INTRODUCTION TO 3-MANIFOLDS

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As we know, a topological n-manifold X is a Hausdorff space such that every point contained in it has a neighborhood (is contained in an open set) homeomorphic to an n-dimensional open ball. We will be focusing on 3-manifolds much the same way we looked at 2-manifolds (surfaces).

A basic example of a 3-Manifold: $\mathbb{R}^3$ is a 3-manifold because every point in $\mathbb{R}^3$ is contained in an open ball in $\mathbb{R}^3$.

Our study of 3-Manifolds will benefit greatly by making sure we have a strong standing in surfaces. All surfaces admit one of three geometries or geometrics structures.

1. Geometric Structures

A geometric structure is defined as a complete and locally homogenous Riemannian manifold. That is, a manifold with a metric defined locally (in the target space) that can be integrated to find lengths of paths.

The line with minimum length, also known as a distance-minimizing path, between two points is called a geodesic.

The three geometries that model all surfaces are Euclidean (flat geometry), spherical, and hyperbolic geometry. These three geometries act as the universal covers of all surfaces.

Spherical and Hyperbolic geometries are infinitesimally Euclidean. That is, in arbitrarily small neighborhoods, these geometries behave like Euclidean geometry. However, on a larger scale these three geometries can be differentiated by several unique attributes. We will look at two.

Euclidean geometry follows Euclids fifth postulate: given any line and a disjoint point, there exists exactly one line containing our point that does not intersect our given line.

Hyperbolic and Spherical geometry do not. In Hyperbolic geometry there exist at least two lines (defined later) disjoint to our given line and containing our point. In Spherical geometry, all lines intersect.

As well, consider a geodesic triangle, three points connected by geodesics, in the three geometries. In Euclidean geometry the sum of angles inside a geodesic triangle, Σ, is always equal to π. In hyperbolic geometry, $0 < \Sigma < \pi$. In spherical geometry, $\pi > \Sigma$.

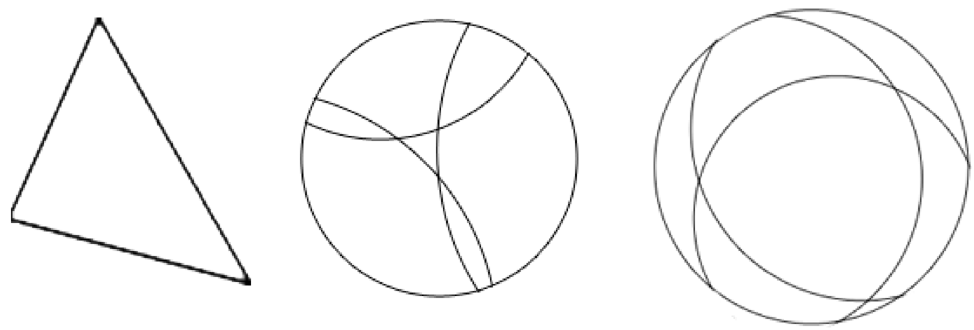

FIGURE 1. L to R, Triangles in Euclidean, Hyperbolic, and Spherical Geometries

1.1. **The Hyperbolic Plane $\mathbb{H}$.** The majority of 3-manifolds admit a hyperbolic structure [Thurston], so we shall focus primarily on the hyperbolic geometry, starting with the hyperbolic plane, $\mathbb{H}$. There are several model spaces of $\mathbb{H}$. By that we mean a way of displaying geometric shapes in an underlying space. We shall focus on the upper-half plane and the disc model, both of which have the complex plane as the underlying space, $\mathbb{C}$.

The upper half plane is defined as:

$\mathbb{H}$={z ϵ $\mathbb{C}$, $Im(z) > 0$ }

with the metric $ds=\frac{|dz|}{Im(z)}$

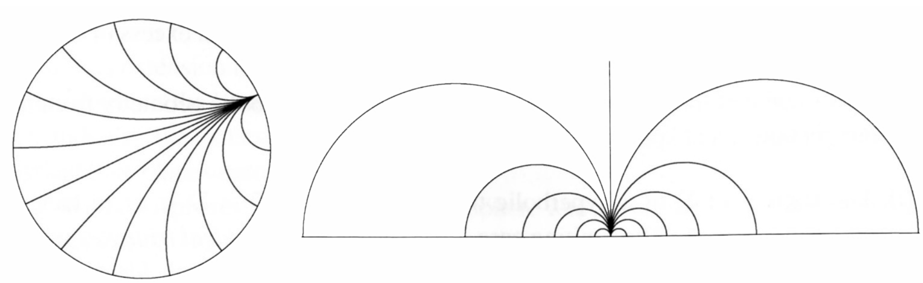

FIGURE 2. Disk and Upper Half Plane Model of $\mathbb{H}$ (Silvio Levy)

There exist two types of lines in the upper half plane. If points x, y have the same real component in $\mathbb{C}$, the line connecting them is perpendicular to real line in $\mathbb{C}$. If x and y do not have the same real component, then a line connecting them is defined as a Euclidean circle centered on the real line.

The disk model is defined as:

$\mathbb{H} = \{z \in \mathbb{C}$ such that $|z| < 1\}$

With the metric $ds = \frac{2|dz|}{1-|z|^2}$

Most lines in the Poincare disk model are arcs of circles that intersect the boundary $\mathbb{S}^1$ orthogonally. There also exist Euclidean straight lines that connect two points opposite each other across the center. Both types are displayed in Figure 2 and Figure 3.

In any n-dimensional hyperbolic space. There exists exactly one geodesic connecting two points. In fact, the uniqueness of lines in the hyperbolic plane is discussed in [Anderson]. As well, in both the hyperbolic plane and its three dimensional analog, as a line approaches the boundary, whether it be the bottom of an upper half model, or the boundary of a ball/disk model, the metrics are defined such that the length of the line increases and is of infinite length if it intersects the boundary. As well, in any dimension of hyperbolic space, angles between lines (and planes) approach zero as the length of sides increase.

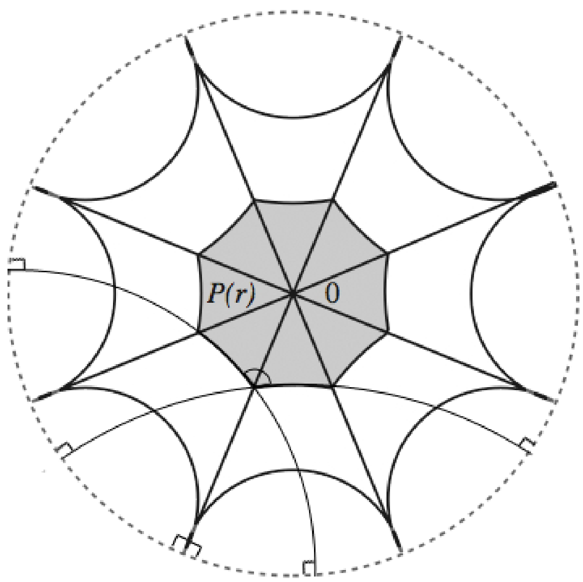

Figure 3. An image of two Octagons in the Hyperbolic Disk Model adapted from [Lackenby]

In Figure 3, we can see this as the length of our sides get larger. If two lines intersect at the boundary, since they both intersect the boundary orthogonally, the angle between them will be zero. Since Hyperbolic geometry is infinitesimally Euclidean, by an application of the intermediate value theorem, we can make our angle anything we desire between its

Euclidean angle and zero. It is worth noting the opposite is true in n-dimensional spherical geometries. As the length of sides of a polygon increase towards their boundary, the angle between them increases towards 180 degrees.

1.2. **Hyperbolic 3-Space- $\mathbb{H}^3$.** Like the Hyperbolic Plane, there exist several model spaces for Hyperbolic 3-Space. The two we will focus on are the analogs to our 2-dimensional examples. The left figure below is an the open ball model with examples of planes. The right diagram is the Upper Half Space model with examples of planes.

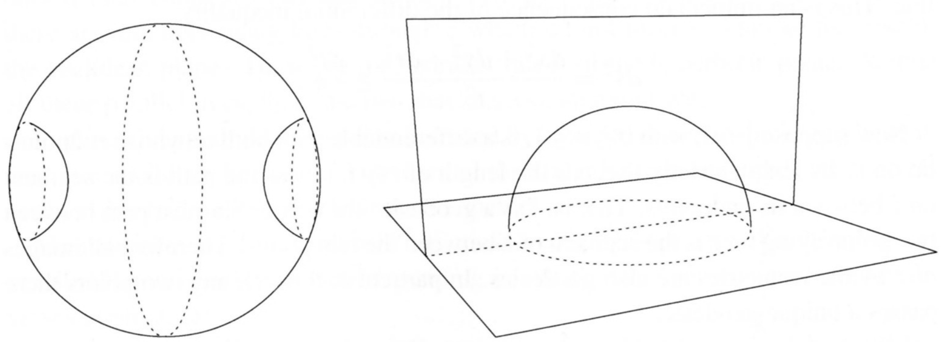

FIGURE 4. Open Ball and Upper Half Space Model (Silvio Levy)

The Upper Half Space is defined as $\mathbb{R}^3$ such that the z-coordinate is greater than zero. Planes exist in the Upper Half Space in two forms. They are either planes that run perpendicular to $\mathbb{R}^2$, or as hemispheres that intersect $\mathbb{R}^2$ orthogonally.

The metric on the Upper Half Space is

$ds = \frac{|\vec{dx}|}{t}$

The open ball model can be thought of as $\mathbb{R}^2 \bigcup \infty$, or simply as an open unit 3-ball. Planes in the open ball model exists as either Euclidean planes that pass through the center of the ball (whose intersection with the boundary $\mathbb{S}^2$ is a great circle), or as a hemisphere whose intersection with the boundary $\mathbb{S}^2$ is orthogonal. More specifically they are fixed point sets of involutions (isometries of order 2).

The metric on the open ball model is

$ds = \frac{2|\vec{dx}|}{1-|\vec{x}|^2}$

2. 3-MANIFOLDS

Now that we are working in the 3-space, lets define one more 3-manifold that should be readily available for our understanding:

The 3-torus is a 3-manifold constructed from a cube in $\mathbb{R}^3$. Let each face be identified with its opposite face by a translation (without twisting). You can imagine this as a direct extension from the 2-torus we are comfortable with. If you were to sit inside of a 3-torus and look straight you would see infinitely many images of yourself. Unlike in a hall of mirrors though, you would see images of the back of your head. In a 3-Torus, you would not only see infinite images of yourself from behind in front of you, but if you were to look up you would see images of you from below repeating off into space. If you were to look to your left you would see the right side of your body, with infinite repetitions behind gradually getting smaller.

This 3-torus is constructed from $\mathbb{E}^2$ because all the faces of a cube can be identified without any overlap of angles. By this we mean the dihedral angles of the cube are 90 degrees. If we take a point on an edge of the cube, a neighborhood of our point is a 90 degree wedge of a 3-ball. After gluing the face containing one of the edges of our 90 degree wedge with it's opposite face, we now have a hemisphere as a neighborhood. One more identification of faces creates a 3-balls neighborhood. If you think of a vertex on the corner of the cube, a neighborhood about that point is one quarter of a sphere with three planar faces. After we glue these faces with their opposites we can realize the neighborhood is actually a 3-ball.

If we are to think of higher dimensional polyhedra, problems arise when gluing faces. A regular dodecahedron in Euclidean space has dihedral angles of approximately 116.6 degrees. If we take a point on an edge of a pentagon, the open neighborhood of that point is a little less than a one third wedge of a 3-ball, with two planar faces. If we are to identify opposite faces with minimal twisting, there is no way to create a complete 3-ball about a point on an edge in Euclidean geometry. We will either under shoot the dihedral sum of 360 degrees, or go way over.

Here we can look at other geometries. We could put our dodecahedron in a Spherical geometric structure and make our dihedral angles equal 120 degrees by increasing it's size. With a 1/10 clockwise twist of opposite faces while gluing, this creates a 3-manifold called the Poincare dodecahedral space.

As well, in hyperbolic geometry we can make the inside angles of polygons as small as we want by making the edges longer. Using this method and a system of twists we can construct something called the Seifert-Weber space in Hyperbolic Space. This happens to be a Hyperbolic 3-Manifold.

Definition: A Hyperbolic 3-Manifold is a quotient manifold $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of orientation preserving isometries of $\mathbb{H}^3$.

An isometry in general is a distance preserving homeomorphism on a space. That is, a homeo $f : (X, d_X) \to (f(X), d_{f(X)})$ such that $d_X(x, y) = d_{f(X)}(f(x), f(y))$ for every x and y in X.

This definition of Hyperbolic 3-manifold follows the same idea as the universal covering of the 2-Torus by the Euclidean plane. The 2-torus is a Euclidean 2-manifold because it is the quotient manifold of E^2 by the isometry group $\Gamma =< t_1, t_2 >$, where $< t_1, t_2 >$ is the normal subgroup of translations in the x and y directions. This was essentially the same as taking a grid of squares and identifying the edges of each square in the $aba^{-1}b^{-1}$ form.

3. Examples of Hyperbolic 3-Manifolds

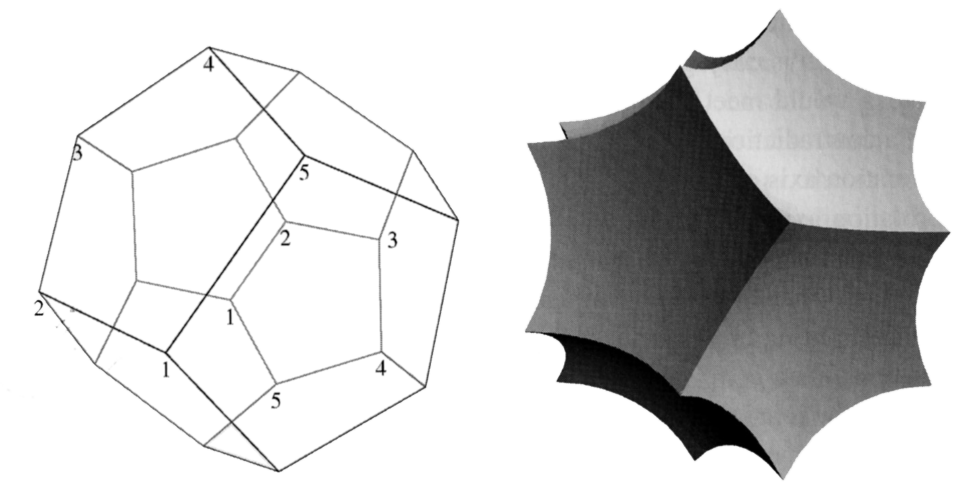

Figure 5. Seifert-Weber Space (Silvio Levy)

3.1. **Seifert-Weber Space.** To construct the Seifert-Weber Dodecahedral Space, take a dodecahedron and identify opposite faces with a 3/10 clockwise twist.

By observing our quotient maps combinatorially, we see that the edges are connected in 6 groups of 5. Thinking back to the problem mentioned above, we know the dihedral angles are of approximately 116.6 degrees in Euclidean space. Therefore to make these 5 wedges line up without overlap we need smaller dihedral angles. From earlier we know by the intermediate value theorem we can place our dodecahedron in Hyperbolic Space and lengthen edges until we get a satisfactory angle. Since edges are glued in groups of five, a dihedral angle 72 degrees would add up to 360 degrees perfectly. This is easiest to see in the open ball model above right.

Now we must check to see if this is actually a 3-manifold after all.

It is easy to see that point in the middle of one of our pentagon faces has a 3-ball neighborhood. Before gluing, the neighborhood is an open hemisphere, and after identification with the opposite face, we complete our 3-ball. (Think of a point sandwiched between two solid walls.)

Points on the edge of a pentagon are a little more difficult to see. Before identifications, a neighborhood of a point was a 72 degree wedge of a 3-ball. However after gluing all opposite faces, edges are glued in groups of 5.

Imagine being on the surface of a 3-ball surrounding our point x, standing on a great circle. When traversing in any direction on the boundary sphere through a wedge of our dodecahedron, you travel inside the wedge until arriving at a face of a pentagon. This pentagon however has been glued to its opposite pentagon via a twisting translation isometry. You then cross a non-existent border and are now walking on the surface of a 3-ball intersect a 72-degree wedge elsewhere in our space. We repeat this process until, after passing through 5 gluings, we are back in our original wedge. We have successfully traversed a great circle on a 3 ball surrounding our point x. Since we arbitrarily chose the direction we traveled in, it is clear the neighborhood of x is in fact a 3-ball.

Vertices of the dodecahedron take a little more effort to visualize. After gluing, it turns out that all vertices are mapped to one point v, and that as well has a 3-ball neighborhood. [Thurston] We are not always this fortunate when dealing with vertices, as we will see with the figure eight knot complement.

Some observations about Seifert-Weber Space:

Imagine you are standing in the SW space. If you stand with your back to one pentagon and you look through the center of the dodecahedron at the opposite face, you will see a slightly smaller image of yourself from behind, with a 3/10 clockwise twist. Beyond that you would see infinitely many more images of you from behind, each twisting 3/10 clockwise from the previous image. You would have to look at the tenth image in the distance to actually see an image of your back with the same orientation as yourself.

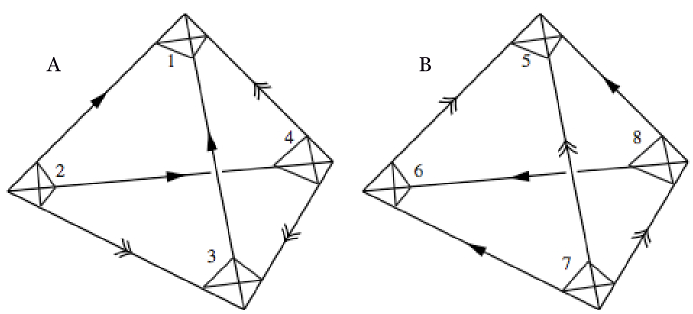

FIGURE 6. The identifications of two tetrahedra needed to make the figure-eight knot complement adapted from [Lackenby]

3.2. Figure Eight Complement. The figure eight knot complement is a classic example of a slightly more complicated 3-manifold. Take the two tetrahedra above and glue them according to the orientations described. Lets call this new figure M. The faces will all be glued in pairs, but all the vertices will be glued together. The neighborhood of a point on a face is homeomorphic to a 3-ball by the same solid-wall sandwich idea. A little bit later we shall prove the same is true in hyperbolic space for any point besides the vertex, v. A small neighborhood of the v however is a cone with a torus boundary.

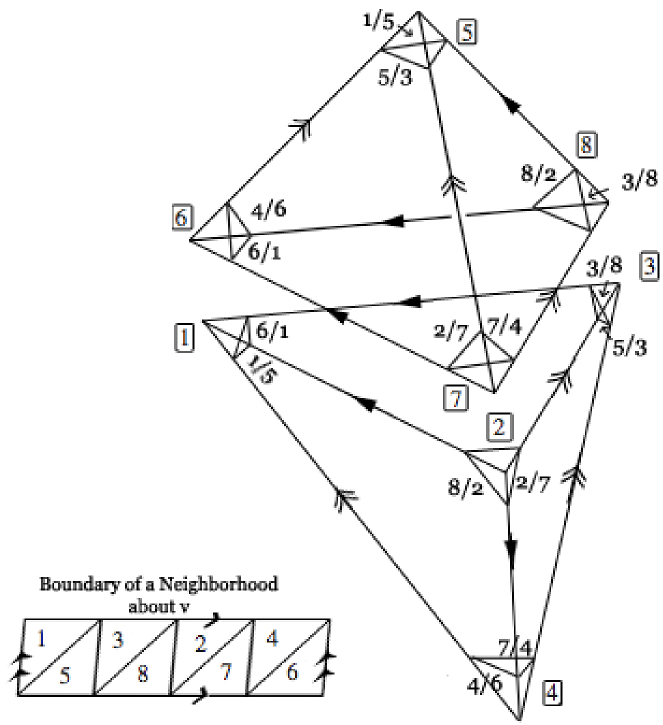

FIGURE 7. Identifications of Tetrahedra adapted from [Lackenby]

You can see in Figure 7 that the 123 triangle is glued to the 678 triangle with 1 onto 6, 2 onto 7, and 3 onto 8. Take a small neighborhood about angle 1. The boundary of that neighborhood can be seen as the triangle that is sitting inside the tetrahedron, not on a face. One side of this triangle is glued to one side of a boundary triangle about a small neighborhood around angle 6.

Walking around the corner of our angle 1 neighborhood boundary triangle we see that the 124 triangle is glued to the 586 triangle, with angle 1 glued to angle 5. The neighborhoods of these two points also share a side on their boundary triangles. We can continue this exercise until we return to a neighborhood about angle 6. From here we almost have the boundary of a neighborhood of v from Figure 7.

When we followed identifications around on the boundary triangle of a neighborhood of v, we neglected a face for each angle. Look at angle 1, we have not declared what the back edge of the boundary triangle (lying in the 134 face) is glued to. If we follow the identification, that triangle edge is identified to the edge of the boundary triangle of angle 5 the lies on face 578. So, a neighborhood about angle 1 and angle 5 are glued together twice, as described in Figure 7. From here we can see that the boundary of a neighborhood about v is a torus.

Since v was our only problematic point, we can take out v every point has a has a 3-ball neighborhood. Therefore $M - v$ is a 3-manifold.

Theorem 1.

$M - v$ is homeomorphic to $S^3 - K$ where K is the figure 8 knot.

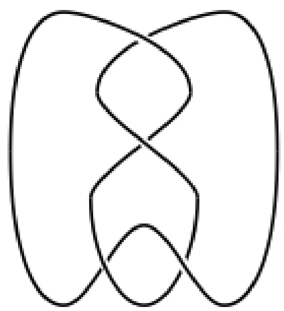

Figure 8. Figure Eight Knot K [Lackenby]

We shall follow the proof from [Lackenby].

Consider the K^1 cell complex on a figure eight knot shown below embedded in S^3.

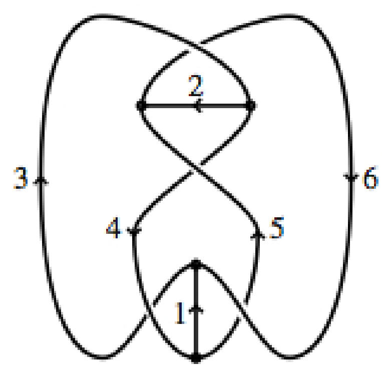

Figure 9. K^1 complex on knot K [Lackenby]

Attach 2-Cells as defined in Figure 10 (next page) giving us a K^2 cell complex embedded in S^3.

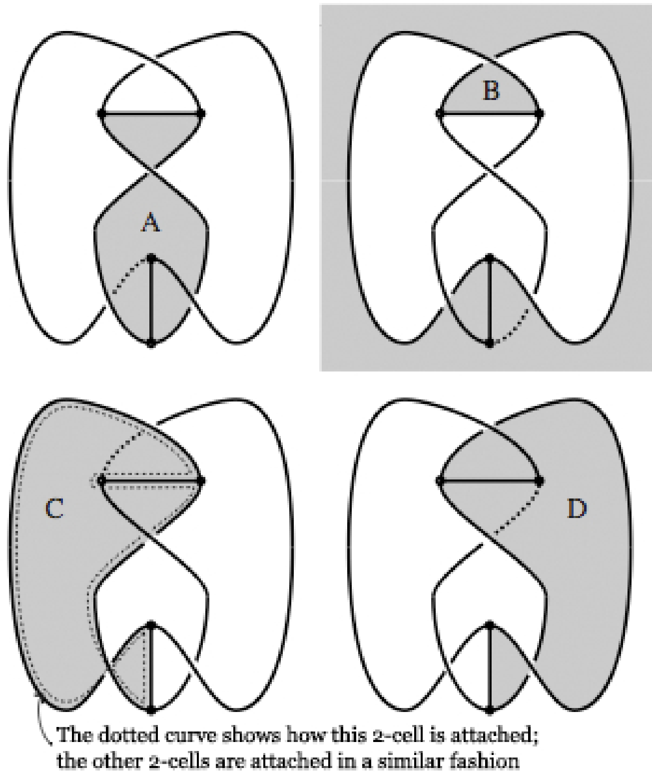

Figure 10. adapted from [Lackenby]

We claim that S^3-K^2 is homeomorphic to two 3-Balls

This is easily follows from the claim there exists a homeomorphism from S^3-K^1 to S^3-K_1^1, where K_1^1 is Figure 11.

The proof of this claim comes directly from [Lackenby] and the idea is demonstrated clearly in Figure 12. Take a fattened neighborhood of 1-Cells 1 and 2. We can then untangle 3 & 6 from 4 & 5 without changing the complement giving us the complex we want. We now have our K^2 cell complex on the flattened and untangled version of the figure-8 knot, K_1^2, filling $\mathbb{R}^2$.

If we embed K_1^2 in S^3 and take the complement, we get something that is homeomorphic to two 3-Balls. Think of K_1^2 embedded in S^3 as $R^2 \bigcup \{\infty\}$, dividing S^3 into two parts. From this idea we can see that the complement is two 3-Balls.

Now, without removing K_1^2 from S^3, we can think of these 3-Balls as the interior 3-cells and extend our K^2 cell complex to a K^3 complex. The boundary of the 3-cells are connected to our K^2 complex as shown by Figure 13.

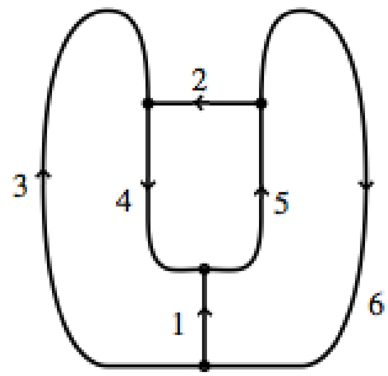

FIGURE 11. K_1^1 adapted from [Lackenby]

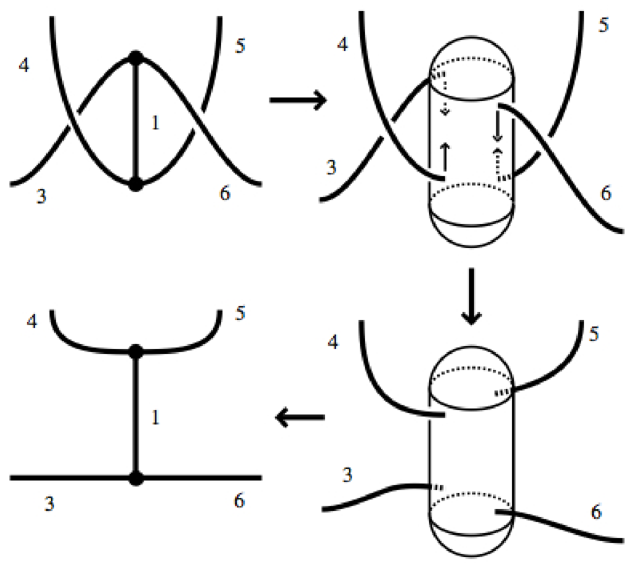

FIGURE 12. adapted from [Lackenby]

The 0-cells and 1-cells 3, 4, 5, and 6 combine to form a figure eight knot K, as can be seen in Figure 14. This is the original figure-8 knot we were given before turning it into a K^1 cell complex by adding 1-cells 1 and 2.

We can then reduce the 0 and 1-cells 3, 4, 5, and 6 to a point, v, and remove v. Since we are working in S^3, this is equivalent to reducing a figure-8 knot to a point and removing it from S^3. All that follows now is showing that the cell complex after collapsing these cells to v is M.

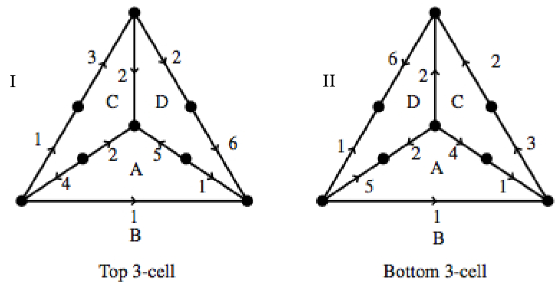

FIGURE 13. [Lackenby]

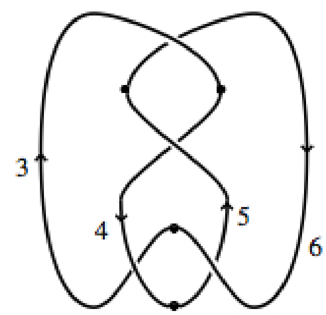

FIGURE 14. adapted from [Lackenby]

If we reduce the 0 cells and 1 cells 3, 4, 5, 6 to point, we are left with one point v, two 1- cells (1 and 2), four 2-cells (A, B, C, D), and two 3-cells. With a little investigating, one sees that the identifications on Figure 13, after collapsing K to a point, match that of Figure 6.

Look at the Tetrahedron A from Figure 6, and imagine it sitting on the 124 face with angle three pointing straight up. Imagine we were to squish A straight down but be able to make out the angles and edges as in Figure 15, and were able to then view our flat structure from underneath. This would look like the K^2 complex I from Figure 13 with 1-cells 3, 4, 5, 6, and the 0-cells reduced to a point. We can do a similar procedure with Tetrahedron B and get a reduced cell complex II.

Since we have a sufficient complex to create $S^3 - K$ from $M - v$, we can determine that this procedure indeed proves they are homeomorphic. $\square$

Finally, since this example has been considered a hyperbolic manifold without any justification, we shall prove that $M - v$ is a hyperbolic manifold.

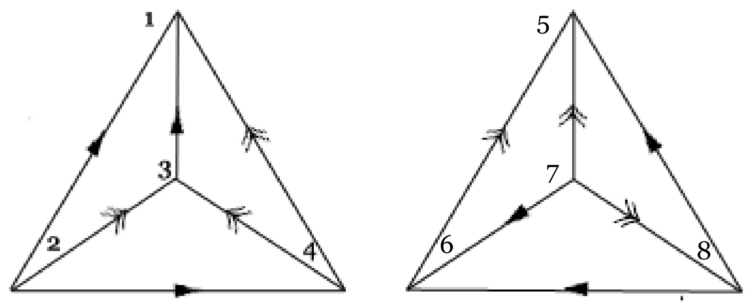

FIGURE 15. L: Crushed Tetrahedra A from Figure 6, R: Crushed Tetrahedra B

Theorem 2. *$M - v$ is a hyperbolic manifold.*

Definition. An ideal tetrahedron is a polyhedron centered at the origin with its four vertices on the boundary S^2 of the open ball model.

Lackenby proves one theorem that we shall use in proving $M - v$ is a hyperbolic manifold.

Theorem 3. *Let M be a structure obtained by gluing faces of hyperbolic polyhedra in pairs via isometries. Suppose that each point $x \epsilon M$ has a neighborhood U_x and an open mapping ϕ_x: $U_x \to B_{\epsilon(x)}(0)$ where $B_{\epsilon(x)}(0)$ is an epsilon ball about $\phi_x(x)$ centered at the origin, and ϕ_x is a homeomorphism which sends x to 0 and restricts to an isometry on each component of U_x that intersects a face of a glued polyhedra. Then M inherits a hyperbolic structure.*

Definition An ideal tetrahedron is regular if, for any permutation of its vertices, there is a hyperbolic isometry which realizes this permutation.

Build a regular tetrahedron, $\triangle$, by constructing a Euclidean tetrahedron centered at the origin of the open ball model with vertices in S^2. Lackenby asserts that $\triangle$ is regular because any permutation of its vertices is realized by an orthogonal map of $\mathbb{R}^3$ which is a hyperbolic isometry.

We then glue two $\triangle$ via the identifications from Figure 6. All we need to do is prove for Theorem 3 to hold is that any point on an edge has a 3-ball neighborhood (we deleted our vertex). Since we constructed our regular ideal tetrahedron from a Euclidean tetrahedron, the dihedral angles between faces is $\pi/3$ (proof in [Lackenby]). Since our edges are glued in groups of 6 (look at Figure 6), our $\pi/3$ wedges add up to a 3-ball.

This fulfills the criterion for Theorem 3. Therefore $M - v$ is a hyperbolic structure maintaining that is is a hyperbolic 3-manifold. $\square$

4. Applications

4.1. **Dehn Surgery.** Dehn surgery is process of using a link or knot inside a 3-manifold to generate a different 3-manifold.

To perform a Dehn surgery, take any link or knot K contained in M, a 3-manifold, hyperbolic or otherwise, and choose a small open tubular neighborhood, avoiding self-intersection. Remove this expanded K_0. The result is a Manifold minus a Torus. No matter how ugly the torus may be, it still has the same fundamental group since we avoided self intersection.

We now take a solid torus, T, and choose two fundamental group generating paths on it. We glue T back into our drilled out section by gluing ∂T onto ∂M with the paths we have chosen on T lining up with a longitudinal and meridian line on what used to be the boundary of K_0.

If we choose the same paths on T as ∂M to glue along, we get the same manifold M back. However, we can choose any two fundamental group generating paths and can generate a different 3-manifold. In fact Lickorish and Wallace proved in the 1960's that:

Theorem 4. Any closed, connected, orientable 3-Manifold may be obtained from the 3-Sphere by Dehn surgery on a link L contained in S^3 [seen in Lackenby]

Dehn surgery on hyperbolic manifolds is unique to 3-manifolds, making the study of 3-manifolds that much more rich than many higher dimensional manifolds. In the 1980's another interesting fact about Dehn surgeries was proven.

Theorem 5. All but finitely many Dehn surgeries result in a hyperbolic manifold. [Thurston]