Triangle reflection groups

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Abstract

We introduce some basics of hyperbolic geometry and prove the angle sum relations for spherical and hyperbolic space. We then work towards proving a theorem regarding the algebraic structure of a group of transformations generated by reflections in the sides of a triangle and a corollary concerning the finiteness of the group based on an algebraic property of one of its presentations.

1 Introduction

The main results of this paper are the introduction and partial proof of a theorem concerning the group of transformations generated by reflections in the sides of a triangle and a corollary concerning the finiteness of the group based on an algebraic property of one of its presentations. The main references, included at the end, are Beardon, [1], Coxeter, [2] and Ratcliffe, [3]. The information on hyperbolic geometry in Beardon is relatively accessible; in Ratcliffe it is treated in generality. The main theorem is treated in quite abstract and general terms in both Beardon and Ratcliffe. It is treated with slightly less generality in two dimensions in Coxeter.

Throughout the paper we consider the metric spaces Euclidean and spherical metric spaces $E^2 = (\mathbb{R}^2, d_E)$ and $S^2 = (\mathbb{S}^2, d_S)$, respectively (here $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$). We also consider the upper half-plane and conformal ball models of the hyperbolic plane: $H^2 = (\{x+iy \in \mathbb{C} : y > 0\}, d_H)$ and $B^2 = (\{z \in \mathbb{C} : ||x|| < 1\}, d_B)$, respectively. d_E is the usual metric on $\mathbb{R}^2$. We defer to Ratcliffe ([3], pp. 37) for the spherical metric and to Beardon ([1], pp. 130, 132) for the hyperbolic metrics. Let $\theta(x, y)$ be the Euclidean angle between x and y, then $d_S(x,y) = \theta(x,y)$. Also, $d_H(z,w) = \log \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$ and $d_B(z,w) = \log \frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}$.

2 Spherical and hyperbolic geometry

Next we describe the triangles of the three spaces. More information on the geometry of these metric spaces is given in Ratcliffe, [3], and in Beardon, [1]. The triangles of E^n are the usual triangles of Euclidean geometry. The triangles of S^n are described as follows: Take three points $x, y, z, \in \mathbb{S}^2$ with x, y, z not co-linear (meaning that there is no great circle containing all three points). Let $H(x,y,z)$ be the half-sphere including the great circle formed by x, y and with z in the interior. Then the triangle with x, y, z as vertices is the set $H(x,y,z) \cap H(y,z,x) \cap H(z,x,y)$.

A lune is defined to be the intersection $H(x_1,y_1,z_1) \cap H(x_2,y_2,z_2)$ of two distinct nonopposite half spheres. Note that, in general, $x_1 = x_2$ or $x_1 \neq x_2$ and similarly for y_1, y_2 and z_1, z_2. However, $H(x_1,y_1,z_1)$ and $H(x_2,y_2,z_2)$ must be distinct and nonopposite. Then, by rotations any lune is congruent to a lune $L(\alpha) = \{(\phi,\theta) : 0 \leq \phi \leq \pi, 0 \leq \theta \leq \alpha\}$. Then $\text{Area}(L(\alpha)) = \int_0^\alpha \int_0^\pi \sin\phi d\phi d\theta = 2\alpha$.

Theorem 2.1. *For any spherical triangle T with angles α, β, γ,*

$$\textit{Area}(T) = (\alpha + \beta + \gamma) - \pi$$

Corollary 2.2. *For any spherical triangle T with angles α, β, γ, $\alpha+\beta+\gamma > \pi$.*

Proof. Extend the sides of the triangle into three great circles. These great circles divide the sphere into eight triangular regions. Two are T and $-T$; the other are A, $-A$, B, $-B$, C, $-C$. Any two of the great circles form the boundary of a lune of angle α, β or γ. The lune with angle α is the union of T and A, so we have $\text{Area}(T) + \text{Area}(A) = 2\alpha$. Similarly, $\text{Area}(T) + \text{Area}(B) = 2\beta$ and $\text{Area}(T) + \text{Area}(C) = 2\gamma$. Also note that $\text{Area}(T) + \text{Area}(A) + \text{Area}(B) + \text{Area}(C) = 2\pi$. Then add the first three equations and subtract the fourth: $2\text{Area}(T) = 2\alpha + 2\beta + 2\gamma - 2\pi$, so $\text{Area}(T) = (\alpha + \beta + \gamma) - \pi$. □

The triangles of H^2 are described next. We first need to know more about the geometry of the space. A geodesic in H^2 is either a Euclidean line orthogonal to the real axis (a line of the form $z = a + it$ for $t \in \mathbb{R}^+$ and for some $a \in \mathbb{R}$) or a circle orthogonal to the real axis (e.g. $\{x + iy \in \mathbb{C} : |x + iy| = 1, y > 0\}$). A ray from z is a geodesic $[z, \alpha)$ for some $\alpha \in \mathbb{R}$ or $\alpha = \infty$ (in which case the ray is a vertical line which, if extended to meet the real axis, would be orthogonal to it).

Consider three non-colinear points x, y, z (so there is no geodesic containing x, y, z). Let L_2 and L_3 be the rays from x to y and x to z respectively. Define an angle at x as the ordered pair of rays (L_2, L_3). L_2 determines a geodesic, say, L^*. $L_3 - \{x\}$ is contained in one connected componenet, say Σ, of $H^2 - L_2^*$. Similarly, $L_2 - \{z\}$ is contained in one connected component, Σ' of $H^2 - L_3^*$. Let the interior of the angle be $A_1 = \Sigma \cap \Sigma'$. Similarly, let A_2 and A_3 be the interior of the angles at point y and z. Then the triangle $T(x, y, z) = A_1 \cap A_2 \cap A_3$.

Theorem 2.3. *For any hyperbolic triangle T with angles α, β, γ,*

$$Area(T) = \pi - (\alpha + \beta + \gamma)$$

Corollary 2.4. *For any hyperbolic triangle T with angles α, β, γ, $\alpha+\beta+\gamma < \pi$.*

Proof. Consider the upper half plane model of the hyperbolic plane. From Beardon, we have that, for a set $E \subset H^2$, $\text{Area}(E) = \int\int_E \frac{1}{y^2} dxdy$ ([1], pp. 132). Consider a triangle T with vertices $c = \infty$ and a and b lie on the unit circle $|z| = 1$. Then

$$\begin{aligned}
\text{Area}(T) &= \int_{\cos(\pi-\alpha)}^{\cos(\beta)} \int_{(1-x^2)^{1/2}}^{\infty} \frac{dy}{y^2} dx \\
&= \int_{\cos(\pi-\alpha)}^{\cos(\beta)} \frac{1}{\sqrt{1-x^2}} dx \\
&= \arcsin(\cos(\beta)) - \arcsin(\cos(\pi - \alpha)) \\
&= \arcsin(\cos(\beta)) + \arcsin(\cos(\alpha)) \\
&= (\frac{\pi}{2} - \arccos(\cos(\beta))) + (\frac{\pi}{2} - \arccos(\cos(\alpha))) \\
&= \pi - (\alpha + \beta)
\end{aligned}$$

In general, any triangle is the difference two such triangles, so the general formula follows.

□

Lastly, we need to describe what the reflections are in each space. The reflections of E^2 are the usual Euclidean reflections. The reflections of S^2 are the reflections of E^2 restricted to S^2. The reflections of B^2 are the the Möbius transformations $f(z) = \frac{az+b}{cz+d}$ that fix the unit circle.

3 Triangle reflection groups

Definition 3.1. *A topological group is a group G that is a also topological space where the group multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are continuous functions.*

Definition 3.2. *A discrete group is a topological group G where every point is an open set.*

Definition 3.3. *A set F is a fundamental domain for a group G of isometries if:*

(1) F *is open and connected*

(2) $X = \bigcup_{g\in G} \overline{g(F)}$

(3) $g_1 \neq g_2 \in G$ *implies* $g_1(F) \cap g_2(F) = \emptyset$

Theorem 3.4. *Let G be the group of transformations generated by the reflection in the sides of a triangle T with angles $\alpha = \frac{\pi}{p}$, $\beta = \frac{\pi}{q}$, $\gamma = \frac{\pi}{r}$. Then:*

(i) G *is discrete*

(ii) T *is a fundamental domain for* G

(iii) $G = \left\langle r_1, r_2, r_3 | r_i^2, (r_1 r_2)^{k_{12}}, (r_2 r_3)^{k_{23}}, (r_1 r_3)^{k_{13}} \right\rangle$

where $\{k_{12}, k_{23}, k_{13}\} = \{p, q, r\}$.

Proof. We prove that (ii) implies (i). For a technical proof of (ii), the reader is referred to Beardon, [1] or Ratcliffe, [3]. For a proof, in two dimensions, that (ii) implies (iii), the reader is referred to Coxeter, [2].

(ii) $\implies$ (i): Note that if G is not discrete, then there exists an injective sequence $(\gamma_n) \in G$ which converges to 1. Thus $\gamma_n(x) \to x$ for every $x \in X$. However, because T is a fundamental domain for G, we know that for any $g_1 \in G$ we have $g_1(T) \cap T = \emptyset$. Therefore, for a neighborhood $N \subset \text{Interior}(T)$ of a point $x \in \text{Interior}(T)$, we have that $g_1(N) \cap N = \emptyset$. Thus we can have no sequence of $(\gamma_n) \in G$ such that $\gamma_n(x) \to x$ and $\gamma_n \to 1$. Thus G is discrete. ∎

Corollary 3.5. *Let* $G = \langle r_1, r_2, r_3 | r_i^2, (r_1 r_2)^{k_{12}}, (r_2 r_3)^{k_{23}}, (r_1 r_3)^{k_{13}} \rangle$. *Then G is finite if and only if* $\frac{1}{k_{12}} + \frac{1}{k_{23}} + \frac{1}{k_{13}} > 1$.

Proof. Let $\frac{1}{k_{12}} + \frac{1}{k_{23}} + \frac{1}{k_{13}} > 1$. Let T be a triangle with angles $\frac{\pi}{k_{12}}$, $\frac{\pi}{k_{23}}$ and $\frac{\pi}{k_{13}}$. Then the reflection group of T is G. Since $\frac{\pi}{k_{12}} + \frac{\pi}{k_{23}} + \frac{\pi}{k_{13}} > \pi$ it follows that T is a triangle in $\mathbb{S}^2$. Since $\mathbb{S}^2$ is compact and the area of T is not 0, it follows that we need only a number of transformations to to cover $\mathbb{S}^2$. Thus G is finite.

Let G be finite and have the presentation above. Then there is a triangle T with angles $\frac{\pi}{k_{12}}$, $\frac{\pi}{k_{23}}$, $\frac{\pi}{k_{13}}$ such that G is the refleciton group of T. Either T is a Euclidean triangle, a hyperbolic triangle or a spherical triangle. Since G is finite, T is neither Euclidean nor hyperbolic. Instead, T is spherical. Thus $\frac{1}{k_{12}} + \frac{1}{k_{23}} + \frac{1}{k_{13}} > 1$. ∎

References

[1] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

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[3] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.