THE DEGREE-GENUS FORMULA FOR NON-SINGULAR COMPLEX ALGEBRAIC CURVES

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1. Introduction

The study of elliptic curves is a fascinating subject. With the definition of a curious group law, elliptic curves become powerful computational devices in number theory. Perhaps more interesting is that, through careful construction of elliptic curves, one can create curves whose group law is identical to that of multiplication or addition. In a sense, all the operations we use in day to day life can be created and studied on elliptic curves, as can some far more exotic ones.

The possible group structures on elliptic curves over the rationals is somewhat well understood. It is known that this group is abelian, and the possible structures of its torsion points is known as well. Finding the way to describe the rank of all these groups, however, remains an open question.

One way that has been postulated to approach this problem is to study the topology of elliptic curves. It turns out that the topology of elliptic curves is relatively easy to describe - they're all isomorphic to tori.

In fact, however, it is possible to make a more general statement. Elliptic curves are a special case of a more general object refered to as a complex algebraic curve. It turns out that all complex algebraic curves are also homeomorphic to connected sums of tori. Proving this will be the goal of this paper.

It is important to stress the word *relatively* in relatively easy. The proof of the degree-genus formula is far from trivial, and requires a modest background in the properties of complex algebraic curves, as well as some results from topology. This paper will be written assuming the reader has an adequate background.

2. One Method of Proof

Recall that a complex algebraic curve is the set of zeroes of a homogeneous polynomial $P(x, y, z) = 0$ realized in the complex projective space P_2. The degree of a complex algebraic curve is simply the degree of the polynomial $P(x, y, z)$ used to define it.

Complex algebraic curves are topologically oriented surfaces, and so have a genus. The degree-genus formula states that

Theorem 1. *Nonsingular comples algebraic curves of degree d are topologically equivalent to a surface of genus g, where*

$$g = \frac{1}{2}(d-1)(d-2).$$

Note that the possible values of g are the triangular numbers.

There are two basic ways to approach proving this theorem. The first is interesting enough to deserve a mention, but the details are very complicated and are omitted.

The first way is to first realize a *singular* complex algebraic curve C as the union of d projective lines, where d is the degree of the curve. Recall that complex projective lines are homeomorphic to S^2. Since d lines intersect in $\frac{1}{2}d(d-1)$ points, this shows that the curve is homeomorphic to d spheres touching at $\frac{1}{2}d(d-1)$ points.

From there, one needs to show that it is possible to perturb the coefficients of the polynomial $P(x, y, z)$ defining the curve C an arbitrarily small amount in order to define a new, *nonsingular* curve, C'. Furthermore, it needs to be shown that topologically, this is identical to taking the points where the d spheres intersect and widening these points of intersection into connecting handles.

This means, then, that topologically, C' is equivalent to d spheres connected by $\frac{1}{2}(d)(d-1)$ handles. $d-1$ of these handles can be used to join the d spheres into one sphere. We then have a sphere with

$$\frac{1}{2}d(d-1) - (d-1) = \frac{1}{2}(d-1)(d-2)$$

handles, meaning it is a surface of genus $g = \frac{1}{2}(d-1)(d-2)$, as desired.

From there, one needs to prove that one can perturb the coefficients of an arbitrary non-singular curve without changing the resulting topology. Finally, one then shows the space of polynomials defining complex algebraic curves of order d is path-connected. Putting all of this together constructs homeomorphisms from any curve of degree d onto the curve C', which we saw has genus g.

3. Another Method of proof

This second method of proof is, perhaps, less elegant and less general. This also means that it is much easier to prove.

Once again, we begin with a curve C of degree d defined by $P(x, y, z) = 0$ in projective space. We can assume $(0, 1, 0) \notin C$, since if it is, we can apply a transformation to change our C and remove this point. If we then set $z = 1$, the resulting $(x, y, 1) = 0$ can be viewed as defining y as a multi-valued function of x. This is since each value of x will define a degree d polynomial in y, which could have up to d different values of y as roots.

Many points x will define d values of y, but some will define less. We call these points "branch points", for reasons that will soon become apparent. More precisely, a point x is a branch point if there is a value of y such that

$$P(x, y, 1) = 0 = \frac{\partial P}{\partial y}(x, y, 1).$$

If we denote the branch points $\{p_1, ..., p_r\}$, then we can cut the complex plane C into the "cut plane" D by cutting along the line segments $[p_1, p_2], [p_2, p_3], ..., [p_{r-1}, p_r]$. For all x values in the cut plane there are d values of y, which we can write $f_1(x), ..., f_d(x)$. We have, then,

$$\{[x, y, 1] \in C | x \in D\} = U_{1 \leq j \leq d}\{[x, y, 1] \in P_2 | x \in D, y = f_j(x)\}$$

This means we can construct C by glueing together d copies of D along the cuts described above, which will result in a sphere with some amount of handles, an amount that will end up being described by the degree-genus formula.

4. Branched Covers

To fill in the details of the idea of the proof above, we need some information on branched covers of P_1. In this section, we will define what, exactly, a branched cover is, show that all complex algebraic curves C are branched covers of P_1, and show that we have $d(d-1)$ ramification points (to be defined later) that we will use in the proof of the degree genus formula.

The setup is familiar. Once again, let C be a complex algebraic curve defined by a homogenous polynomial $P(x, y, z) = 0$ of degree d. We can once again assume that $(0, 1, 0) \notin C$. Define $\phi : C \to P_1$ as

$$\phi(x, y, z) = (x, z).$$

Definition 1. *The ramification index $v_\phi(a, b, c)$ of ϕ at a point $(a, b, c) \in C$ is the order of the zero of the polynomial $P(a, y, c)$ at $y = b$. The point (a, b, c) is called a ramification point of ϕ if $v_\phi(a, b, c) > 1$.*

A few basic, but key, observations are that
A) $v_\phi(a, b, c) > 0$ iff $(a, b, c) \in C$,
B) $v_\phi(a, b, c) > 1$ iff $P(a, b, c) = 0 = \frac{\partial P}{\partial y}(a, b, c)$
and C) $v_\phi(a, b, c) > 2$ iff $P(a, b, c) = 0 = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial^2 P}{\partial y^2}(a, b, c)$.

Also simple, but a bit less so, is the following:

Lemma 1. *The inverse image $\phi^{-1}((a, c))$, of any (a, c) in P_1 under ϕ has*

$$d - \sum_{p \in \phi^{-1}((a,c))} (v_\phi(p) - 1)$$

points.

Proof: Since by assumption, the coefficient of y^d in $P(x, y, z)$ is not 0, we have that each point $x = a$ defines a degree d polynomial in y, $f(y) = P(a, y, c)$. Factoring, we can write $f(y) = \Pi(y - b_i)^{m_i}$, with $\sum m_i = d$.

But then $\phi^{-1}([a, c]) = \{[a, b_i, c]\}$, and $v_\phi[a, b_i, c] = m_i$, which proves the lemma.

With this background in place, we can define what a branched map is. Let R be the set of ramification points of ϕ.

Definition 2. *The image $\phi(R)$ of R under ϕ is called the branch locus of ϕ.*

Definition 3. *$\phi : C \to P_1$ is called a branched cover of P_1.*

There is a branched cover ϕ of P_1 for every C. In fact, we can say more.

Lemma 2. *ϕ has at most $d(d-1)$ ramification points. In fact, if $v_\phi(a, b, c) \leq 2$ for all (a, b, c), then ϕ has exactly $d(d-1)$ ramification points.*

To prove this, we need to recall:

Theorem 2. Weak form of Bezout's Theorem. *If two complex algebraic curves C and D with degrees m, n have no common component, then they intersect in at most mn points.*

Proof: See (1), Pg.54.

We can now prove lemma 2.

Proof of Lemma 2: Since, by assumption, $(0,1,0) \notin C$, the coefficient of y^d is not zero. This means $P'(x,y,z) = \frac{dP}{dy}(x,y,z) \neq 0$. Let D be the complex algebraic curve defined by $P'(x,y,z) = 0$.

Since C is non-singular, we know the polynomial defining it, $P(x,y,z)$, is irreducible. But since $P'(x,y,z)$ is a non-zero polynomial of degree less than the degree of $P(x,y,z)$, $P(x,y,z)$ and $P'(x,y,z)$ share no common component. We can then apply Theorem 2, which gives us that D and C interesect in at most $d(d-1)$ points. But the points of intersection of C and D are precisely the ramification points.

By another corollary to Bezout's theorem, to prove the second part of the lemma it suffices to show that if $(a,b,c) \in C \cap D$, then (a,b,c) is a nonsingular point on D and the tangent lines to C and D at (a,b,c) are distinct.

Suppose this were not the case. Then $P(a,b,c) = 0 = P_y(a,b,c)$ since $(a,b,c) \in C \cap D$, and $(P_{xy}(a,b,c), P_{yy}(a,b,c), P_{zy})(a,b,c)$ is a scalar multiple of $(P_x(a,b,c), P_y(a,b,c), P_z(a,b,c)$.

But then $P(a,b,c) = 0 = P_y(a,b,c) = c_1 P_{yy}(a,b,c)$, so $P_{yy}(a,b,c) = 0$. But as we saw earlier, this means $v_\phi(a,b,c) > 2$, a contradiction.

This proves the lemma. $\square$.

The reason we are interested in the case $v_\phi(a,b,c) \leq 2$ is because we can, in fact, apply a projective transformation to C to ensure that $v_\phi(a,b,c) \leq 2$ for all $(a,b,c) \in C$. This is because there are only a finite number of inflection points of C.

5. Proof of the Degree Genus Formula.

With this background, we are ready to tackle the proof of the degree genus formula. We assume that the reader knows what a triangulation of a surface is. Recall that the Euler Characteristic of a triangulation is $\chi = \#V - \#E + \#F$, and is related to the genus of the corresponding surface by $g = \frac{1}{2}(2 - \chi)$.

We will need two last preliminary results.

Lemma 3. *Let $\{p_1, ..., p_r\}$ be any set of at least three points in P_1. Then there is a triangulation of P_1 with $p_1, ..., p_r$ as its vertices and with $3r - 6$ edges and $2r - 4$ faces.*

Proof: By induction on r. The case $r = 3$ is easy to check. Realize P_1 as $C \cup \{\infty\}$ and consider the unit circle in C. If we place three vertices on this circle, we have a triangulation of P_1 with three verticies, three edges, and two faces, as desired.

Now, suppose we have a triangulation with $r = n$ vertices. To create a triangulation with $r = n + 1$ vertices, we can simply add a vertex to the existing triangulation.

If we add a vertex into the middle of a triangle, then simply connect the vertex to the vertices of the triangle containing it. This adds one vertex, three edges, and two sides, satisfying our condition.

If instead we add the vertex to the middle of an edge, then simply connect the vertex to the two vertices of neighboring triangles to which it is not yet connected. This adds one vertex, three edges, and two sides, once again satisfying our condition.

Lemma proved. $\Box$.

Lemma 4. *Let $C = \{(x, y, z) \in P_2 | P(x, y, z) = 0\}$ be a nonsingular curve containing $(0, 1, 0)$, and let $\phi : C \to P_1$ be the branched cover defined by $\phi(x, y, z) = (x, z)$. Suppose that (V, E, F) is a triangulation of P_1 such that the set of vertices V contains the branch locus $\phi(R)$ of ϕ. Then there is a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C with $\phi((\tilde{V}, \tilde{E}, \tilde{F})) = (V, E, F)$. Furthermore, if $v_\phi(p)$ is the ramification index of ϕ at p and d is the degree of C then*

$$\#\tilde{V} = d\#V - \sum_{p \in R} (v_\phi(p) - 1),$$

$$\#\tilde{E} = d\#E,$$

$$\#\tilde{F} = d\#.F$$

Proof: The first part of this lemma follows from realizing that $C \to P_1$ is a covering space outside the branch locus. The rest of the lemma follows from Lemma 1.

With this in place, we are ready, finally, to prove the degree-genus formula. Instead, however, we will prove a stronger result that will imply the degree-genus formula.

Theorem 3. *Let C be a nonsingular projective curve of degree d in P_2. If r is a positive integer and $r \geq d(d - 1)$ and $r \geq 3$, then C has a triangulation with $rd - d(d - 1)$ vertices, $3(r - 2)d$ edges and $2(r - 2)d$ faces.*

Proof:Let $P(x, y, z) = 0$ be a homogeneous polynomial of degree d defining the curve C. By applying an appropriate projective transofrmation, we can assume $(0, 1, 0) \notin C$, so $\phi(x, y, z) = (x, z)$ is well defined, and that $v_\phi(x, y, z) \leq 2$ for all $(x, y, z) \in C$.

By Lemma 2, ϕ has exactly $d(d-1)$ ramification points. By Lemma 3, there exists a triangulation (V, E, F) of P_1 with $V \supset \phi(R)$, $\#V = r$, $\#E = 3r - 6$, $\#F = 2r - 4$.

By Lemma 4, the, there exists a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C with

$$\#\tilde{E} = d\#E = 3(r-2), d$$

$$\#\tilde{F} = 2(r-2)d,$$

and

$$\#\tilde{V} = d\#V - \sum_{p \in R}(v_\phi(p) - 1).$$

Since $\#R = d(d-1)$ and $v_\phi(p) = 2$ for all $p \in R$, we have this is

$$\#\tilde{V} = rd - d(d-1). \quad \square$$

The degree-genus formula follows immediately.

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7. Biblography

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