consider the vectors \( \vec{u} = (1, 2, 1) \) and \( \vec{v} = (2, -1, 1) \). (a) Are \( \vec{u} \) and \( \vec{v} \) orthogonal? (b) Find a vector orthogonal to both \( \vec{u} \) and \( \vec{v} \). (c) Find an equation for the plane through the origin containing \( \vec{u} \) and \( \vec{v} \).

(a) \( \vec{u} \cdot \vec{v} = 1 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1 = 1 \) so no, \( \vec{u} \) and \( \vec{v} \) are not orthogonal.

(b) We can either take a general vector \( \vec{w} = (x, y, z) \) and write down the 2 equations saying that \( \vec{u} \cdot \vec{w} = 0 \) or \( \vec{v} \cdot \vec{w} = 0 \), or much quicker, remember that \( \vec{u} \times \vec{v} \) is orthogonal to both \( \vec{u} \) and \( \vec{v} \).

\[
\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+1 \\ -(1-2) \\ 1-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}
\]

(c) This plane can be described as the set of all vectors \( \vec{w} = (x, y, z) \) which are orthogonal to \( \vec{u} \times \vec{v} \), so it has equation:

\[3x + y - 5z = 0.\]