Thesis abstract

The setting of this thesis is the still rather unexplored area of discrete groups of isometries of complex hyperbolic space, in particular that of lattices in $PU(2,1)$.

The study of discrete subgroups of semisimple Lie groups is a now classical and well-developed subject (see for instance the books [Rag] and [Mar2] or the introductory expositions [Mos3] and [Pan]). A fundamental difference exists between discrete subgroups of finite covolume, or *lattices*, and those of infinite covolume, concerning the questions which arise as well as the methods to explore them (the *covolume* of a subgroup Γ of a Lie group G is the Haar measure of the quotient G/Γ). Typically, one can expect a classification of lattices which are much less abundant than their infinite covolume counterparts, a crucial notion being that of *arithmeticity*. In our particular case of interest where the associated symmetric space has non-negative sectional curvature, lattices are isolated points in a big space (more precisely, Mostow's strong rigidity theorem states that, in that case, representations of a given group Γ whose image is a lattice in G are isolated points of the representation space $Hom(\Gamma, G)/G$). In contrast, discrete representations with infinite covolume can in general be deformed, and the study of these deformations is a rich and fascinating subject. The general theory of these deformations can be found for instance in Weil's articles ([W1], [W2]) or the memoir [LM]; see also [GM] for the complex hyperbolic case. The case of representations of surface groups is especially rich, as can be seen in [Gol1], [Gol2] as well as [H], [Lab] for representations in $SL(n,\mathbb{R})$ and [Tol], [GKL] for those in $PU(n,1)$. Another case of special interest in $PU(2,1)$ is that of (complex) reflection triangle groups, the study of which started with the ideal triangles of [GolPar] and continues to inspire much research, see the survey [Sz]. We will be more concerned with lattices, but the geometric methods by which we hope to produce new examples of discrete groups (see the last chapter) may a priori yield examples from both types.

There exists a general construction of lattices in a semisimple Lie group, due to Borel and Harish-Chandra, which generalizes the classical situation of the lattice $SL(n,\mathbb{Z}) \subset SL(n,\mathbb{R})$ (see the original article [BHC] or the introductory treatise [WM] for the exact construction and various examples). A lattice which can be obtained by those means is called *arithmetic*; this approach proves among other things that every semismple Lie group contains lattices, cocompact and non cocompact (Γ is said to be cocompact if G/Γ is compact). Other classical examples of arithmetic lattices include Bianchi's groups $SL(2,\mathcal{O}_d) \subset SL(2,\mathbb{C})$ comprising matrices whose entries are all in an imaginary quadratic ring of integers $\mathcal{O}_d$, such as the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[e^{2i\pi/3}]$; these are very similar to Picard's first examples in $PU(2,1)$ which we will mention below.

For our first examples of nonarithmetic lattices, consider the reflection triangle groups in the hyperbolic plane $H^2_{\mathbb{R}} \simeq H^1_{\mathbb{C}}$ (or Poincaré plane). These groups are generated by the reflections in the sides of a geodesic triangle of this plane; it is not difficult to see that if the angles of this triangle have measures $\pi/p, \pi/q, \pi/r$ with $p, q, r \in \mathbb{Z} \cup \{\infty\}$ (Coxeter's conditions) then the group in question is a discrete subgroup of $PSL(2,\mathbb{R}) \simeq SO_0(2,1)$ and the triangle is a fundamental domain for its action on the plane. Now we know, by the condition on the angles of a triangle in nonnegative curvature, that such a triangle exists in $H^2_{\mathbb{R}}$ provided that $1/p + 1/q + 1/r < 1$; we thus have a countable family of such non-isometric triangles whose reflection groups are non-conjugate in $SL(2,\mathbb{R})$. However, there is only a finite number of arithmetic lattices among these groups (see for instance [Tak]), the most famous example being the modular group $PSL(2,\mathbb{Z})$ which is contained with index 2 in the triangle group with $(p,q,r) = (2,3,\infty)$. There is thus a countable collection of nonarithmetic lattices in $PSL(2,\mathbb{R})$.

It so happens that this situation is very rare, but not because of phenomena related to low dimension. Margulis proved in 1974 a conjecture of Selberg (refined by Piatetski-Shapiro, see [Mos3]) which states that **if the real rank of G is greater than 2** then all irreducible lattices in G are arithmetic (in fact, Margulis proved that these lattices have the stronger property of *superrigidity*, see [Mar1] or [GP]). The real rank of G is the maximal dimension of an $\mathbb{R}$-split torus (a subgroup consisting of matrices which can simultaneously be written in diagonal form over $\mathbb{R}$); geometrically, it is the maximal dimension of an isometrically embedded Euclidian space with simply-connected image in the associated symmetric space (see [WM]). Thus the groups with real rank 0 are the compact groups (whose discrete subgroups are the finite subgroups); as for simple connected groups of real rank one, these are, up to finite index, by Elie Cartan's classification:

$$SO(n,1) \qquad SU(n,1) \qquad Sp(n,1) \qquad F_4^{-20}$$

(the latter being a form of the exceptional group F_4), acting by isometries on the following symmetric spaces:

$$H^n_{\mathbb{R}} \qquad H^n_{\mathbb{C}} \qquad H^n_{\mathbb{H}} \qquad H^2_{\mathbb{O}}$$

which are hyperbolic spaces of dimension n over the reals, complex numbers, quaternions respectively, and the hyperbolic plane over the Cayley octonions. Margulis' superrigidity results wer extended to the two latter cases by the work of Corlette ([Cor]) and Gromov-Schoen ([GS]); remain only the cases of real and complex hyperbolic geometry.

In the case of real hyperbolic geometry, we know how to construct discrete groups generated by reflections in a manner analogous to that of the previous triangle groups of the plane. The reason for this is that there are in $H^n_{\mathbb{R}}$ hyperplanes (i.e. totally geodesic real hypersurfaces) with associated reflections; considering a finite collection $H_1, ..., H_k$ of such hyperplanes and the group generated by their reflections, one can show as above that if the dihedral angles $(\widehat{H_i, H_j})$ all have measure π/n_{ij} with $n_{ij} \in \mathbb{Z} \cup \{\infty\}$, then the associated reflection group is discrete and has as a fundamental domain the polyhedron cut out by the hyperplanes (such a polyhedron is called a *Coxeter polyhedron*). Unfortunately, this construction only yields lattices in small dimensions. The first examples in dimension 3 (in $SO(3,1)$) are due to Makarov, who constructed non-cocompact lattices among which some are nonarithmetic (see [Mak]). Vinberg then initiated a systematic study of such polyhedra (see [Vin1] and [Vin2] pp. 198–210) which revealed the following remarkable facts:

- (Vinberg) Compact Coxeter polyhedra do not exist in $H^n_{\mathbb{R}}$ for $n \geqslant 30$.
- (Prokhorov-Khovanskij) Coxeter polyhedra of finite volume do not exist in $H^n_{\mathbb{R}}$ for $n \geqslant 996$.

The bounds given by these results are wide (especially the second); known examples are in dimensions up to 8 for the first case, and 21 for the second.

On the other hand, Gromov and Piatetski-Shapiro have given a general construction, which they call *hybridation* of two hyperbolic manifolds (see [GPS] as well as [Mar2], [Vin2] and [WM]), which yields examples of nonarithmetic lattices in $SO(n,1)$ in all dimensions, cocompact and non-cocompact.

As for the case of $SU(n,1)$, all the questions we have raised remain wide open, even in the smallest dimensions. This is mostly due to the fact that there are no totally geodesic real hypersurfaces in complex hyperbolic geometry, and in particular no natural notion of polyhedra or reflection groups as in the real hyperbolic (or Euclidian) case. This situation makes it difficult to construct not only discrete subgroups of $SU(n,1)$, but also fundamental polyhedra for such groups. Possible substitutes in $H^n_{\mathbb{C}}$ are *complex reflections* (or $\mathbb{C}$-reflections) which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $H^{n-1}_{\mathbb{C}} \subset H^n_{\mathbb{C}}$), and *real reflections* (or $\mathbb{R}$-reflections) which are antiholomorphic involutions fixing pointwise a Lagrangian subspace (a copy of $H^n_{\mathbb{R}} \subset H^n_{\mathbb{C}}$). Most examples up to now are generated by complex reflections, or are arithmetic constructions. The use of $\mathbb{R}$-reflections and of the corresponding Lagrangian planes is very recent (it was introduced by Falbel and Zocca in [FZ] in 1999) and is one of the guiding principles of this thesis; we will mention this again below.

The first constructions of lattices in $SU(2,1)$ go back to the end of the XIXth century and are due to Picard; it is striking that they are still almost the only known examples. Picard's first examples are of an arithmetic nature, in the spirit of the Bianchi groups; namely he considered in [Pic1] the groups $SU(2,1,\mathcal{O}_d) \subset SU(2,1)$ comprising the matrices whose entries are all in the imaginary quadratic ring of integers $\mathcal{O}_d$ (d is square-free and negative), such as the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[e^{2i\pi/3}]$. It is not difficult to see that such a subgroup is a non-cocompact lattice (see [McR]); however it is interesting to note that the explicit determination of a fundamental domain for these groups (providing for instance a presentation) is delicate and is the subject of contemporary research even in the most classical cases of the Gaussian and Eisenstein integers.

More surprising is Picard's second construction coming from monodromy groups of the so-called *hypergeometric* functions, which are meromorphic functions of a complex variable (see [Pic2] and [DM] for a more modern language), which produces a subgroup of $PU(2,1)$ generated by three complex reflections. We now consider $PU(2,1)$ (a quotient of order 3 of $SU(2,1)$) which is more natural geometrically, as the group of holomorphic isometries of

$H^n_{\mathbb{C}}$. This construction is related to the moduli space of quintuples of points on the Riemann sphere $\mathbb{C}P^1$, as well as to Euclidian cone metrics on the sphere studied by Thurston in [Th] (see also [Par]). Picard stated conditions sufficient to ensure dicreteness of the corresponding group (conditions which he obtained through a geometric analysis); his student Le Vavasseur enumerated all the parameters saisfying these conditions (in [LeV]), which amount essentially to 27 cases. The 27 corresponding lattices are now known as the *Picard lattices*; among them, 7 are nonarithmetic. This observation is due to Deligne and Mostow (the notion was not even burgeoning in Picard's time), when they rewrote and made systematic Picard's hypergeometric approach a century later in [DM]. They obtained by this method lattices in $PU(n,1)$ for $n \leqslant 5$; it is interesting to note that in dimension 2 they find exactly the 27 Picard lattices (although using a weaker discreteness condition) and that the only other nonarithmetic lattice of their list is a non-cocompact example in $PU(3,1)$.

In the meantime, at the end of the 1970's, Mostow had studied new examples of subgroups of $PU(2,1)$ generated by three complex reflections of order p ($p = 3,4,5$), which he denoted by $\Gamma(p,t)$ (t being a real parameter). His investigation [Mos1] is based on a detailed analysis of the geometric action of the group on $H^2_{\mathbb{C}}$, from which he infers discreteness conditions as well as the description of a fundamental domain when this is the case. These examples, still poorly understood, are the other guiding principle of this thesis. Mostow's original approach is the construction of a fundamental domain by Dirichlet's method, which consists in the consideration, for a "central" point p_0, of the set of points closer to p_0 than to any other point in its orbit. This approach has the advantages of generality and simplicity of principle, but it is very hard to use in practice. Namely, the basic object in this construction is the set of points equidistant from two given points (p_0 and one of its translates), which Mostow called a *bisector*. These objects enjoy remarkable geometric properties, such as the double foliation by $\mathbb{C}$-planes and $\mathbb{R}$-planes (see [Gol3]), but their intersections are difficult to understand. For instance, one of the goals of Goldman's book [Gol3] is to understand these intersections; see also the third chapter of this thesis (in particular its introduction) for problems related to this construction.

This is one of the aspects of the difficulty of constructing polyhedra in $H^2_{\mathbb{C}}$, in particular their codimension-1 faces. As we have mentioned, the major problem is the absence of hyperplanes (or totally geodesic real hypersurfaces), which have no natural substitute. Different constructions of hypersurfaces have appeared, the nature of which is linked to the type of the group in question. Schwartz has described some different examples (in 2002) in his survey [Sz]; an idea shared by all these constructions is the use of $\mathbb{C}$-planes and $\mathbb{R}$-planes (and their boundaries, the $\mathbb{C}$-circles and $\mathbb{R}$-circles of the Heisenberg group $H^3 = \partial H^2_{\mathbb{C}}$) to foliate the hypersurfaces as in the case of bisectors.

This is where the $\mathbb{R}$-reflections which appear in the group play a crucial role, according to the general principle by which the most natural fundamental domain for a group rests upon the fixed-point loci of certain elements of the group. We thus gain some geometric information by decomposing the generators of a group of isometries as products of $\mathbb{R}$-reflections, which is a recurring theme in this thesis. A classic example of this situation is that of triangle groups of the plane (hyperbolic, Euclidian, or spherical), generated by two rotations which one decomposes as a product of reflections to determine the order of the product (and the third angle of the triangle bounded by the reflecting lines, which is a fundamental domain for the group). Another example arises in our investigation of Mostow's lattices in chapter 3 (see [DFP]), where we notice that a group slightly bigger than his lattice is generated by two isometries which decompose as a product of three $\mathbb{R}$-reflections. The corresponding $\mathbb{R}$-planes could be seen by transparency in Mostow's article (typically, certain quintuples of vertices of his domain were contained in one of these $\mathbb{R}$-planes), whithout his noticing them (see for this aspect figure 14.2 on p. 239 of [Mos1]). This allowed us to simplify the structure of the fundamental polyhedron by introducing some 2-faces contained in these $\mathbb{R}$-planes.

The thesis is organized as follows. Each chapter is conceived to be self-contained, with its own introduction and bibliography. After a short chapter of geometric preliminaries in $H^2_{\mathbb{C}}$ (for which the reader may also refer to Goldman's book [Gol3]), we investigate the simplest kind of discrete groups, namely finite groups. In our setting, these groups are conjugate to subgroups of $U(2)$; it is thus a linear question in $\mathbb{C}^2$, but which already contains some rich geometric aspects. One of the motivations for this study was to understand the elementary building block of Mostow's lattices, the finite groups generated by two of the three fundamental complex reflections (this is the example of the group 3[3]3 in Coxeter's notation, which we study in detail). We describe precisely the role of $\mathbb{R}$-planes in that setting, and construct fundamental domains in the boundary of the unit ball of $\mathbb{C}^2$ based upon arcs of the corresponding $\mathbb{R}$-circles. We detail the conditions under which a given $\mathbb{R}$-reflection decomposes an elliptic isometry of $H^2_{\mathbb{C}}$ (in the sense where the latter can be written as the product of the given $\mathbb{R}$-reflection with another $\mathbb{R}$-reflection), and infer among other things the following result:

Theorem 0.1 *Every finite subgroup of $U(2)$ is of index 2 in a group generated by $\mathbb{R}$-reflections. More precisely:*

- *Every two-generator subgroup of $U(2)$ is of index 2 in a group generated by 3 $\mathbb{R}$-reflections.*
- *The exceptional finite subgroups of $U(2)$ which are not generated by two elements are of index 2 in a group generated by 4 $\mathbb{R}$-reflections.*

This second chapter is a joint work with E. Falbel and has been published in *Geometriae Dedicata* ([FPau]).

The next chapter consists in a detailed analysis of Mostow's lattices $\Gamma(p,t)$, based on the aforementioned observation that these lattices naturally contain $\mathbb{R}$-reflections. We construct a new fundamental domain Π which is simpler than Mostow's, but mostly which allows the use of synthetic geometric arguments which spare the resort to massive computer use for the proofs as in [Mos1]. Further motivations and ramifications can be found in the detailed introduction to that chapter, as well as some notation. $\tilde{\Gamma}(p,t)$ denotes the group generated by one of the complex reflections, say R_1 and the isometry J which cyclically permutes these three reflections; it contains $\Gamma(p,t)$ with index 1 or 3. We can sum up our results in the following:

Theorem 0.2 *The group $\tilde{\Gamma}(p,t) \subset PU(2,1)$, for $p = 3, 4, \text{or} 5$ and $|t| < \frac{1}{2} - \frac{1}{p}$, is discrete if $k = (\frac{1}{4} - \frac{1}{2p} + \frac{t}{2})^{-1}$ and $l = (\frac{1}{4} - \frac{1}{2p} - \frac{t}{2})^{-1}$ are in $\mathbb{Z}$. In that case Π is a fundamental domain with side pairings given by J, R_1, R_2, R_2R_1, R_1R_2 and the cycle relations give the following presentation of the group*

$$\tilde{\Gamma}(p,t) = \langle J, R_1, R_2 \mid J^3 = R_1^p = R_2^p = J^{-1}R_2JR_1^{-1} = R_1R_2R_1R_2^{-1}R_1^{-1}R_2^{-1}$$

$$= (R_2R_1J)^k = ((R_1R_2)^{-1}J)^l = I\rangle.$$

This third chapter is a joint work with M. Deraux and E. Falbel and has been published in *Acta Mathematica* ([DFP]).

The last chapter is a first step in the search for new discrete groups in a family which we will call *elliptic triangle groups*. These are the groups generated by two elliptic isometries A and B (i.e. each having a fixed point inside $H^2_{\mathbb{C}}$) whose product AB is also elliptic. In the same way that we use the characterization of triangles of the plane (hyperbolic, Euclidian, or spherical) to understand the groups generated by two rotations, we will need the characterization of which conjugacy classes the product AB can be in when A and B are each in a fixed conjugacy class. Elliptic conjugacy classes in $PU(2,1)$ are characterized by an unordered angle pair, so that the question is the determination of the image in the surface $\mathbb{T}^2/\mathfrak{S}_2$ of the map $\tilde{\mu}$, which is the composite of the group product (restricted to the product of fixed conjugacy classes), followed by projection from the group to its conjugacy classes. This is an occurrence of a momentum map associated with a quasi-Hamiltonian group action on a symplectic manifold, generalizing the classical Hamiltonian setting and defined in [AMM] (see also [Sf] for more details). The chapter is devoted to the explicit determination of the image, as well as the counterpart in terms of $\mathbb{R}$-planes, namely the question of classifying triples of $\mathbb{R}$-planes intersecting pairwise inside $H^2_{\mathbb{C}}$. We obtain among other things the following result, which rests upon a detailed description of the collection W_{red} of *reducible walls* (i.e. those points which are images of pairs (A,B) generating a reducible linear group), and is reminiscent of the classical convexity theorem of Atiyah-Guillemin-Sternberg (see [A], [GS1], [GS2]):

Theorem 0.3 *Let C_1 and C_2 be two elliptic conjugacy classes in $PU(2,1)$, at least one of which is not a class of complex reflections. Then the image of the map $\tilde{\mu}$ in $\mathbb{T}^2/\mathfrak{S}_2$ is a union of chambers containing the locally convex hull of the reducible walls W_{red}, except possibly around self-intersections of W_{red}.*

Note that these questions were also motivated by the study of Mostow's lattices in the respect that each of his families $\Gamma(p,t)$ with fixed p is part of our setting, sitting as a segment inside of the momentum polygon. This distinguishes a larger geometric family to which these lattices belong, and among which we hope to find other examples. We finish by suggesting some explicit candidates.

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