Adding up these last inequalities, we get $\sum_{n=1}^{\infty} t_n \leqslant b_1 \leqslant f(1)$ and so $\int_0^1 h < \infty$. The proof of Theorem 1 is now complete.

Below s_n denotes the partial sum $s_n = \sum_{m=1}^{n} a_m$ of a series $\sum_{n=1}^{\infty} a_n$.

THEOREM 2. *Let F be a positive function on $\mathbb{R}^+$. The following conditions are equivalent*:

(1′) *For any divergent positive series $\sum a_n$, the series $\sum a_n F(s_n)$ is convergent.*
(2′) *There exist a $\delta > 0$ and a decreasing function f on $[\delta, \infty)$ such that*

$$F(x) \leqslant (f(y) - f(x))/(x - y) \text{ for all } \delta \leqslant y < x.$$

(3′) *There exists a decreasing function h, integrable on (δ, ∞), $\delta > 0$, which majorizes F: $F \leqslant h$.*

The idea of the proof is similar to that in Theorem 1. If (1′) holds, there exists a $\delta > 0$, perhaps large, such that the inequality $s_m \geqslant \delta$ implies $\sum_{n=m+1}^{\infty} a_n F(s_n) \leqslant 1$. The function f is now defined on $[\delta, \infty)$ by $f(x) = \sup \sum_{n=m+1}^{\infty} a_n F(s_n)$, the supremum being taken over all divergent positive series $\sum a_n$ with $\sum_{n=1}^{m} a_n = x$. If (2′) holds, we put

$$g(x) = \inf_{\delta \leqslant y < x} ((f(y) - f(x))/(x - y)),$$

and then the function $h(x) = \sup_{x \leqslant t} g(t)$ is decreasing and integrable on (δ, ∞). All other details of the proof can be written as before. We leave them to the reader.

Acknowledgement. The author wishes to thank Professor C. Ryll-Nardzewski for suggesting a possible generalization of the Dini theorems.

Reference

1. Konrad Knopp, Theory and Application of Infinite Series, Blackie and Son, London, 1966.

CLASSIFICATION OF SURFACES

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We use "surface" synonymously with "compact connected 2-manifold without a boundary", i.e., a surface is a compact connected metric space that is locally homeomorphic with the Euclidean plane E^2. The purpose of this note is to prove the following classification theorem, first discovered in the early part of the twentieth century. (See comments and references in [**3**, p. 53].)

THEOREM 1. *Every surface is homeomorphic with a space obtained by removing a finite number of disjoint disks from a 2-sphere and replacing each of them with a Möbius band or a punctured torus.*

The first step of the proof is to model the given surface with a polygonal disk D whose edges are identified in pairs. The existence of such a model depends on the fact that every surface can be triangulated. (See comments and references in [**3**, p. 52].) The remainder of the proof usually rests on a tedious process of cutting and pasting operations, performed on D, that produce another disk D' so that the given surface results from identifying, in pairs, the edges of D' in such a manner that all vertices of D' are identified at a single point and any two identified edges are adjacent on the boundary of D'. (Discussions of this procedure can be found in any of the four references.)

In this note, we give an alternative proof by using induction on the number of edges of the disk D. For this, it will be convenient to use the concept of "a connected sum of two surfaces". (See pp. 8–10 in [**3**] and the first definition below.)

DEFINITIONS AND NOTATION. Let M_1 and M_2 be two disjoint surfaces and D_1 and D_2 be disks in M_1 and M_2, respectively. Let $M = (M_1 - \operatorname{Int} D_1) \cup (M_2 - \operatorname{Int} D_2)$, where $M_1 - \operatorname{Int} D_1$

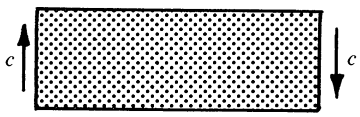

FIG. 1.1

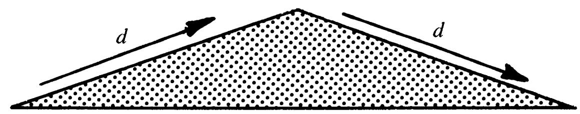

FIG. 1.2

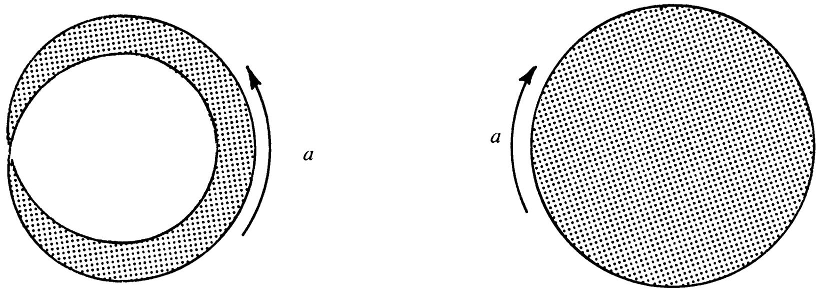

FIG. 2

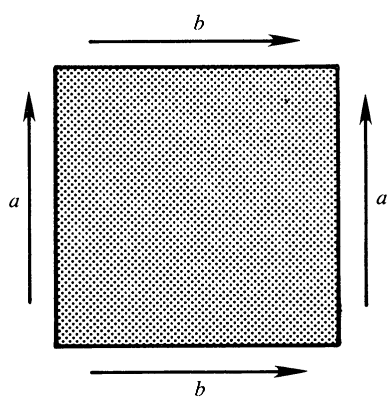

FIG. 3

and $M_2 - \operatorname{Int} D_2$ are identified (sewn together) on their boundaries, i.e., for some homeomorphism h of $\operatorname{Bd} D_1$ onto $\operatorname{Bd} D_2$, M is the quotient space

$$[(M_1 - \operatorname{Int} D_1) \cup (M_2 - \operatorname{Int} D_2)]/\{(x, h(x)) | x \in \operatorname{Bd} D_1\}.$$

The surface M is called the connected sum of M_1 and M_2 and is denoted by $M_1 \# M_2$. (It is well known that $M_1 \# M_2$ is independent of the choice of D_1 and D_2 and that

$$M_1 \# (M_2 \# M_3) = (M_1 \# M_2) \# M_3,$$

but it is not our purpose in this note to elaborate on these properties.)

A 2-**sphere**, denoted by S, is a space that is homeomorphic with the graph of $x^2 + y^2 + z^2 = 1$ in E^3.

A **Möbius band** is a space obtained by identifying, or sewing, two opposite edges of a rectangular disk as indicated in Fig. 1.1. Equivalently, a Möbius band is obtained by identifying two adjacent edges of a triangular disk as indicated in Fig. 1.2. To see that a Möbius band results in 1.2, draw a diagonal in the rectangle of 1.1, form the Möbius band by identifying the two ends as indicated, and then cut along the simple closed curve that results from identifying the end points of this diagonal.

A **projective plane**, denoted by P, is a space obtained by sewing a Möbius band and a disk

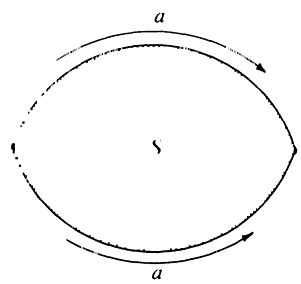

FIG. 4.1

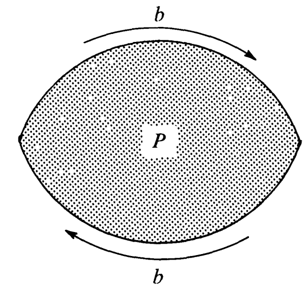

FIG. 4.2

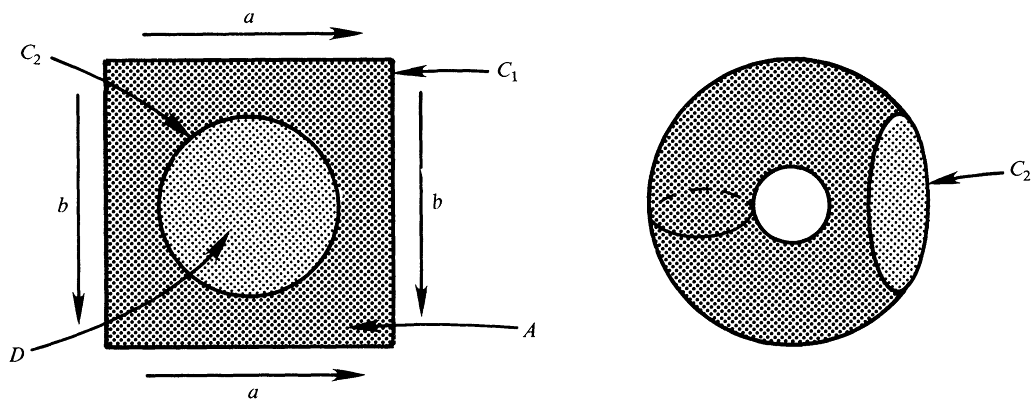

FIG. 5

together on their boundaries, as in Fig. 2. Equivalently, a projective plane is obtained with the identification indicated in Fig. 4.2.

A **torus**, denoted by T, is a space homeomorphic with the Cartesian product $S^1 \times S^1$, where S^1 denotes a circle. A torus is obtained by identifying the four edges of the square disk as indicated in Fig. 3.

Let $D(n)$ denote a disk with n edges on its boundary, where n is even, and let $M(n)$ denote a surface that is obtained by identifying, in pairs, the edges of $D(n)$.

Two identified edges of a disk are called a **twisted pair** if the identification involves the same direction for the two edges around the boundary of D. The pairs labeled with c and d in Fig. 1 and with a in Fig. 6.1 are illustrations of twisted pairs.

Two pairs of identified edges of a disk D are called **separated pairs** if the two edges in one pair separate the two in the other pair on the boundary of D. The pairs labeled with a and b in Figs. 3 and 7.1 and with b and c in Fig. 6.1 are illustrations of separated pairs that are nontwisted.

The second step in the proof of Theorem 1 is to present two lemmas that are used in the inductive procedure described in the fourth step.

LEMMA 1. *If $M(2)$ is a surface obtained by identifying the two edges of a disk $D(2)$, then $M(2)$ is either a sphere or a projective plane.*

There are two different identifications as indicated in Fig. 4, with the two resulting surfaces—the sphere and the projective plane—labeled S and P.

LEMMA 2. *If A is an annulus with n edges (n even) on one component C_1 of its boundary, then any space obtained by identifying these n edges in pairs is homeomorphic with a punctured $M(n)$, i.e., there is a disk D in $M(n)$ such that the resulting space is homeomorphic with $M(n) - \operatorname{Int} D$.*

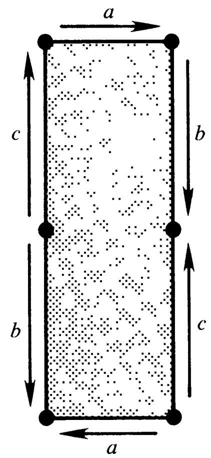

FIG. 6.1

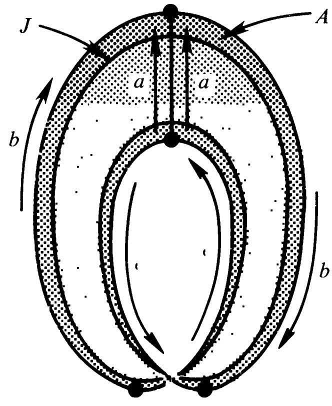

FIG. 6.2

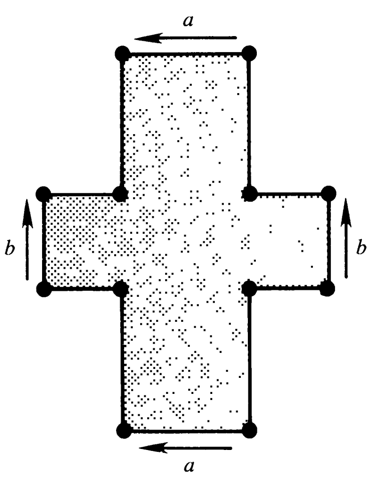

FIG. 7.1

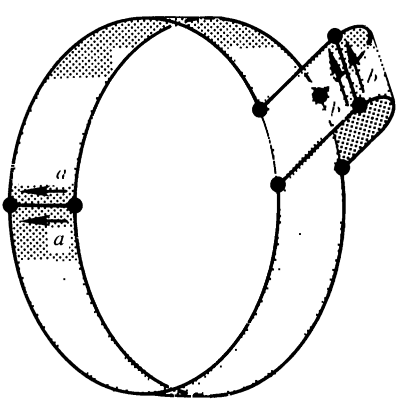

FIG. 7.2

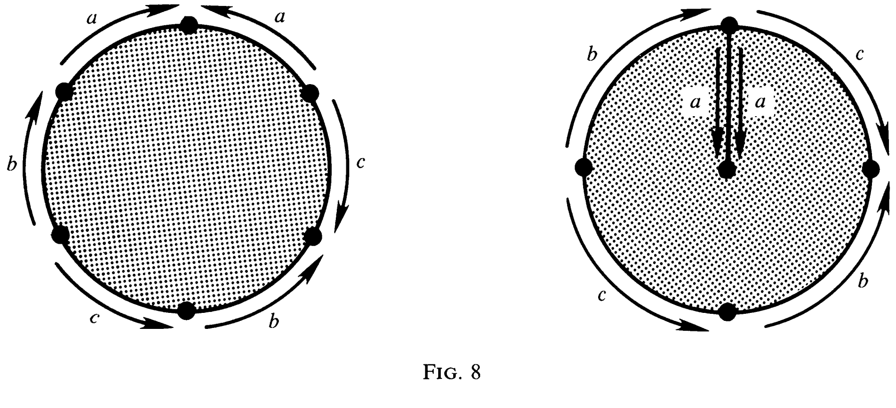

FIG. 8

This is easily seen by identifying the boundary of a disk D with the other component C_2 of the boundary of A to obtain a disk D' with boundary C_1, where $D' = D \cup A$. Identify the edges of D', in pairs, to obtain a surface $M(n)$ that contains D. The special case where the identification of the edges of D' produces a torus is illustrated in Fig. 5.

The third step in the proof is to notice that Theorem 1 can be re-stated as follows:

ALTERNATIVE STATEMENT OF THEOREM 1. *If M is a surface, different from a sphere, then $M = M_1 \# M_2 \# \cdots \# M_j$, where for each i, M_i is either a projective plane or a torus.*

The fourth, and final, step is a proof of this alternative statement, using induction on the number of edges in the disk D obtained in the first step. Let n be a positive even integer such that for any even integer k, where $2 \leqslant k < n$, any surface $M(k)$ resulting from identifying, in pairs, the edges of a disk $D(k)$ is a connected sum as required in the conclusion of the alternative statement of Theorem 1. (See the description above of the notation $D(k)$ and $M(k)$.) We wish to show that any surface $M(n)$ is such a connected sum. Let $D(n)$ be a disk with the edges identified to obtain $M(n)$. We assume, by Lemma 1, that $n \geqslant 4$. The inductive argument is separated into four cases, with some overlapping among them.

CASE 1. There is a twisted pair in the identification of the edges of $D(n)$. Identify the two edges in such a twisted pair to obtain a Möbius band B with $n - 2$ edges on its boundary. There is an annulus A in B such that one component of the boundary of A is the boundary of B. (See Fig. 6 for an illustration of this where $n = 6$.) Let J denote the other component of the boundary of A. Notice that the closure of $B - A$ is a Möbius band and that J is its boundary. Identify the edges in the boundary of B as specified for $D(n)$. By Lemma 2, it is now easy to see that $M(n) = P \# M(n - 2)$, where J becomes the identified boundaries of the two disks that are removed to obtain the connected sum of P and $M(n - 2)$.

CASE 2. There are two separated pairs of edges of $D(n)$ that are nontwisted. If $n = 4$, then $M(n)$ is a torus. (See Fig. 3.) If $n \geqslant 6$, it is easy to see, by use of Fig. 7, that a punctured torus results from identifying the edges in two separated pairs of edges that are nontwisted. By Lemma 2, $M(n) = T \# M(n - 4)$.

CASE 3. There is a nontwisted pair of adjacent edges in $D(n)$. A disk $D(n - 2)$ is obtained by identifying these two adjacent edges as indicated in Fig. 8. Thus $M(n) = M(n - 2)$.

CASE 4. There is a nontwisted pair of nonadjacent edges of $D(n)$ that does not separate any other identified pair. An annulus A is obtained by identifying the edges in some such nontwisted pair, as indicated in Fig. 9. It is easy to see that $n \geqslant 6$. There are two positive even integers h and l such that $h + l = n - 2$, where h denotes the number of edges in one component of the boundary of A and l the number in the other component. Each edge in each of these two

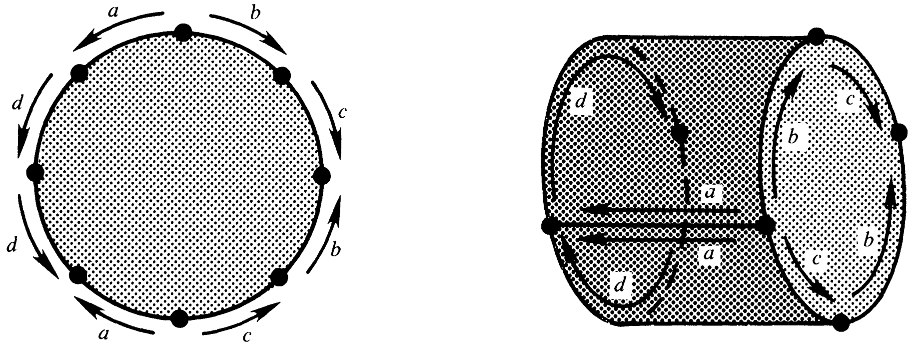

FIG. 9

components must be identified with an edge in the same component. By Lemma 2,

$$M(n) = M(h) \# M(l) = M(n-2).$$

REMARKS. The one-sided property of a Möbius band can be used to show that the connected sum of three projective planes is topologically the same as the connected sum of a torus and a projective plane. This can be used, as in [**3**, pp. 26–29], to obtain the following more specific classification of surfaces.

THEOREM 2. *Any surface different from a sphere is either a connected sum of a finite number of tori or a connected sum of a finite number of projective planes.*

The inductive procedure in the proof of Theorem 1, combined with Theorem 2, furnishes the following information about the number and types of surfaces that can be obtained from the various identifications of the edges of a given disk.

If k is odd, then $(3k+1)/2$ topologically distinct surfaces can be obtained from $D(2k)$. The connected sum of k projective planes is the only such surface that cannot be obtained from $D(2k-2)$, where $k > 1$.

If k is even, then $(3k+2)/2$ topologically distinct surfaces can be obtained from $D(2k)$. In this case, both the connected sum of k projective planes and the connected sum of $k/2$ tori can be obtained from $D(2k)$ but not from $D(2k-2)$.

References

1. Michael Henle, A Combinatorial Introduction to Topology,,Freeman, San Francisco, 1979, Chapt. 4.

2. Donald W. Kahn, Topology: An Introduction to Point-set and Algebraic Areas, Williams and Wilkins, Baltimore, 1975, Chapt. 7.

3. William S. Massey, Algebraic Topology: An Introduction, Harcourt, Brace and World, New York, 1967, Chapt. 1.

4. H. Seifert and W. Threlfall, A Textbook of Topology (translated by Michael A. Goldman), Academic Press, New York, 1980, Chapt. 6.

AN INEQUALITY CONCERNING MINORS OF A SEMIDEFINITE MATRIX

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Let $S = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a Hermitian n by n matrix of complex numbers, where A, B, and C are its submatrices of size m by m, m by $(n-m)$, and $(n-m)$ by $(n-m)$, respectively $(0 < m < n)$. Here M^* denotes the complex conjugate of the transpose of a matrix M.

THEOREM. *If S is semidefinite (either positive or negative) then*

$$|\det S| \leqslant |\det A \cdot \det C|. \tag{1}$$

For definite S equality in (1) *holds if and only if* $B = 0$.

Inequality (1) was proved originally by E. F. Beckenbach and R. Bellman in [**1**], Chapter 2, § 10 and 14, but their proof is based on a representation of the determinant as a multiple integral. In this note we give an alternative proof of (1) which is purely algebraic.

REMARK. The example of $S = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ with $m = 2$ shows that the word "definite" in the last assertion of the Theorem cannot be replaced by "semidefinite".

Using the Theorem for $m = 1$, and proceeding inductively, one derives the following well-known corollary.