On Poincaré−Hopf Index Theorem

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1 Motivation

The Euler Characteristic of a surface S, $\chi(S)$, as a combinatorial invariant on its 2-complex sheds light on surface's global structure. Even highly complicated surfaces admit Euler Characteristic as an invariant by using a triangulation of the surface, but as only combinatorial properties of the triangulations are used in realizing this invariant, it is interesting to question whether there exist analytic properties of the surface which somehow escape the Euler Characteristic. This aspect is informed by a result in differential topology due to Henri Poincaré (who proved the result in two dimensions) and Heinz Hopf (due to whom there exist generalizations in higher dimensions) known as the Poincaré−Hopf Index Theorem. The Poincaré−Hopf Index Theorem relates vector fields on compact surfaces to the Euler Characteristic, thus tying together objects with analytic knowledge of the surface with another that is dependent on the structure of its 2-complex.

2 Preliminaries

Before introducing the statement of the Poincaré−Hopf Index Theorem some definitions related to vector fields are provided.

Definition 2.1. If $A \subset \mathbb{R}^n$, then a vector field $\mathbf{v}$ on A is a continuous function $\mathbf{v} : A \to \mathbb{R}^n$.

Definition 2.2. If $\mathbf{v}(\mathbf{x}) = \mathbf{0}$ then $\mathbf{x}$ is defined as a critical point of $\mathbf{v}$, and is isolated if there exists a neighborhood of $\mathbf{x}$ that contains no other critical points of $\mathbf{v}$.

Definition 2.3. A curve homeomorphic to a circle S^1 enclosing a region homeomorphic to a disc D^2 is defined as a Jordan Curve.

2.1 The Winding Number $w_{\mathbf{v}}(C)$

Given a Jordan curve C and an arbitrary point $\mathbf{x_0}$ on the curve, consider the direction of a vector field $\mathbf{v}(x)$ defined everywhere (and non-zero) on C as x is moved along C. On moving over the curve, the vector field may change direction at each point, but still on returning to $\mathbf{x_0}$ the direction has to return to the initial direction $\mathbf{v}(\mathbf{x_0})$, implying that the vector field must have swept out an angle that is an integral multiple of 2π as it went around C (of course, the usual convention of considering the angles swept counter clockwise as positive and those swept clockwise as negative is adopted here as well). This integral (*positive or negative*) multiple of 2π radians is defined as the winding number of the vector field $\mathbf{v}$ over C, $w_{\mathbf{v}}(C) = \theta/(2\pi)$ (where θ is the angle swept by the vector field over C).

Assuming that $\mathbf{v}(x) = 0$ never holds for the vector field $\mathbf{v}$ on the loop C, a more technical definition of $w_{\mathbf{v}}(C)$ can be formulated which allows one to easily establish that the winding number in well-defined (independent of which loop is used to calculate the winding number). First consider the following notion of a *degree* of a map f from S^1 to S^1 in terms of the first homology group (the abelianized fundamental group and in case of S^1 simply $\mathbb{Z}$) of S^1 (since the aim is to handle only the dimension 2 case, all arguments presented will be implicitly for a 2-Manifold, although some can be extended to higher dimensional cases).

Definition 2.4. If $f : S^1 \to S^1$ is a continuous map then f induces a homomorphism $f_1 : H_1(S^1) \cong \mathbb{Z} \to H_1(S^1) \cong \mathbb{Z}$ implying that f_1 is multiplication by some integer k. Define the $deg(f) = k$.

This lends the following definition of the winding number.

Definition 2.5. If $D \subset \mathbb{R}^2$ is a homeomorphic to B^2 (2-dimensional ball) with boundary $C = \partial(D)$ and homeomorphism $f : S^1 \to C$ along with a $\mathbf{v}$ a vector field on D with no zeroes on C then define $\bar{\mathbf{v}} : C \to S^1$ as $\bar{\mathbf{v}}(x) = \mathbf{v}(x) \cdot ||\mathbf{v}(x)||^{-1}$. The winding number of $\mathbf{v}$ on C is defined as $w_{\mathbf{v}}(C) = deg(\bar{\mathbf{v}} \circ f)$

Using this one can define the *index* of an isolated critical point of a vector field.

Definition 2.6. If $\mathbf{v}$ is a vector field on a surface S with an isolated critical point x and D is a neighborhood of x such that x is the only critical point in $D \cup \partial D$ then the index of x is defined by $I_{\mathbf{v}}(x) = w_{\mathbf{v}}(\partial D)$.

That is, the index of an isolated critical point of $\mathbf{v}$ is the winding number of the vector field $\mathbf{v}$ about a loop enclosing that critical point and no other. With these defined Poincaré−Hopf Index Theorem can now be stated for a disc D^2.

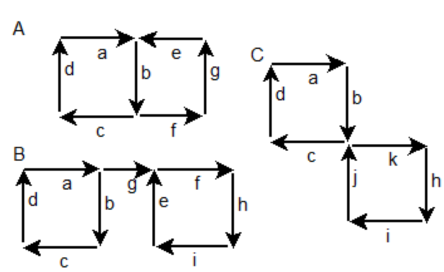
Theorem 2.7 (The Poincaré−Hopf Index Theorem on Disc D^2)**.** *If D^2 is homeomorphic to 2-ball with $C = \partial(D^2)$ and $\mathbf{v}$ is continuous vector field on D^2 with only isolated critical points $x_1, x_2 ...$ x_k on $Int(D^2)$ then $w_{\mathbf{v}}(C) = \sum_{i=1}^{k} I_{\mathbf{v}}(x_i)$.*

2.2 Comments, Corollaries and Essential Results

Before building the proof of the The Poincaré−Hopf Index Theorem in the special case of D^2, the following essential results are established.

Lemma 2.8. *If $\mathbf{v}$ is a vector field on $\mathbb{R}^2$ and C_1 and C_2 are 1−cycles such that $\mathbf{v}$ is never zero on either one then $w_{\mathbf{v}}(C_1 + C_2) = w_{\mathbf{v}}(C_1) + w_{\mathbf{v}}(C_2)$.*

Proof. There are three possible cases −



i) C_1 and C_2 share an edge as in A) − This yields $w_{\mathbf{v}}(C_1{+}C_2) = [\theta_a{+}\theta_b{+}\theta_c{+}\theta_d{+}\theta_b{+}\theta_f{+}\theta_g{+}\theta_e]/2\pi$ where θ_{edge} is the angle swept by the field while travesing that edge of the cycle. Rearranging, $w_{\mathbf{v}} = [\theta_a + \theta_b + \theta_c + \theta_d]/2\pi + [\theta_b + \theta_f + \theta_g + \theta_e]/2\pi$ but this is exactly the angle swept by the field over C_1 and C_2 individually $\Rightarrow w_{\mathbf{v}}(C_1 + C_2) = w_{\mathbf{v}}(C_1) + w_{\mathbf{v}}(C_2)$.

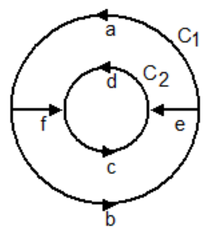
$ii)$ C_1 and C_2 are disjoint − Add an edge that links both cycles as in B) this makes $w_{\mathbf{v}}(C_1 + C_2) = w_{\mathbf{v}}(a + b + c + d + g + f + h + j + e - g)$, that is g is traversed in both directions, thus canceling out and leaving the disjoint union of C_1 and C_2. Now $w_{\mathbf{v}}(C_1 + C_2)$ becomes $[\theta_a + \theta_b + \theta_c + \theta_d + \theta_f + \theta_h + \theta_i + \theta_e]/2\pi = [\theta_a + \theta_b + \theta_c + \theta_d]/2\pi + [\theta_f + \theta_h + \theta_i + \theta_e]/2\pi$. As in $i)$ this too is sum of the angles swept individually over C_1 and $C_2 \Rightarrow w_{\mathbf{v}}(C_1+C_2) = w_{\mathbf{v}}(C_1)+w_{\mathbf{v}}(C_2)$.

$iii)$ C_1 and C_2 share a vertex as in C) − This yields $w_{\mathbf{v}}(C_1 + C_2) = [\theta_a + \theta_b + \theta_c + \theta_d + \theta_k + \theta_h + \theta_i + \theta_j]/2\pi$. Rearranging, as in $i)$ and $ii)$ $w_{\mathbf{v}}(C_1 + C_2) = w_{\mathbf{v}}(C_1) + w_{\mathbf{v}}(C_2)$.

□

Proposition 2.9. *The index $I_{\mathbf{v}}(x)$ is well defined.*

Proof. Consider the cycles C_1 and C_2 as illustrated −



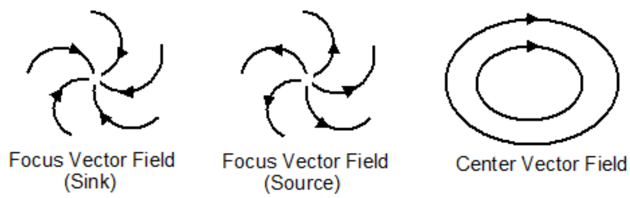
Assume both C_1 and C_2 enclose the same critical point of the field and it is the only one they enclose. Now by Lemma 2.8 $w_{\mathbf{v}}(C_1) - w_{\mathbf{v}}(C_2) = w_{\mathbf{v}}(C_1 - C_2)$. Now by the decomposition of each of the cycles C_1 and C_2 into edges, $w_{\mathbf{v}}(C_1 - C_2)$ becomes $w_{\mathbf{v}}(a + b - (c + d))$. Writing this as a composition of the boundaries ∂A and ∂B, $w_{\mathbf{v}}(C_1) - w_{\mathbf{v}}(C_2) = w_{\mathbf{v}}((a+f-d-e)+(b+e-c-f))$, again by Lemma 2.8 $w_{\mathbf{v}}((a+f-d-e)+(b+e-c-f)) = w_{\mathbf{v}}((a+f-d-e))+w_{\mathbf{v}}((b+e-c-f)) = w_{\mathbf{v}}(\partial A) + w_{\mathbf{v}}(\partial B) = 0 + 0 = 0$ since regions bounded by ∂A and ∂B contain no critical points making the winding numbers around ∂A and ∂B *zero* for $\mathbf{v}$. □

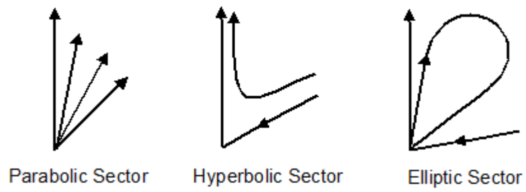
Lemma 2.10. *If $C \subset \mathbb{R}^{n+1}$ is homeomorphic to S^n such that C bounds a region D homeomorphic to B^{n+1} then for a vector field $\mathbf{v}$ with no zeros on C or inside D, $w_{\mathbf{v}}(C) = 0$.*

Proof. As $\mathbf{v}$ is never zero on $D \cup C$, the map $\bar{\mathbf{v}} : D \to S^n$ such that $\bar{\mathbf{v}}(x) = \mathbf{v}(x) \cdot ||\mathbf{v}(x)||^{-1}$ is well defined and continuous. Now if f was the homeomorphism from $B^{n+1} \to D$ then the composition $\bar{\mathbf{v}} \circ f$ is a continuous on B^{n+1}. Since B^{n+1} deformation retracts to a point, therefore the map $\bar{\mathbf{v}} \circ f$ is homotopic to the constant map. Obviously the degree of the constant map is 0 as the induced homomorphism maps everything to the same element. Since homotopic maps have the same degree, degree of $(\bar{\mathbf{v}} \circ f) = 0$, thus $w_{\mathbf{v}}(C) = 0$. □

A couple of comments now, first, a compact surface can only have a finite number of isolated critical points, since if it had an infinite number of critical points then there would be an open cover of the surface consisting of neighborhoods of each of the critical points with the restriction that each critical point is contained in exactly one of the open sets in the cover. This cover has no finite sub-cover contradicting that the surface is compact.

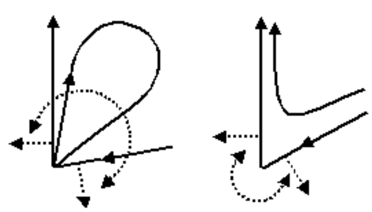
Second, it should be noted that all types of vector fields except a focus and a center (illustrated below) can be broken down into sectors where their behavior is one of the three types − Parabolic, Hyperbolic, or Elliptic.





Theorem 2.11 (Poincaré–Bendixson Theorem). *If* $\mathbf{v}$ *is a vector field with an isolated critical point* x *which is not a center or focus, with* e *Elliptic sectors and* h *Hyperbolic sectors around* x *then* $I_{\mathbf{v}}(x) = 1 + (e - h)/2$.

Proof. The proof is rather geometric, and revolves around realization that angles swept by parabolic sector of measure θ is θ, while elliptic and hyperbolic sectors of measure θ yield spans of $\theta + \pi$ and $\theta - \pi$ respectively. This is clear in the following phase diagrams by considering the angles swept out by a normal assigned to each flow vector according to the right hand rule.



Angle swept by elliptic and hyperbolic sectors

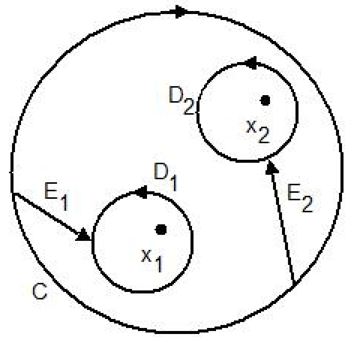
Consider the winding number of x about an ϵ circle, S_ϵ, that contains only x as a critical point and since Indices are independent of loops (which contains only one critical point of the field), therefore $I_{\mathbf{v}}(x) = w_{\mathbf{v}}(S_\epsilon)$. Computing $w_{\mathbf{v}}(S_\epsilon)$ as total angle swept normalized by 2π, $w_{\mathbf{v}}(S_\epsilon) = [p \cdot \theta_{parabolic} + e \cdot (\theta_{elliptic} + \pi) + h \cdot (\theta_{hyperbolic} - \pi)]/2\pi$ which becomes $[p \cdot \theta_{parabolic} + e \cdot \theta_{elliptic} + h \cdot \theta_{hyperbolic} + e \cdot \pi - h \cdot \pi]/2\pi$ but as $p \cdot \theta_{parabolic} + e \cdot \theta_{elliptic} + h \cdot \theta_{hyperbolic} = 2 \cdot \pi$, since it is the total angle measure of S_ϵ, $\Rightarrow I_{\mathbf{v}}(x) = 1 + (e - h)/2$.

□

With these results available, a proof for The Poincaré–Hopf Index Theorem in special case of D^2 can be constructed.

2.3 The Poincaré–Hopf Index Theorem (D^2 case)

Proof. As D^2 is compact, therefore it can only have a finite number of isolated critical points $\{x_i\}_{i=1}^k$. For each x_i, define D_i to be a neighborhood homeomorphic to a disc which encloses only the one critical point x_i and no other critical point lies inside or on its boundary $C_i = \partial D_i$. Demand that D_i's be disjoint. This gives $w_{\mathbf{v}}(C_i) = I_{\mathbf{v}}(x_i)$. Now connect each C_i to $C = \partial D^2$ via edges E_i, again since there are only a finite number of critical points in D^2, the E_i's can be chosen so as to miss all critical points.



Now note that $\hat{D} = D^2 - [\cup_{i=1}^k (D_i \cup E_i)]$ is homeomorphic to an open disc and contains no critical points of $\mathbf{v}$, hence $w_\mathbf{v}(\partial\hat{D}) = 0$. Now Consider winding number for each piece of boundary consisting of C_i and corresponding the E_i traversed in both directions, $w_\mathbf{v}(C_i + E_i - E_i) = w_\mathbf{v}(C_i) + w_\mathbf{v}(E_i) - w_\mathbf{v}(E_i) = w_\mathbf{v}(C_i) \equiv I_\mathbf{v}(x_i)$ (by Lemma 2.8). Now $w_\mathbf{v}(\partial\bar{D}) = w_\mathbf{v}(C) - \sum_{i=1}^k w_\mathbf{v}(C_i) = 0 \Rightarrow w_\mathbf{v}(C) = \sum_{i=1}^k w_\mathbf{v}(C_i) = \sum_{i=1}^k I_\mathbf{v}(x_i)$. □

3 The Poincaré–Hopf Index Theorem on 2-Manifolds

3.1 The Poincaré–Hopf Index Theorem

Before addressing the general statement and furnishing a proof, certain definitions are given.

Definition 3.1. A smooth 2−manifold is an 2−manifold $M \subset \mathbb{R}^2$ such that every point $x \in M$ has a disc neighborhood U relative to M which is diffeomorphic to a disc $D^2 \subset \mathbb{R}^2$ via $f : D^2 \to U$.

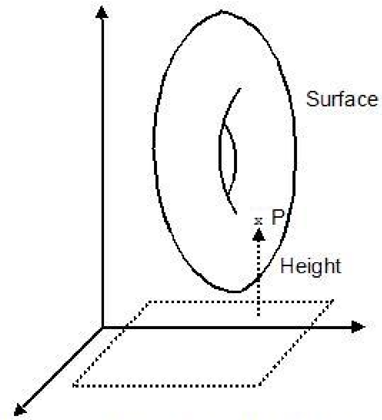
Definition 3.2. For $M \subset \mathbb{R}^2$ a smooth 2-manifold, a tangent vector field $\mathbf{v}$ on M is a continuous function $\mathbf{v} : M \to \mathbb{R}^2$ (under the identification of tangent space with $\mathbb{R}^2$) such that $\mathbf{v}(x) \in T_x(M)$ for every $x \in M$. (That is, $\mathbf{v}$ associates with every $x \in M$ a tangent vector $\mathbf{v}(x)$ in the tangent plane).

Given this, the statement of The Poincaré–Hopf Index Theorem can be introduced.

Theorem 3.3 (The Poincaré–Hopf Index Theorem)**.** *If* $\mathbf{v}$ *is a tangent vector field on a smooth, compact surface* S *with only isolated critical points* $x_1, x_2 ... x_k$ *then* $\sum_{i=1}^k I_\mathbf{v}(x_i) = \chi(S)$.

Proof. The idea of the proof is to first establish it in case of orientable surfaces and then extend it to the case of non-orientable surfaces, first to a single $\mathbb{P}^2$ using a orientable double cover, then via surgery to connected sums of projective planes. And since by the Classification Theorem, these are exactly all the non-orientable surfaces, the proof will be complete.

Starting with a given compact, connected, orientable surface S, consider a special vector field $\mathbf{v}$ which assigns to every point p the gradient of the distance function *"height"*) from a fixed plane as illustrated.



Reference Plane - Height of Point P on Surface

The claim is that given any arbitrary vector field **U** (with only isolated critical points) on S, the sum of indices of critical points of **U** is same as that of **v** which will be demonstrated to be $\chi(S)$, the Euler Characteristic of S. Note that since there are only a finite number of critical points **v** can be chosen (by changing the reference plane) in a way that none of the critical points of **v** and **U** coincide.

Since the gradient is the direction of steepest ascent as such this field will have no elliptic sectors (as the field cannot complete a loop while continuously ascending, this also precludes center type vector fields, while focus type fields are not possible as because in a focus type field the field never reaches the critical point, but ascending continuously without bound (for all time) would make the surface unbounded, thus not compact) implying the index of a critical point x will simply be $1-h/2$, h being the number of hyperbolic sectors around x. Now consider a triangulation of S with critical points of **v** included in the set of vertices such that no two vertices of the same triangle take on the same height over the reference plane, again this is possible as the surface being compact admits a finite triangulation. This implies that one vertex of each triangle has a height between that of the other two, hence is defined as the *middle vertex*. The triangulation can further be refined such that corresponding to each hyperbolic sector around a vertex x_t there exists exactly one triangle that contains x_t as a middle vertex.

Now $\sum_{i=1}^{k} I_{\mathbf{v}}(x_i) = \sum_{i=1}^{k} 1 - h/2$, but as the index of a non-critical point is 0 (that is, a non critical point must have exactly 2 hyperbolic sectors), this sum can be extended to sum over all vertices in the triangulation. Since for each i, h is same as number of triangles with x_i as a middle vertex, therefore summing h over the entire triangulation is simply the number of faces of in the triangulation (as each triangle has exactly one middle vertex). Therefore, $\sum_{i=1}^{k} I(x_i) = \sum_{x\in\mathbb{T}} I_{\mathbf{v}}(x) = V - F/2 = V - E + F = \chi(S)$ (using $3F = 2E \Rightarrow F = 2E - 2F$).

What needs to be shown now is that sum of indices over the given **U** field is the same as that of **v**. To this end some combinatorial results about triangulations are required.

Lemma 3.4. *Given a triangle with vertices labeled A, B, C then on subdivion, with the restriction that each new vertex be labeled A or B if it is created on edge AB, B or C if on BC, C or A if on AC and one of A, B, C on the interior, the triangle will yield at least one triangle with vertices A, B, C.*

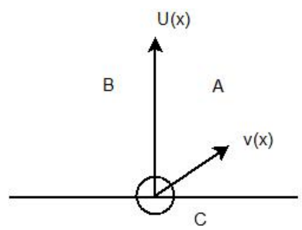
Proof. Consider the side AB of the original triangle. If a is the number of edges AA, b the number of edges AB and c the number of vertices labeled A in the interior of the side. Then $2a+b = 2c+1$

$\Rightarrow b$ is *odd*. Now if d is the number of triangles ABA or BAB, with e number of triangles ABC and f the edges AB (in the interior only) then $2d+e = 2f+b$ implying e must be odd and positive. $\square$

Lemma 3.5. *If an oriented surface with boundary $\bar{S}$ is triangulated with triangle vertices carrying labels A,B or C then number of complete triangles (triangles carrying all three labels) counted with orientation equals the number of edges AB on the boundary counted with orientation.*

Proof. Define $\phi(x) = +1$ if x is edge AB, -1 if $x = BA$, 0 otherwise. Then $\sum_{t \in \partial \mathbb{T}} \phi(t) = \sum_{t \in \mathbb{T}} \phi(t)$ (that is number of edges AB or BA on the boundary must equal those in the entire triangulation) since any edge AB if an interior edge of a triangle will be counted as edge BA of the triangle adjacent to it, thus canceling. But this sum must equal the number of complete triangles, as any triangle missing A or B would have $\phi(x) = 0$ for all of its edges and any triangle missing a C must have form ABA or BAB, again the number of AB would equal the number of BA and the triangle would not contribute to the sum. $\square$

Returning to the main claim, from the surface S, create the surface $\tilde{S}$ by removing a neighborhood of each of the critical points of $\mathbf{U}$ and $\mathbf{v}$. Triangulate $\tilde{S}$ and assign vertex labels according to the following convention −



That is, in the local coordinate system formed at point p by $\mathbf{U}(p)$ as the y-axis, label the vertex A if $\mathbf{v}(p)$ lies in first quadrant, B if in second C otherwise. This is where the orientability of $\tilde{S}$ and thus of S is essential, since otherwise this labeling would not be consistent − with any choice of a y-axis, to get the x-axis, a unique normal to the surface is needed.

Now note that this triangulation would not yield any complete triangles, since if it did then, by Lemma 3.4, each complete triangle on subdivison would yield a strictly smaller complete triangle. Iterating this process and using the fact that since these triangles are compact such a descending sequence of sets would have a limit point (well, triangle). This limit triangle will put the labels A, B, C arbitrarily close to each other. But this would imply by continuity of the field that the field must vanish over the limit triangle. This contradicts the construction of $\tilde{S}$.

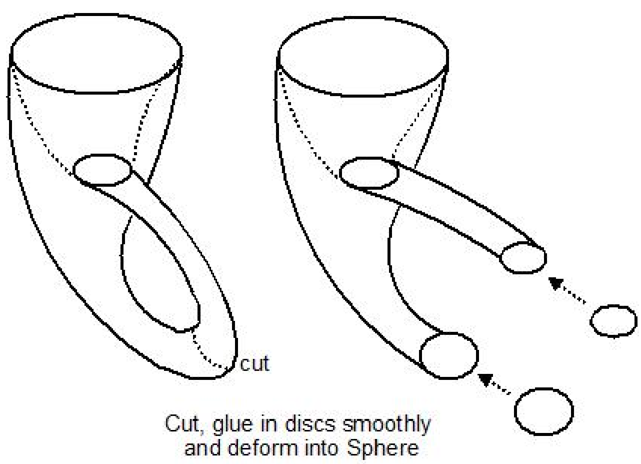
As there are no complete triangles, by Lemma 3.5 the sum of edges AB counted with orientation must be 0. Note that the boundary of $\tilde{S}$ which is exactly the boundary of neighborhoods removed at each of the critical points, and thus contains the boundary of neighborhoods of $\mathbf{v}$ that were removed. For a triangulation fine enough, $\mathbf{U}$ would remain essentially constant about the deleted neighborhoods containing a critical point of $\mathbf{v}$ making the coordinate system fixed on such neighborhoods. Traversing this boundary of the deleted disc about the critical point of $\mathbf{v}$ would be equivalent to computing the index at that critical point, since each AB marks the field $\mathbf{v}$ crossing from the first quadrant into the second with respect to the now fixed coordinate system.

Now consider the deleted neighborhoods of critical points of $\mathbf{U}$ where $\mathbf{v}$ is essentially constant, thus $\mathbf{v}$ can used as y-axis to give a coordinate system that holds on the entire neighborhood. Relabel the vertices in the triangulation as before but with roles of $\mathbf{U}$ and $\mathbf{v}$ interchanged. This exchanges labels As and Bs while preserving Cs implying that the orientation of traversal around the critical point is opposite to the one used with $\mathbf{U}$ fixed. Taking this into account, and using the same argument as made for deducing that number of ABs was equal to sum of indices of critical points of $\mathbf{v}$, here (due to the opposite orientation) the number of ABs is equal to negative the sum of indices of $\mathbf{U}$, therefore $\#AB = \sum_{i=1}^{k} I_{\mathbf{v}}(x_i) - \sum_{i=1}^{K} I_{\mathbf{U}}(x_i) = 0, \Rightarrow \sum_{i=1}^{k} I_{\mathbf{U}}(x_i) = \sum_{i=1}^{K} I_{\mathbf{v}}(x_i) = \chi(S)$. This concludes the proof for orientable surfaces.

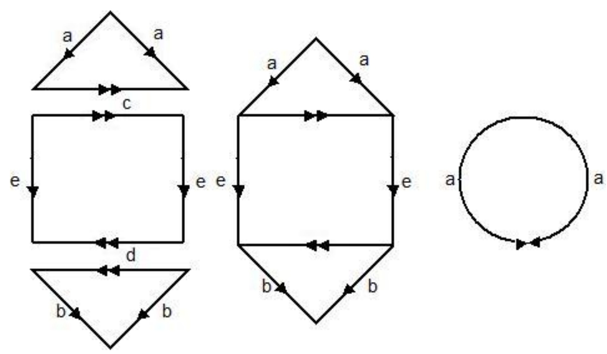
The case for non-orientable surfaces is handled via the Classification Theorem on surfaces by demonstrating a surgery on $m\mathbb{P}^2$ which gives the the Euler Characteristic in terms of the indices as required. The case for $m\mathbb{P}^2$ is built inductively. To establish the base case for induction first note that $\mathbb{P}^2$ can be thought of as $\mathbb{S}^2$ with antipodal points identified. That is, any point on $\mathbb{P}^2$ can be written as $p(x)$ with $x \in \mathbb{S}^2$ with $p(x) = p(-x)$.

Given a tangent field (with isolated critical points) $\mathbf{v}_{\mathbb{P}^2}$ on $\mathbb{P}^2$, define a tangent field $\mathbf{v}_{\mathbb{S}^2}$ by $\mathbf{v}_{\mathbb{S}^2}(x) = \mathbf{v}_{\mathbb{P}^2}(p(x))$ (the composition of vector field on $\mathbb{P}^2$ with the map p). If $p(x_{\mathbb{P}^2,i})$ (with $i \in \{1, 2..K\}$) are the critical points of $\mathbf{v}_{\mathbb{P}^2}$ then critical points of $\mathbf{v}_{\mathbb{S}^2}$ are $x_{\mathbb{P}^2,i}$ and $-x_{\mathbb{P}^2,i}$, using this and the fact that the Poincaré−Hopf Theorem has been established for orientable surfaces, $\sum_{i=1}^{2K} I_{\mathbf{v}_{\mathbb{S}^2}}(x_{\mathbb{P}^2,i}) = \sum_{i=1}^{K}[I_{\mathbf{v}_{\mathbb{S}^2}}(x_{\mathbb{P}^2,i}) + I_{\mathbf{v}_{\mathbb{S}^2}}(-x_{\mathbb{P}^2,i})] = 2$. But $I_{\mathbf{v}_{\mathbb{S}^2}}(x_{\mathbb{P}^2,i}) = I_{\mathbf{v}_{\mathbb{S}^2}}(-x_{\mathbb{P}^2,i}) = I_{\mathbf{v}_{\mathbb{P}^2}}(p(x_{\mathbb{P}^2,i}))$, this follows from definition of $\mathbf{v}_{\mathbb{S}^2}$ in terms of $\mathbf{v}_{\mathbb{P}^2}$, giving $2\sum_{i=1}^{K} I_{\mathbf{v}_{\mathbb{P}^2}}(p(x_{\mathbb{P}^2,i})) = 2$ and finally $\sum_{i=1}^{K} I_{\mathbf{v}_{\mathbb{P}^2}}(p(x_{\mathbb{P}^2,i})) = 1 = \chi(\mathbb{P}^2)$.

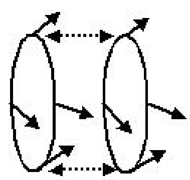
The machinery that is used to allow induction on the genus of the surface uses the result that $3\mathbb{P}^2 = \mathbb{T}^2\#\mathbb{P}^2$, and so the case for $2\mathbb{P}^2 = \mathbb{K}^2$ (the Klein bottle) must handled separately. Consider a vector field $\mathbf{v}$ on the $\mathbb{K}^2$. Assume that $\{x_i\}_{i=1}^{k}$ are the critical points, the interest here is $\sum_{i=1}^{k} I_{\mathbf{v}}(x_i)$. Since the there are only a finite number of critical points, it is possible to cut along the handle (see illustration) of $\mathbb{K}^2$ while avoiding all x_i.



Two discs D_1 and D_2 can be glued smoothly to the surface to close it off. It's not obvious from the cutting that what remains is actually homeomorphic to $\mathbb{S}^2$. The following planar diagram for this gluing makes this clear.



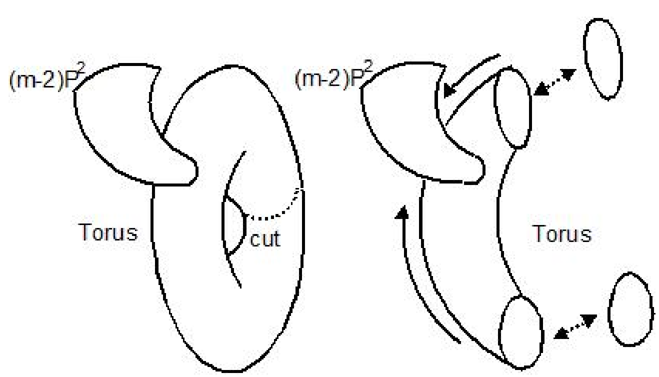
A new field can be introduced on this surface by extending the field $\mathbf{v}$ over D_1 and D_2, define this field as $\hat{\mathbf{v}}$. Assume that $\{\hat{x}\}_{i=1}^{K}$ new critical points are induced on D_1 and D_2, in addition to the critical points the new field inherits from $\mathbf{v}$. Now the sum of indices for the field is $\sum_{i=1}^{k} I_{\hat{\mathbf{v}}}(x_i) + \sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i)$, which is $\sum_{i=1}^{k} I_{\mathbf{v}}(x_i) + \sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) = \chi(\mathbb{S}^2) = 2$. Now consider the discs by themselves. Note that the fields on the boundaries on D_2 and D_1 agree - $\hat{\mathbf{v}}$ on boundary of each is just $\mathbf{v}$ along the cut on the handle, as such D_1 and D_2 can be glued along the boundary to give $\mathbb{S}$ (see illustration), thus $\sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) = \chi(\mathbb{S}^2) = 2$.



This yields $\sum_{i=1}^{k} I_{\mathbf{v}}(x_i) = \chi(\mathbb{S}^2) - \sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) = \chi(\mathbb{S}^2) - \chi(\mathbb{S}^2) = 0 = \chi(2\mathbb{P}^2)$.

Stating the induction hypothesis now, $\sum_{x_{critical}} I_{\mathbf{v}}(x_i) = 2 - m = \chi(m\mathbb{P}^2)$ where as before $\mathbf{v}$ is a tangent vector field with only isolated critical point on the compact surface $m\mathbb{P}^2$. Assume that this holds for all integers less than m.

Using $3\mathbb{P}^2 = \mathbb{T}^2\#\mathbb{P}^2$, $m\mathbb{P}^2$ (for $m > 2$) can be written as $\mathbb{T}^2\#(m-2)\mathbb{P}^2$. Now the $\mathbb{T}^2$ is cut open along the handle avoiding all critical points (possible as there are only a finite number of them), and two discs D_1 and D_2 glued smoothly to close off the open ends. Extend the vector field on this surface to the glued in discs (as before), define this as $\hat{\mathbf{v}}$.



Cut, glue in discs smoothly
and deform into Sphere leaving
behind (m-2) Projective Plane

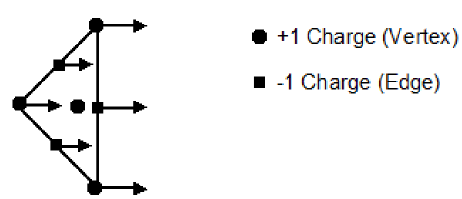
Now what is left is diffeomorphic to simply $(m-2)\mathbb{P}^2$, as the $\mathbb{T}^2$ in the connected sum is reduced to $\mathbb{S}^2$. Let $\{\hat{x}_i\}_{i=1}^{K}$ be the new critical points created on the two discs. Then if $\{x_j\}_{j=1}^{k}$ were the critical points on the original surface (which are all the critical points of the field $\mathbf{v}$) then the sum of indices of the new vector field is $\sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) + \sum_{j=1}^{k} I_{\hat{\mathbf{v}}}(x_j)$ which by induction hypothesis must be $\chi(m-2\mathbb{P}^2) = 2-(m-2)$. Again the two disc D_1 and D_2 can be glued together on the boundary to form an $\mathbb{S}^2$, this identification with $\mathbb{S}^2$ by another application of Poincaré–Hopf Theorem yields $\sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) = \chi(\mathbb{S}^2) = 2$. Now $\sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) + \sum_{j=1}^{k} I_{\hat{\mathbf{v}}}(x_j) = \sum_{i=1}^{K} I_{\hat{\mathbf{v}}}(\hat{x}_i) + \sum_{j=1}^{k} I_{\mathbf{v}}(x_j) = 2-(m-2) \Rightarrow 2 + \sum_{j=1}^{k} I_{\mathbf{v}}(x_j) = 4-m$, therefore $\sum_{j=1}^{k} I_{\mathbf{v}}(x_j) = 2-m = \chi(m\mathbb{P}^2)$. This completes the proof.

□

3.2 Sketch of Thurston's proof

There's a very elegant proof by William Thurston of the Poincaré–Hopf Index Theorem. The basic idea is that given a cell complex for a 2-surface S (orientable, connected, compact), a $+1$ charge is attached at every vertex and at the center of every face, while a -1 charge is placed at the midpoint of every edge. Note that if there are e edges, f faces and v vertices, then there are $e+f$ positive charges and v negative charges, and the sum of all charges is $f-e+v = \chi(S)$. Now given a vector field on this surface with only isolated zeros, away from the critical points the surface can be triangulated such that at none of the edges the vector field is tangential - the field is always transverse at each edge. Critical points can be enclosed within polygons with the field again transverse on the boundary (it is not always possible to enclose critical points within triangles with the field transverse on the edges). This forms a cell complex for S. Note that the cell maps are required to be differentiable so as to be able to admit tangent vector fields. Attach $+1$ to faces and vertices and -1 to edges.

Consider what happens when charges flow according to the vector field on the surface. Away from the critical points the triangulation can be made fine enough that on each triangle the field is nearly constant. Therefore if a charge on a vertex moves right so must all the charges on the triangle. As described below, after the operation of the flow the charges inside the triangle would add up to zero (no charge would stay on the boundary as that would require the vector field to be tangential to the edge) –

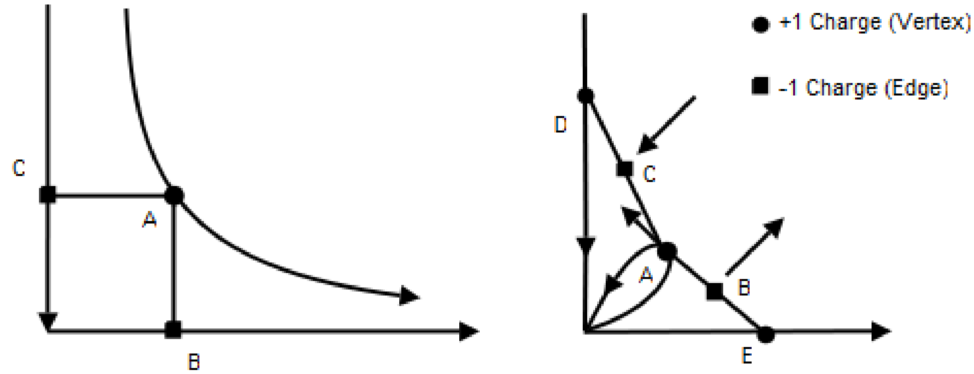


Charge flow directions on a
triangle with nearly constant vector field
(Note the charge at the center)

But the sum of charges over the entire complex is the $\chi(S)$ and since the sum of charges for each triangle away from a critical point is zero, $\chi(S)$ must be equal to the sum of charges inside the polygons enclosing critical points of the field after the flow.

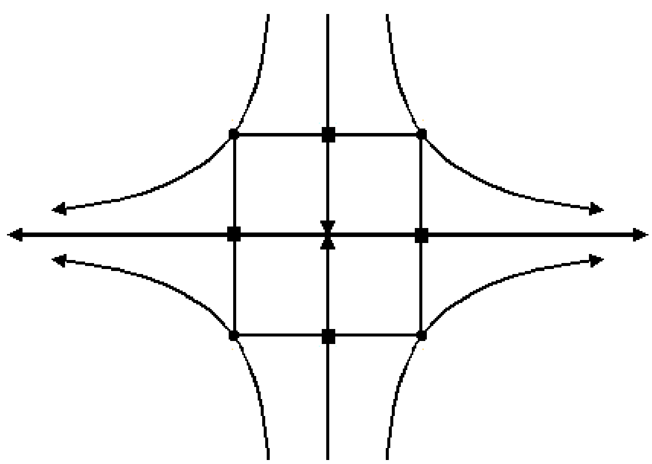
Thurston's big claim (and one that he does not elaborate) is that sum of charges inside a polygon after the flow is the index for the critical point enclosed by that polygon. To establish this, consider the breakdown of the vector field into sectors of hyperbolic, parabolic and elliptic type. Note that a parabolic sector in the polygon will force all charges in or out, thus not contributing to the sum.

As for hyperbolic and elliptic sectors, consider the following schemes of defining the boundary of the polygon in the respective sectors −

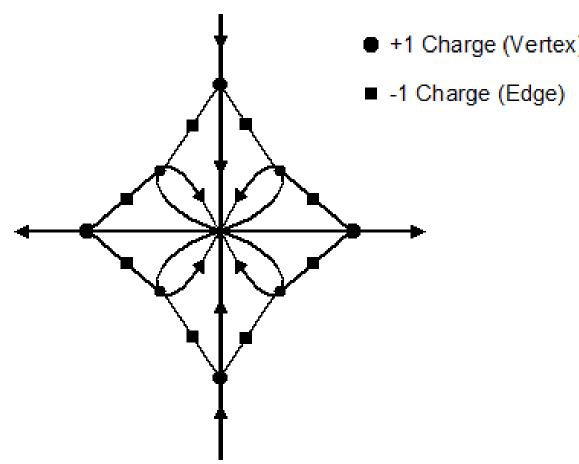


Polygon boundaries and charge flow directions over Elliptic and Hyperbolic sectors

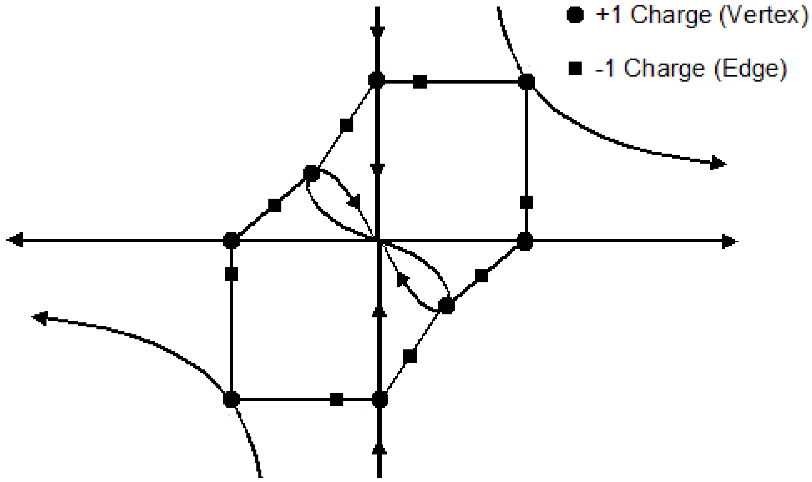
In the case of the hyperbolic sector A is the vertex, while C and B are midpoints of edges that extend into the adjacent sectors (figures that follow demonstrate this). A carries $+1$ charge while B and C carry -1 each. The flow pushes the charges at A and B out while charge at C is driven inside. Thus at the end of the flow the hyperbolic sector contains a total charge of -1. The elliptic case is a little more complicated, since an elliptic sector cannot be enclosed in the sector with boundary of the polygon being convex in that region. The simplest way to achieve this is as illustrated. Here A, D, E are vertices and have $+1$ charge while B and C are midpoints of edges AD and AE with -1 charge. The flow moves charges at A, C and D inside while B and E leave the polygon. The charge remaining in elliptic sector at the end of the flow is $+1$. Care needs to be taken in extending this line of reasoning to the entire polygon, since there is possible sharing of charges across the common boundary of two adjacent sectors. The following illustrate some possibilities where this might happen.



4 Hyperbolic Sectors



4 Elliptic Sectors



Alternate Elliptic / Parabolic Sectors

The subtlety is that when the two sectors of same type are adjacent, they share a charge along the common edge. It's a negative charge in case of parabolic sectors and positive in hyperbolic. This cause the total charge in the two adjacent sectors to be offset by ± 1. Two hyperbolic sectors now contribute -1 to the total charge and while elliptic ones contribute a $+1$.

In the general case where there are h hyperbolic and e elliptic sectors, charges from $min(e, h)$ sectors of each kind will cancel each other out no matter how they are arranged. This is because the only patterns of arrangement are that sectors of the two different types pair up in which case $+1$ from elliptic cancels -1 from the hyperbolic, if sectors are paired up with another of the same type, then since there are $min(e, h)$ sectors of both kinds, each pairing of hyperbolic sectors has a corresponding pairing of elliptic sectors, the charge contributions again cancel, lastly if sectors don't belong to any pairs (say if e and h are odd) then too the contributions from hyperbolic and elliptic sectors add to zero. Therefore, the only contribution to the total charge is from the $|e - h|$ remaining sectors. Now as a consequence of the Poincaré−Bendixson Theorem $|e - h|$ must even (as the index $I_{\mathbf{v}}(x) = 1 + (e - h)/2$ must be an integer). Since the $|e - h|$ sectors are of the same type and $|e - h|$ is even, sectors pair up, sharing charges along their boundaries. Thus, if $|e - h|$ sectors are all elliptic then the total charge contribution is $|e - h|/2$, and if hyperbolic then it is $-|e - h|/2$. Combining these two and taking into account the positive charge at the center of the polygon, the sum of charges becomes $1 + (e - h)/2$ which by the Poincaré−Bendixson Theorem is exactly the index of the critical point enclosed in the polygon. Therefore the total charge over all polygons is the sum of the indices of the critical points they enclose. But as no charge is contributed by any of the triangles in the complex away from the zeros, this is just the total charge of the 2-complex which is $v + f - e = \chi(S)$. Hence, the sum of indices of critical points matches the Euler Characteristic.

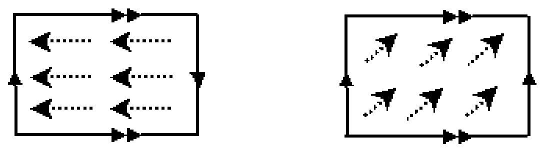
3.3 Hairy Ball Theorem

An interesting application of the Poincaré−Hopf Index Theorem which finds application in meteorology is the "Hairy Ball Theorem" (Luitzen Brouwer, 1912). It states that there does not exist a non-vanishing continuous tangent vector field on a sphere - *you can't comb a hairy ball straight.* It follows as an immediate corollary of the Index-Euler Characteristic, since as a non-vanishing field would have no critical points, the sum of indices would be trivially zero and so must be the Euler Characteristic of the surface, but $\chi(\mathbb{S}^2) = 2$, contradiction.

With respect to meteorology, wind velocity is often modeled as a two dimensional vector lying in the tangent plane at that point, the vertical component of wind velocity is negligible compared to the radius of earth and can be ignored. Hairy Ball Theorem now implies that at all time there must be at least one point of the surface of earth such that wind velocity - the vector field - becomes zero. This corresponds to there being a cyclone or an anti-cyclone on earth's surface at all times.

3.4 Non-Vanishing Vector Fields

Surfaces with Euler Characteristic of zero - $\mathbb{T}^2$ and $\mathbb{K}^2$ do allow non-vanishing continuous tangent vector fields, for instance : −



Non-vanishing Vector Fields on a Klein Bottle and a Torus

In the case of $\mathbb{T}^2$ the non-vanishing field follows from the differentiable group structure (that is, inversion and translation maps are differentiable) that it admits. This allows a group action of translation within the tangent bundle, and hence parallel transportation of tangent vectors. Given this a constant non-zero vector can be translated to every point on the surface, the vector field generated does not vanish anywhere on the surface. Another example is $\mathbb{S}^1$ which allows a non-vanishing vector field as it inherits a differentiable structure from the complex plane $\mathbb{C}$ in which it embeds into the multiplicative subgroup of elements of unit norm. This generalizes to higher dimensions as well, for example $\mathbb{S}^3$ and $\mathbb{S}^7$ (which are again embeddings in the multiplicative subgroup of elements of unit norm in Quaternions $\mathbb{H}$ and Octonions $\mathbb{O}$ respectively) are n-manifolds that allow non-vanishing vector fields since they too get a differentiable structure from the overlying spaces $\mathbb{H}$ and $\mathbb{O}$. This is also why all Lie groups too allow a non-vanishing vector field.

3.5 References

The proofs presented here take ideas from multiple sources, particularly Edward Early's paper "On Euler Characteristic" (available at http://www-math.mit.edu/phase2/UJM/vol1/EFEDUL 1.PDF) which presents a much more concise treatment of the orientable surfaces case. The case for non-orientable surfaces is based on sketch of the proof by Christine Kinsey in "Topology of Surfaces," while Thurston's elegant proof is from "Three-Dimensional Geometry and Topology, Volume I" by William Thurston and Silvio Levy where it is presented very tersely. Other references include Allan Sieradski's "Introduction to Topology and Homotopy" (contains a treatment of surgery on the torus), Michael Henle's "A Combinatorial Introduction to Topology" and wikipedia.com.