

POTENTIAL AUTOMORPHY OF GSpin_{2n+1} -VALUED GALOIS REPRESENTATIONS

STEFAN PATRIKIS, SHIANG TANG

ABSTRACT. We prove a potentially automorphy theorem for suitable Galois representations $\Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{F}}_p)$ and $\Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$, where Γ_{F^+} is the absolute Galois group of a totally real field F^+ . We also prove results on solvable descent for $\mathrm{GSp}_{2n}(\mathbb{A}_{F^+})$ and use these to put representations $\Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ into compatible systems of $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l)$ -valued representations.

1. INTRODUCTION

Given a connected reductive group G defined over a number field F , the Langlands program predicts a connection between suitably algebraic automorphic representations of $G(\mathbb{A}_F)$ and geometric p -adic Galois representations $\mathrm{Gal}(\overline{F}/F) \rightarrow {}^L G(\overline{\mathbb{Q}}_p)$ into the L-group of G . Striking work of Kret-Shin ([26]) constructs the automorphic-to-Galois direction when G is the group GSp_{2n} over a totally real field F^+ , and π is a cuspidal automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A}_{F^+})$ that is essentially discrete series at all infinite places and is a twist of the Steinberg representation at some finite place. In this paper we will establish a partial converse, proving a potential automorphy theorem, and some applications, for suitable GSpin_{2n+1} -valued Galois representations. Before discussing our main results, we will put the work of Kret-Shin in context.

Their construction builds on two monumental works. First, it depends on the construction of automorphic Galois representations when $G = \mathrm{GL}_{2n+1}/F^+$, and π is cuspidal, regular algebraic, essentially self-dual, and square-integrable at some finite place: extending work of Kottwitz, Clozel ([13]) constructed the relevant Galois representations, and the essential properties for the purposes of [26] were proven by Harris-Taylor ([22]) and Taylor-Yoshida ([39]).¹ Second, it requires Arthur's endoscopic classification of representations ([2]), which among other things describes the discrete automorphic spectrum of $\mathrm{Sp}_{2n}(\mathbb{A}_{F^+})$ in terms of self-dual discrete automorphic representations of $\mathrm{GL}_{2n+1}(\mathbb{A}_{F^+})$. These two marvelous developments allow (e.g., [26, Theorem 2.4]) the construction of SO_{2n+1} -valued Galois representations associated to cuspidal automorphic representations of $G = \mathrm{Sp}_{2n}$ that are discrete series at infinity and Steinberg at some finite prime (and indeed more generally).

Taking into account all of those advances, there remains a significant gap between the theorem of [26] for GSp_{2n} and the previously-known result for Sp_{2n} . We briefly summarize the difficulty. Let $\Gamma_{F^+} = \mathrm{Gal}(\overline{F^+}/F^+)$, and fix an isomorphism $\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. A theorem

¹There is an extensive and deep literature devoted to eliminating the square-integrability condition: we mention as a sampling work of Bellaïche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, and Taylor, which in turn relies on other deep automorphic advances, particularly work of Waldspurger and Ngô.

of Tate shows that any representation $\Gamma_{F^+} \rightarrow \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_p)$ lifts to a $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ -valued representation, which is then determined up to a central twist. Thus one hopes to start with the automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{F^+})$ under consideration, consider an automorphic component π of its restriction to $\mathrm{Sp}_{2n}(\mathbb{A}_{F^+})$, construct the Galois representation $r_{\pi,\iota}: \Gamma_{F^+} \rightarrow \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_p)$ associated to π , and then choose the “right” lift to $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$. As it stands, there is no technique internal to the theory so far described that allows one to do this: even if one can choose a lift with Clifford norm matching the central character of $\tilde{\pi}$, there remains at each place v such that $\tilde{\pi}_v$ is unramified a ± 1 ambiguity in whether the constructed Galois representation matches the Satake parameter of $\tilde{\pi}_v$ (not to speak of the other finite places). No elementary twisting argument can resolve this, and Kret-Shin address the problem by realizing the composition $\mathrm{spin}(r_{\tilde{\pi},\iota})$ of the desired $r_{\tilde{\pi},\iota}$ with the (faithful) spin representation $\mathrm{spin}: \mathrm{GSpin}_{2n+1} \rightarrow \mathrm{GL}_{2n}$ inside the cohomology of a Shimura variety for a suitable inner form of GSp_{2n} . The principal challenge of their paper consists of the subtle analysis of the cohomology of this Shimura variety.

We now return to the setup of our paper. The deepest inputs for our potential automorphy theorems are the potential automorphy theorem of Barnet-Lamb, Gee, Geraghty, Taylor of [3] (and, for the strongest statement, a recent improvement due to Calegari, Emerton, and Gee in [11]) and Arthur’s work ([2]). We develop these and their relationships with the construction of [26] to prove the following two potential automorphy results, one for mod p and one for p -adic representations. Here and throughout the paper, we let $\mathrm{std}: \mathrm{GSpin}_{2n+1} \rightarrow \mathrm{GL}_{2n+1}$ denote the standard representation, and we let $N: \mathrm{GSpin}_{2n+1} \rightarrow \mathbb{G}_m$ denote the Clifford norm.

Theorem 1.1 (See Theorem 4.8). *Let p be a prime, $p \geq 2n + 4$, and let $\bar{r}: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying the following hypotheses:*

- *The restriction $\mathrm{std}(\bar{r})|_{\Gamma_{F^+(\mu_p)}}$ is irreducible.*
- *\bar{r} is odd (see Hypothesis 3.1).*

Then there exist a totally real Galois extension L^+/F^+ and a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$, satisfying the hypotheses (St) and (L-coh) of [26], such that a suitable GSpin_{2n+1} -conjugate of the representation $r_{\tilde{\pi},\iota}: \Gamma_{L^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ constructed by [26] reduces mod p to $\bar{r}|_{\Gamma_{L^+}}$.

We note that this theorem makes no “Steinberg” local hypothesis; the first step in its proof is to show (Theorem 3.4) that \bar{r} admits a geometric lift $r: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ that at some auxiliary finite place looks like the Langlands parameter of the Steinberg representation (up to twist). Theorem 3.4 relies on using the lifting method of Khare-Wintenberger ([23]) and a comparison of deformation rings for the groups GSpin_{2n+1} , SO_{2n+1} , and GL_{2n+1} , ultimately invoking a finiteness result from [3] that itself comes from combining $R = T$ theorems with the method of [23]. Having constructed this lift r , we can run the method of the proof of our p -adic potential automorphy theorem, which we now state. For details of the terminology, see §2 and [11].

Theorem 1.2 (Theorem 4.9). *Let $r: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ be a continuous representation whose Clifford norm is a geometric character $N(r): \Gamma_{F^+} \rightarrow \overline{\mathbb{Z}}_p^\times$. Assume the following:*

- $p > 2(2n + 1)$.
- \bar{r} is odd.
- For some finite place v_{St} of F^+ , the Frobenius semisimple Weil-Deligne representation associated to $r|_{\Gamma_{F^+_{v_{\mathrm{St}}}}}$ is equivalent to a twist of the Steinberg-type Langlands parameter (equivalently, $P(r|_{\Gamma_{F^+_{v_{\mathrm{St}}}}})$ is a Steinberg parameter for $\mathrm{Sp}_{2n}(F^+_{v_{\mathrm{St}}})$).
- There exist a quadratic CM extension F/F^+ and a character $\mu: \Gamma_{F^+} \rightarrow \overline{\mathbb{Z}}_p^\times$ such that
 - F does not contain ζ_p , and $\mathrm{std}(\bar{r})|_{\Gamma_{F(\zeta_p)}}$ is irreducible.
 - $(\mathrm{std}(r)|_{\Gamma_F}, \mu)$ is polarized, and for some (any) choice of prolongation

$$\rho(\mathrm{std}(\bar{r})|_{\Gamma_F}, \mu): \Gamma_{F^+} \rightarrow \mathcal{G}_{2n+1}(\overline{\mathbb{Z}}_p),$$

$$\rho(\mathrm{std}(r)|_{\Gamma_{F^+}}, \mu)$$
 is globally realizable for each $v|p$.

Then there exist a totally real Galois extension L^+/F^+ and a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ satisfying the hypotheses (St) and (L-coh) of [26] such that $r_{\tilde{\pi}, \iota}$ is equivalent to $r|_{\Gamma_{L^+}}$ as $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ -representations.

For instance, the hypothesis on places above p , which we should note forces the Hodge-Tate cocharacters of $r|_{\Gamma_{F^+_{v^+}}}$ to be regular, includes the cases where $\mathrm{std}(r)|_{\Gamma_{F^+_{v^+}}}$ is Hodge-Tate regular and potentially diagonalizable in the sense of [3]. In that case the theorem statement admits a straightforward modification with no need for an auxiliary extension F or mention of prolongations.

The strategy here is to combine the potential automorphy theorem of [11] with [2] to realize the projection $P(r): \Gamma_{F^+} \rightarrow \mathrm{SO}_{2n+1}(\overline{\mathbb{Z}}_p)$, after restriction to some L^+ , as the representation $r_{\pi, \iota}$ associated to a cuspidal automorphic representation of π of $\mathrm{Sp}_{2n}(\mathbb{A}_{L^+})$. Then we show that π can be extended to a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ that satisfies the hypotheses of the main theorem of [26], and finally we check that a suitable twist of $\tilde{\pi}$ in fact corresponds to our original $r|_{\Gamma_{L^+}}$.

In §5, we discuss an application of our potential automorphy theorem. Recall that for GL_n , a method due to Taylor ([38]) based on combining potential automorphy theorems with Brauer's theorem from finite group theory allows one to realize a representation $r: \Gamma_{F^+} \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, assumed to satisfy the hypotheses of the GL_n potential automorphy theorem, as part of a (strictly) compatible system of ℓ -adic Galois representations. We apply Theorem 4.9 to prove an analogous result for $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ -representations. This turns out to be the subtlest part of our work. Taylor's method relies on solvable descent, which depends on a combination of cyclic (prime degree) descent and knowledge that in the relevant cases the descended automorphic representations themselves have associated Galois representations. We establish instances of solvable descent for $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$, but a number of complications intervene, most essentially that non-trivial global L-packets on Sp_{2n} and GSp_{2n} mean that invariance of a Galois representation $r_{\tilde{\pi}, \iota}$ need not imply invariance of the automorphic representation $\tilde{\pi}$ (a fact further complicated by not knowing full local-global compatibility for the representations constructed by [26]). Theorem 5.3 is enough for our application; it in turn requires the relevant cases of cyclic prime degree descent, which are established in Proposition

5.2. The other difficulty in extending Taylor’s argument is that of course no Brauer argument can be made directly with GSpin_{2n+1} -valued Galois representations. The particular possibilities for the algebraic monodromy groups of the representations $r_{\tilde{\pi}, \iota}$ constructed by [26] make possible a case-by-case analysis, with the “generic” case of full GSpin_{2n+1} relying on applying Brauer’s theorem to $\mathrm{spin}(r)$. (That said, we emphasize that the potential automorphy theorem being applied is still for $r: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$, not for $\mathrm{spin}(r)$; indeed, the latter will typically not satisfy the hypotheses of GL_{2n} potential automorphy theorems.) Theorem 5.1 combines these ideas to construct the desired compatible system:

Theorem 1.3. *Let $r: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ satisfy the hypotheses of Theorem 1.2. Then for all primes ℓ and choices of isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ there is a continuous representation $r_{\iota_\ell}: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ such that:*

- *For all primes v at which r is unramified, the semisimple conjugacy class of $\iota^{-1}r(\mathrm{Fr}_v)^{\mathrm{ss}}$ agrees with that of $\iota_\ell^{-1}r_{\iota_\ell}(\mathrm{Fr}_v)^{\mathrm{ss}}$.*
- *For all primes $v|\ell$, r_{ι_ℓ} is de Rham, and its Hodge-Tate cocharacters are determined up to conjugacy by those of r : for each embedding $\tau: F^+ \rightarrow \overline{\mathbb{Q}}_\ell$, determining a place $v|\ell$ of F^+ , and inducing $\iota_\ell^{-1}\tau: F^+ \rightarrow \overline{\mathbb{Q}}_p$ and a corresponding place $\iota_\ell^*(v)|p$ of F^+ , we have the equality $\iota_\ell^{-1}\mu(r_{\iota_\ell}|_{\Gamma_{F_v^+}}, \tau) = \iota^{-1}\mu(r|_{\Gamma_{F_{\iota_\ell^*(v)}^+}}, \iota_\ell^{-1}\tau)$ of conjugacy classes of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$ -valued cocharacters.*

Finally, we remark on what we cannot prove at present. While our mod p potential automorphy result imposes no Steinberg local condition, the p -adic potential automorphy theorem does, and we do not know how to circumvent this. Certainly a corresponding improvement in the results of [26], constructing $r_{\tilde{\pi}, \iota}$ for all L-cohomological cuspidal automorphic representations of $\mathrm{GSp}_{2n}(\mathbb{A}_{F^+})$, would allow us to extend these results to that generality. Even with such an extension, generalizing the results of §5 would be a more serious task, requiring at the very least a more flexible cyclic prime degree descent result for GSp_{2n} .

Acknowledgments: S.P. was supported by NSF grants DMS-1700759 and DMS-1752313. S.P. thanks Wushi Goldring for enjoyable discussions related to the subject of this paper, and S.T. thanks Patrick Allen for many helpful conversations and for answering many of his questions.

2. NOTATION AND CONVENTIONS

Let F be a field. Fix an algebraic closure \overline{F} of F and write Γ_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F . If F is a number field, then for each place v of F , we fix an embedding $\overline{F} \rightarrow \overline{F}_v$ into an algebraic closure of F_v , which gives rise to an injective group homomorphism $\Gamma_{F_v} \rightarrow \Gamma_F$. For any finite place v , let k_v be the residue field of v and let $\mathrm{Fr}_v \in \Gamma_{k_v}$ be the arithmetic Frobenius. If H is a group (typically the points over a finite field or a p -adic field of a reductive algebraic group), and there is a continuous group homomorphism $r: \Gamma_F \rightarrow H$, we will sometimes write $r|_v$ for $r|_{\Gamma_{F_v}}$, the restriction of r to the decomposition group Γ_{F_v} . If H acts on a finite-dimensional vector space V , we write $r(V)$ for the Γ_F -module induced by precomposing this action with r . (Typically H will be a reductive algebraic group and V will be its Lie algebra equipped with the adjoint action of H .) Let $\kappa: \Gamma_F \rightarrow \overline{\mathbb{Z}}_p^\times$ be the

p -adic cyclotomic character and \bar{k} be its reduction modulo p . We will always assume $p \neq 2$, and our main theorems will make stronger hypotheses on p .

Let $n \geq 1$ be an integer, let SO_{2n+1} be the odd orthogonal group of rank n , which for convenience we will take to be defined by $\mathrm{SO}_{2n+1}(R) = \{g \in \mathrm{SL}_{2n+1}(R) : {}^t g \cdot g = 1\}$ for any ring R . In fact, we will always be free to enlarge the ring of definition and consider this group to be defined over the ring of integers \mathcal{O} in a sufficiently large finite extension of \mathbb{Q}_p , and in particular we will assume this is in fact the split form of the group. Let GSpin_{2n+1} be the corresponding general spin group, so there is an exact sequence of algebraic groups

$$1 \rightarrow Z \rightarrow \mathrm{GSpin}_{2n+1} \xrightarrow{P} \mathrm{SO}_{2n+1} \rightarrow 1$$

where $Z \cong \mathbb{G}_m$ is the center of GSpin_{2n+1} . Let $N : \mathrm{GSpin}_{2n+1} \rightarrow \mathbb{G}_m$ be the Clifford norm. Note that $\mathrm{GSpin}_{2n+1}(\mathbb{C})$ is the Langlands dual of GSp_{2n} . Let $\mathrm{std} : \mathrm{GSpin}_{2n+1} \rightarrow \mathrm{GL}_{2n+1}$ be the composition of P with the standard representation of SO_{2n+1} . Let $\mathrm{spin} : \mathrm{GSpin}_{2n+1} \rightarrow \mathrm{GL}_{2n}$ be the spin representation. For a homomorphism $r : \Gamma \rightarrow \mathrm{GSpin}_{2n+1}$, we will frequently consider the composites $P(r) : \Gamma \rightarrow \mathrm{SO}_{2n+1}$ and $\mathrm{std}(r) : \Gamma \rightarrow \mathrm{GL}_{2n+1}$.

We recall the definition of the Clozel-Harris-Taylor group scheme \mathcal{G}_n over \mathbb{Z} which is defined as the semidirect product $(\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \{1, j\}$ where $j(g, a)j = (a({}^t g)^{-1}, a)$, and the similitude character $\nu : \mathcal{G}_n \rightarrow \mathrm{GL}_1$ given by $\nu(g, a) = a$ and $\nu(j) = -1$. The groups \mathcal{G}_n will arise in the following setting. Suppose we have a homomorphism $r : \Gamma_F \rightarrow \mathrm{GSpin}_{2n+1}(R)$ for some field F and some ring R . Let F'/F be a quadratic extension of F , and define $\rho(r) : \Gamma_F \rightarrow \mathcal{G}_{2n+1}(R)$ as the composite

$$\Gamma_F \xrightarrow{P(r) \times \mathrm{res}_{F'}} \mathrm{SO}_{2n+1}(R) \times \mathrm{Gal}(F'/F) \rightarrow \mathcal{G}_{2n+1}(R),$$

where the last map sends $g \in \mathrm{SO}_{2n+1}(R)$ to its image in $\mathrm{GL}_{2n+1}(R)$ and sends the nontrivial element of $\mathrm{Gal}(F'/F)$ to j . By our choice of definition of $\mathrm{SO}_{2n+1}(R)$, $\rho(r)$ is a well-defined homomorphism. We will refer to $\rho(r)$ as the standard prolongation of $\mathrm{std}(r)$ with respect to the extension F'/F (compare [11, §1.4]).

Let T be a maximal torus of a reductive group G over an algebraically closed field. A cocharacter $\mu \in X_*(T)$ is said to be regular if $\langle \mu, \alpha \rangle \neq 0$ for all roots α of T in G .

We assume that the reader is familiar with Galois deformation theory and will use the standard terms and results of Galois deformation theory freely throughout this paper. The reader may refer to [3, Section 1] and [5] for the details. We recall here some deformation-theoretic terminology. Given a topologically finite-generated profinite group Γ , a finite extension E/\mathbb{Q}_p with ring of integers \mathcal{O} and residue field k , a reductive algebraic group G defined over \mathcal{O} and a continuous homomorphism $\bar{r} : \Gamma \rightarrow G(k)$, let $R_{\mathcal{O}, \bar{r}}^{\square}$ be the universal lifting ring representing the functor sending a complete local noetherian \mathcal{O} -algebra R with residue field k to the set of lifts $r : \Gamma \rightarrow G(R)$ of \bar{r} . We will always leave the \mathcal{O} implicit, writing only $R_{\bar{r}}^{\square}$, and at various points in the argument we enlarge \mathcal{O} ; see [3, Lemma 1.2.1] for a justification of (the harmlessness of) this practice. We write $R_{\bar{r}}^{\square} \otimes \overline{\mathbb{Q}}_p$ for $R_{\mathcal{O}, \bar{r}}^{\square} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$ for any particular choice of \mathcal{O} , and again by [3, Lemma 1.2.1], $R_{\bar{r}}^{\square} \otimes \overline{\mathbb{Q}}_p$ is independent of the choice of E .

Suppose moreover that $\chi : G \rightarrow A$ is a morphism to an abelian group scheme over \mathcal{O} , and $\theta : \Gamma \rightarrow \mathcal{O}^{\times}$ is a continuous character whose mod p reduction equals $\chi(\bar{r}) : \Gamma \rightarrow k^{\times}$. Then we write $R_{\bar{r}}^{\square, \theta}$ (again, leaving \mathcal{O} implicit) for the quotient of $R_{\bar{r}}^{\square}$ representing the functor sending a complete local noetherian \mathcal{O} -algebra R with residue field k to the set of liftings

$r : \Gamma \rightarrow G(R)$ of \bar{r} with fixed similitude character θ , i.e. $\chi(r) = \theta$. We will apply this convention when the character χ is the Clifford norm $N : \mathrm{GSpin}_{2n+1} \rightarrow \mathbb{G}_m$, and when it is the character $\nu : \mathcal{G}_{2n+1} \rightarrow \mathbb{G}_m$ defined above.

When K/\mathbb{Q}_p is a finite extension and Γ is Γ_K , we moreover consider quotients of $R_{\bar{r}}^{\square, \theta}$ having fixed inertial type and p -adic Hodge type. The fundamental analysis here is due to Kisin ([24]), and the state of the art, and our point of reference, is [5], and we refer there for details. We will index p -adic Hodge types of deformations $r : \Gamma_K \rightarrow G(\mathcal{O})$ by collections $\underline{\mu}(r) = \{\mu(r, \tau)\}_{\tau : K \rightarrow \overline{\mathbb{Q}_p}}$ of (conjugacy classes of) Hodge-Tate co-characters, and write $R_{\bar{r}}^{\square, \theta, \underline{\mu}(r)}$ for the \mathbb{Z}_p -flat quotient of $R_{\bar{r}}^{\square, \theta}$ whose points in finite local E -algebras are precisely those of $R_{\bar{r}}^{\square, \theta}$ that are moreover potentially semi-stable with p -adic Hodge type $\underline{\mu}(r)$. We likewise consider the quotients with fixed inertial type σ , $R_{\bar{r}}^{\square, \theta, \underline{\mu}(r), \sigma}$, referring to [5, §3.2] for details.

We fix, *once and for all*, a field isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$. This will be used in associating p -adic Galois representations to automorphic representations. We will typically note the use of this isomorphism explicitly, with one exception: for type A_0 Hecke characters $\chi : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times$ of a number field F , we will continue to write $\chi : \Gamma_F \rightarrow \overline{\mathbb{Q}_p}^\times$ for the p -adic Galois character associated to χ (and similarly for characters of localizations F_v^\times). Thus, we leave the isomorphism ι implicit and the global (or local) reciprocity map implicit; we have done this to simplify the notation, since we do not expect it to cause any confusion. These local and global reciprocity maps are normalized as in [26], which is to say geometric Frobenii correspond to uniformizers.

3. A LIFTING THEOREM

We prove a geometric lifting theorem for $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{F}_p})$ -valued representations analogous to [3, Theorem 4.3.1] or [11, Theorem 4.2.11]. We will use a special case of this theorem in §4 as a first step in establishing our mod p potential automorphy theorem. In this section we will prove something rather more general than what is needed for §4, although we have not attempted to optimize the results. The lifting theorem has global hypotheses—that the mod p representation be odd and irreducible, essentially—and then a local hypothesis on the existence of suitable de Rham lifts at places above p . A similar theorem, with less restrictive local hypotheses, follows from [16, Theorem A], but the methods of that paper cannot yield the minimal lifts produced here via potential automorphy (nor of course can it show they are potentially automorphic).

We will freely make use of the definition of “globally realizable” components of potentially semistable local deformation rings from [11, Definition 2.1.9], and we refer the reader there for details. Let K^+ be a totally real field, and let $\bar{r} : \Gamma_{K^+} \rightarrow \mathrm{GSpin}_{2n+1}(k)$ be a continuous representation, where k is the residue field of the ring of integers \mathcal{O} in a finite extension E of \mathbb{Q}_p (which we will feel free to enlarge without comment). Fix a geometric lift $\theta : \Gamma_{K^+} \rightarrow \overline{\mathbb{Z}_p}^\times$ of $N \circ \bar{r} : \Gamma_{K^+} \rightarrow \overline{\mathbb{F}_p}^\times$.

Hypothesis 3.1. Assume that \bar{r} satisfies the following hypotheses. There exists a CM quadratic extension K/K^+ , not containing ζ_p , such that

- (1) The restriction $\mathrm{std}(\bar{r})|_{\Gamma_{K(\mu_p)}}$ is irreducible.²
- (2) The prime p satisfies $p > 2(2n + 1)$.
- (3) The representation \bar{r} is odd, meaning that for all infinite places v of K^+ , with corresponding conjugacy class of complex conjugations $c_v \in \Gamma_{K^+}$, $\bar{r}(c_v)$ is a Chevalley involution in GSpin_{2n+1} , i.e. $\dim \mathfrak{g}^{\mathrm{Ad}(\bar{r}(c_v))=1} = (\dim \mathfrak{g})/(\dim \mathfrak{b})$, where $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ is the commutator subalgebra of the Lie algebra of GSpin_{2n+1} , equipped with an action of Γ_{K^+} induced by \bar{r} and the adjoint representation, and $\mathfrak{b} \subset \mathfrak{g}$ is any Borel subalgebra of \mathfrak{g} .
- (4) For each $v|p$, there is a lift $r_v : \Gamma_{K_v^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ of $\bar{r}|_{K_v^+}$ such that

- $N \circ r_v = \theta|_{\Gamma_{K_v^+}}$.
- r_v is de Rham, with some p -adic Hodge type $\underline{\mu}(r_v)$ and some inertial type σ_v , and there is an irreducible component $\mathcal{C}(r_v)$ of $R_{\bar{r}}^{\square, \theta, \underline{\mu}(r_v), \sigma_v}[1/p]$ containing r_v that, under the map of lifting rings

$$\mathrm{Spec} R_{\bar{r}}^{\square, \theta, \underline{\mu}(r_v), \sigma_v} \rightarrow \mathrm{Spec} R_{\rho(\bar{r})}^{\square, \delta_{K/K^+}, \underline{\mu}(\rho(r_v)), \rho(\sigma_v)}$$

induced by the standard prolongation (with respect to K/K^+), maps to a globally realizable component $\mathcal{C}(\rho(r_v))$.

- In particular, the previous item implies that, for each embedding $\tau : K_v^+ \rightarrow \overline{\mathbb{Q}}_p$, the Hodge-Tate cocharacters $\mu(\mathrm{std}(r_v), \tau)$ are regular, and consequently the Hodge-Tate cocharacters $\mu(r_v, \tau)$ are regular.

Remark 3.2. If we were basing our arguments on the less general results of [3], then here we would instead assume $\mathrm{std}(r_v)$ is regular and potentially diagonalizable in the sense of [3, §1.4]. We would then have a version of the above hypotheses without mention of the extension K , and we would choose K not containing ζ_p such that $\mathrm{std}(\bar{r})|_{\Gamma_{K(\mu_p)}}$ remained irreducible, and such that all finite primes in S split in K/K^+ . We would also only need to assume $p \geq 2n + 4$, in order to apply the theorems of [3] and Lemma 3.5. Then we would obtain a version of Theorem 3.4 below by invoking [3, Theorem 4.3.1] instead of [11] in the arguments below. To see that this potentially diagonalizable case is contained in our current arguments, we would choose $\mathcal{C}(r_v)$ to be a potentially crystalline component (if $\mathrm{std}(r_v)$ is potentially crystalline, then so is r_v , but r_v could conceivably lie at an intersection of crystalline and semi-stable components), mapping into a potentially crystalline component containing the potentially diagonalizable point $\rho(r_v)$. Any potentially diagonalizable point lies on some globally realizable component ([11, Remark 2.1.14]), which (by the argument given to justify this assertion) is moreover a potentially crystalline component. Since potentially crystalline deformation rings are regular, there are no other potentially crystalline components passing through $\rho(r_v)$. Hence, whatever potentially crystalline component $\mathcal{C}(r_v)$ we chose must map to a globally realizable component $\mathcal{C}(\rho(r_v))$, namely the unique potentially crystalline component passing through $\rho(r_v)$.

²We note that this is strictly stronger than assuming $\bar{r}|_{\Gamma_{K(\mu_p)}}$ is irreducible. Compare [17].

Let S be a finite set of primes of F^+ containing the primes above p and all primes where \bar{r} or θ is ramified. To complement our assumption on the existence of r_v for $v|p$, we have the following lemma:

Lemma 3.3. *For all places $v \in S$ not above p , there is a lift $r_v: \Gamma_{K_v^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ of $\bar{r}|_{\Gamma_{K_v^+}}$ having $N(r_v) = \theta|_{\Gamma_{K_v^+}}$.*

Proof. Since $p \neq 2$, the natural map of lifting rings $R_{P(\bar{r})|_{\Gamma_{K_v^+}}}^{\square} \rightarrow R_{\bar{r}|_{\Gamma_{K_v^+}}}^{\square, \theta}$ is an isomorphism (see Lemma 3.5 for more details), so it suffices to find a lift of $P(\bar{r}|_{\Gamma_{K_v^+}})$. Such a lift exists by Booher's thesis ([6]). \square

From now on, we will fix some such lift r_v . We reiterate that we let \mathcal{O} be the ring of integers in a finite extension E of \mathbb{Q}_p , enlarged if necessary so that all of the above data are defined over \mathcal{O} . We now state the main result of this section:

Theorem 3.4. *Retain the assumptions of Hypothesis 3.1. Let S be a finite set of places of K^+ such that S contains all places above p and all places where θ or \bar{r} is ramified. Fix lifts r_v and components $\mathcal{C}(r_v)$ for $v|p$ as in Hypothesis 3.1, and fix lifts r_v for $v \in S \setminus \{v|p\}$ as in Lemma 3.3. Then \bar{r} has a lift $r: \Gamma_{K^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ unramified outside S such that*

- $N(r) = \theta$.
- For each $v \in S$ not lying above p , $r|_v := r|_{\Gamma_{K_v^+}}$ and r_v lie on the same irreducible component of $\mathrm{Spec}(R_{\bar{r}|_v}^{\square, \theta} \otimes \overline{\mathbb{Q}}_p)$.
- For each $v|p$, $r|_v$ lies on $\mathcal{C}(r_v)$, and in particular is potentially semistable with the given Hodge-Tate cocharacters and inertial types.

The rest of this section will be dedicated to the proof of Theorem 3.4, using a version of the lifting argument of Khare-Wintenberger ([23]). We will define global deformation conditions for \bar{r} , $P(\bar{r})$, and the standard prolongation $\rho(\bar{r})$ with respect to K/K^+ . Recall that $\rho(\bar{r}): \Gamma_{K^+} \rightarrow \mathcal{G}_{2n+1}(\overline{\mathbb{F}}_p)$ is the composite

$$\Gamma_{K^+} \xrightarrow{P(\bar{r}) \times \mathrm{res}_K} \mathrm{SO}_{2n+1}(\overline{\mathbb{F}}_p) \times \mathrm{Gal}(K/K^+) \rightarrow \mathcal{G}_{2n+1}(\overline{\mathbb{F}}_p).$$

Then we will show that there are natural finite maps between the corresponding deformation rings and use the \mathbb{Z}_p -finiteness of the deformation ring for the group \mathcal{G}_{2n+1} to conclude that the deformation ring for the group GSpin_{2n+1} is \mathbb{Z}_p -finite; this, combined with a standard calculation of its Krull dimension (as in [5]), will imply that it has a $\overline{\mathbb{Q}}_p$ -point.

Recall from the discussion in §2 that for each $v \in S$ we can consider the lifting rings $R_{\bar{r}|_v}^{\square, \theta}$, $R_{P(\bar{r})|_v}^{\square}$, and $R_{\rho(\bar{r})|_v}^{\square, \delta_{K/K^+}}$. For $v \in S$ not above p , choose an irreducible component $\mathcal{C}(r_v)$ (resp. $\mathcal{C}(P(r_v))$, $\mathcal{C}(\rho(r_v))$) of $R_{\bar{r}|_v}^{\square, \theta} \otimes \overline{\mathbb{Q}}_p$ (resp. $R_{P(\bar{r})|_v}^{\square} \otimes \overline{\mathbb{Q}}_p$, $R_{\rho(\bar{r})|_v}^{\square, \delta_{K/K^+}} \otimes \overline{\mathbb{Q}}_p$) containing r_v (resp. $P(r_v)$, $\rho(r_v)$) such that under the natural maps

$$\mathrm{Spec} R_{\bar{r}|_v}^{\square, \theta} \otimes \overline{\mathbb{Q}}_p \rightarrow \mathrm{Spec} R_{P(\bar{r})|_v}^{\square} \otimes \overline{\mathbb{Q}}_p \rightarrow \mathrm{Spec} R_{\rho(\bar{r})|_v}^{\square, \delta_{K/K^+}} \otimes \overline{\mathbb{Q}}_p,$$

$\mathcal{C}(r_v)$ maps to $\mathcal{C}(P(r_v))$ and $\mathcal{C}(P(r_v))$ maps to $\mathcal{C}(\rho(r_v))$.

Similarly, for $v|p$, the fixed inertial type and fixed p -adic Hodge type of r_v induces corresponding data for $P(r_v)$ and $\rho(r_v)$, and the fixed component $\mathcal{C}(r_v)$ maps to components $\mathcal{C}(P(r_v))$ of $R_{P(\bar{r})|_v}^{\square, \underline{\mu}(P(r_v)), P(\sigma_v)} \otimes \overline{\mathbb{Q}}_p$ and $\mathcal{C}(\rho(r_v))$ of $R_{\rho(\bar{r})|_v}^{\square, \delta_{K/K^+}, \underline{\mu}(\rho(r_v)), \rho(\sigma_v)} \otimes \overline{\mathbb{Q}}_p$ (the latter as in the theorem hypotheses).

We now define three global deformation rings, for \bar{r} , $P(\bar{r})$, and $\rho(\bar{r})$, by considering lifts that locally lie on the irreducible components we have just specified. More precisely, following the formalism of [5, §4.2], we let $R_{\mathrm{GSpin}}^{\mathrm{univ}}$ be the quotient of the universal, unramified outside S , fixed Clifford norm θ , deformation ring for \bar{r} corresponding to the fixed set of components $\{\mathcal{C}(r_v)\}_{v \in S}$. We similarly define $R_{\mathrm{SO}}^{\mathrm{univ}}$, corresponding to the local components $\{\mathcal{C}(P(r_v))\}$ and $R_{\mathrm{GL}}^{\mathrm{univ}}$ corresponding to the local components $\{\mathcal{C}(\rho(r_v))\}$ (and fixed polarization δ_{K/K^+}). These rings all exist by absolute irreducibility of the respective residual representations, and by the discussion in [5, §4.2] ([5, Lemma 3.4.1] plays a key role here). By construction, there are natural \mathcal{O} -algebra maps

$$R_{\mathrm{GL}}^{\mathrm{univ}} \rightarrow R_{\mathrm{SO}}^{\mathrm{univ}} \rightarrow R_{\mathrm{GSpin}}^{\mathrm{univ}}.$$

Lemma 3.5. *The map $R_{\mathrm{GL}}^{\mathrm{univ}} \rightarrow R_{\mathrm{SO}}^{\mathrm{univ}}$ is surjective, and the map $R_{\mathrm{SO}}^{\mathrm{univ}} \rightarrow R_{\mathrm{GSpin}}^{\mathrm{univ}}$ is an isomorphism.*

Proof. The tangent space of $R_{\mathrm{GL}}^{\mathrm{univ}}$ is a subspace of $H^1(\Gamma_{K^+, S}, \rho(\bar{r})(\mathfrak{gl}_{2n+1}))$ and the tangent space of $R_{\mathrm{SO}}^{\mathrm{univ}}$ is a subspace of $H^1(\Gamma_{K^+, S}, P(\bar{r})(\mathfrak{so}_{2n+1}))$. For $p \geq 2n + 4$, \mathfrak{gl}_{2n+1} is by [35, Proposition 2] a semisimple SO_{2n+1} -module, so *a fortiori* $P(\bar{r})(\mathfrak{so}_{2n+1})$ is a $\Gamma_{K^+, S}$ -direct summand of $P(\bar{r})(\mathfrak{gl}_{2n+1})$. It follows that the natural map $H^1(\Gamma_{K^+, S}, P(\bar{r})(\mathfrak{so}_{2n+1})) \rightarrow H^1(\Gamma_{K^+, S}, \rho(\bar{r})(\mathfrak{gl}_{2n+1}))$ is injective; the dual map is surjective, and we conclude by Nakayama's lemma that the map on universal deformation rings without local conditions is surjective. It then follows immediately that $R_{\mathrm{GL}}^{\mathrm{univ}} \rightarrow R_{\mathrm{SO}}^{\mathrm{univ}}$ is surjective as well.

The canonical map $R_{\mathrm{SO}}^{\mathrm{univ}} \rightarrow R_{\mathrm{GSpin}}^{\mathrm{univ}}$ is induced by a corresponding map on global deformation rings without local conditions; the latter is an isomorphism because it induces isomorphisms $H^i(\Gamma_{K^+, S}, \bar{r}(\mathfrak{so}_{2n+1})) \xrightarrow{\sim} H^i(\Gamma_{K^+, S}, P(\bar{r})(\mathfrak{so}_{2n+1}))$ (one of these copies of \mathfrak{so}_{2n+1} is the derived Lie algebra of GSpin_{2n+1} , while the other is the Lie algebra of SO_{2n+1} ; the natural map between them is an isomorphism since p is not 2), which for $i = 1$ and $i = 2$ means that it identifies both the tangent spaces and obstruction spaces for the two deformation functors. The local lifting rings without local conditions are similarly isomorphic, and this implies that the specified components of the local lifting rings are isomorphic. That $R_{\mathrm{SO}}^{\mathrm{univ}} \rightarrow R_{\mathrm{GSpin}}^{\mathrm{univ}}$ is an isomorphism now follows. \square

We now complete the proof of Theorem 3.4:

Proof of Theorem 3.4. By our assumptions on \bar{r} , the representation $\rho(\bar{r})$ (strictly speaking, the pair $(\mathrm{std}(\bar{r})|_{\Gamma_K}, \delta_{K/K^+})$ with its prolongation $\rho(\bar{r})$) satisfies the assumptions of [11, Theorem 4.2.11], so $(\mathrm{std}(\bar{r})|_{\Gamma_K}, \delta_{K/K^+})$ lifts to a potentially automorphic compatible system $(\{s_\lambda\}_\lambda, \delta_{K/K^+})$ (here λ is indexed over places of a suitable number field, or over field isomorphisms $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_l$). The method of proof of *loc. cit.* implies that the deformation ring $R_{\mathrm{GL}}^{\mathrm{univ}}$ is \mathcal{O} -finite, since that is the means by which lifts are shown to exist. Not to impose on the reader to have to inspect the proof, we can simply take the conclusion of *loc. cit.* and then invoke [11, Lemma 1.4.28].

Lemma 3.5 then implies that $R_{\mathrm{GSpin}}^{\mathrm{univ}}$ is \mathcal{O} -finite. We claim that $R_{\mathrm{GSpin}}^{\mathrm{univ}}$ has Krull dimension at least one. Indeed, this follows from [5, Theorem B]: the assumptions there are satisfied since we have assumed that \bar{r} is odd, $\mathrm{std}(\bar{r})|_{K(\mu_p)}$ is irreducible (which under our assumption on p implies that $H^0(\Gamma_{K^+}, \bar{r}(\mathfrak{so}_{2n+1})(1))$ vanishes), and the Hodge-Tate cocharacters are regular. Thus $R_{\mathrm{GSpin}}^{\mathrm{univ}}$ has a $\overline{\mathbb{Q}}_p$ -point, which proves our theorem. \square

4. POTENTIAL AUTOMORPHY OF MOD p AND p -ADIC REPRESENTATIONS

In this section we will prove two potential automorphy theorems, one for $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{F}}_p)$ -valued representations, and one for $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ -valued representations. For the first we will begin by applying a version of Theorem 3.4, but in fact we can and do get away with a somewhat simpler result, relying on [3] rather than the refinement in [11]. Our p -adic potential automorphy theorem will however require [11] for its most general statement.

4.1. The mod p case. Let F^+ be a totally real field, and let $\bar{r} : \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{F}}_p)$ be a continuous mod p representation satisfying the following slight relaxation of the conditions (1), (2), and (3) of Hypothesis 3.1 (we replace K^+ by F^+ since eventually we will apply the result of §3 to some finite extension K^+ of this F^+):

Hypothesis 4.1. Assume that \bar{r} satisfies the following:

- (1) $\mathrm{std}(\bar{r})|_{\Gamma_{F^+(\mu_p)}}$ is absolutely irreducible.
- (2) $p \geq 2n + 4$.³
- (3) \bar{r} is odd.

In essence, we retain the irreducibility and oddness hypotheses from §3, but not those on the existence of sufficiently nice local lifts. In this section, we use the results in [3] and [26] to prove that \bar{r} is potentially automorphic in the sense that over a finite totally real extension of F^+ , \bar{r} is the mod p reduction of the Galois representation attached in [26] to a suitable automorphic representation of GSp_{2n} .

Let S be a finite set of finite places of F^+ including the places above p , the places where \bar{r} is ramified, and one other auxiliary place v_{St} . Choose a CM quadratic extension F of F^+ that is linearly disjoint from $F^+(\bar{r}, \mu_p)$, and in which all primes in S split. For each $v \in S$, let E_v/F_v^+ be a finite extension over which $\bar{r}|_v$ becomes trivial. We may and do also assume that when $v = v_{\mathrm{St}}$ the residue field of E_v has order q^f where f is even (this is inessential but convenient). By [14, Lemma 4.1.2], there is a finite totally real extension K^+/F^+ linearly disjoint from $F(\bar{r}, \mu_p)$ such that for each place v of S and each place w of K^+ above v , the extension E_v/F_v^+ is isomorphic to K_w^+/F_v^+ . Set $K = FK^+$. Continuing to write S for the finite places of K^+ above S , we have the following result:

³The potential automorphy theorem [3, Theorem 4.3.1] we rely on needs the following hypothesis, somewhat weaker than the assumption in [11]: let d be the maximal dimension of an irreducible subrepresentation of the restriction of $\mathrm{std}(\bar{r})$ to the closed subgroup of Γ_{F^+} generated by all Sylow pro- p -subgroups, and assume that $p \geq 2(d+1)$. In Lemma 3.5 above we use the slightly stricter requirement $p \geq 2n+4$.

Proposition 4.2. *There is a choice of lift $\theta: \Gamma_{K^+} \rightarrow \overline{\mathbb{Z}}_p^\times$ of $N \circ \bar{r}|_{\Gamma_{K^+}}$ and a lift $r: \Gamma_{K^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ of $\bar{r}|_{\Gamma_{K^+}}$ with $N \circ r = \theta$ such that*

- *For all places $v|v_{\mathrm{St}}$, the point corresponding to $r|_{\Gamma_{K_v^+}}$ lies on the same connected component of $R_{\bar{r}|_v}^{\square, \theta}$ as a representation whose associated Weil-Deligne representation is an unramified twist of the local Langlands parameter of the Steinberg representation.⁴*
- *For all $v|p$, the restriction $r|_{\Gamma_{K_v^+}}$ is potentially crystalline with regular Hodge-Tate cocharacters, and $\mathrm{std}(r|_{\Gamma_{K_v^+}})$ is potentially diagonalizable.*

Proof. The hypotheses of Theorem 3.4 are clear except for the local conditions at finite primes. Note that by the choice of K^+ , $\bar{r}|_{\Gamma_{K^+}}$ is everywhere unramified and is trivial at the places in S . Consider a place $v|p$ of K^+ . Choose a set of integers $\{s_1, s_2, \dots, s_n\}$ such that for all $i \neq j$, we have $s_i \neq \pm s_j$, and all s_i are non-zero. Let T be a maximal torus of GSpin_{2n+1} (say, the preimage of the diagonal torus in SO_{2n+1}), so that we can write a based root datum for GSpin (with respect to T and a suitable choice of Borel; see, e.g., [34, §2.8]) as

$$\begin{aligned} X^\bullet(T) &= \bigoplus_{i=1}^n \mathbb{Z}\chi_i \oplus \mathbb{Z}(\chi_0 + \frac{\sum_{i=1}^n \chi_i}{2}) \subset \bigoplus_{i=0}^n \mathbb{Q}\chi_i \\ \Delta^\bullet &= \{\chi_i - \chi_{i+1}\}_{i=1}^{n-1} \cup \{\chi_n\} \\ X_\bullet(T) &= \bigoplus_{i=1}^n \mathbb{Z}(\lambda_i + \frac{\lambda_0}{2}) \oplus \mathbb{Z}\lambda_0 \\ \Delta_\bullet &= \{\lambda_i - \lambda_{i+1}\}_{i=1}^{n-1} \cup \{2\lambda_n\}, \end{aligned}$$

where the pairing is given by $\langle \chi_i, \lambda_j \rangle = \delta_{i,j}$ for all i, j .

Now define a $T(\mathbb{Z}_p)$ -valued lift $r_v: \Gamma_{K_v^+} \rightarrow \mathrm{GSpin}_{2n+1}(\mathbb{Z}_p)$ of $\bar{r}|_v$ by

$$r_v = \prod_{i=1}^n (\lambda_i + \frac{\lambda_0}{2}) (\kappa^{s_i} \cdot [\bar{\kappa}]^{-s_i})$$

where $[\bar{\kappa}]$ denotes the Teichmüller lift of $\bar{\kappa}$ (as $v|p$ varies, we may take the same integers s_i ; but what is essential is that $s = \sum_{i=1}^n s_i$ be independent of $v|p$). Then we see easily that r_v lifts \bar{r}_v , is potentially crystalline with regular Hodge-Tate cocharacters, $\mathrm{std}(r_v)$ is potentially diagonalizable, and $N(r_v) = (\kappa \cdot [\bar{\kappa}]^{-1})^{\sum_{i=1}^n s_i}$. Set $s = \sum_{i=1}^n s_i$. We define $\theta: \Gamma_{K^+} \rightarrow \mathcal{O}^\times$ by $\theta = \kappa^s \cdot [\kappa]^{-s} \cdot [N(\bar{r})]$. In particular, $N(r_v) = \theta|_{\Gamma_{K_v^+}}$ for all $v|p$.

Now for each $v|v_{\mathrm{St}}$ we define a lift r_v of $\bar{r}|_{\Gamma_{K_v^+}}$ (which is trivial) with the desired properties, including now that the norm be the character θ just defined. Namely, let σ_v be an element of $\mathrm{Gal}(K_v^{+, \mathrm{tame}}/K_v^+)$ that maps down to the arithmetic Frobenius in $\mathrm{Gal}(K_v^{+, \mathrm{unr}}/K_v^+)$, and let τ_v be a topological generator of the pro-cyclic group $\mathrm{Gal}(K_v^{+, \mathrm{tame}}/K_v^{+, \mathrm{unr}})$ (so we have $\sigma_v \tau_v \sigma_v^{-1} = \tau_v^{q^f}$). Define $\rho_v: \mathrm{Gal}(K_v^{+, \mathrm{tame}}/K_v^+) \rightarrow \mathrm{SL}_2(\mathcal{O})$ by $\rho_v(\sigma_v) = \begin{pmatrix} q^{f/2} & 0 \\ 0 & q^{-f/2} \end{pmatrix}$ (recall

⁴By this we just mean the principal parameter in the sense of [21, §6]; it is the expected local Langlands parameter of the Steinberg representation.

that f is even), and $\rho_v(\tau_v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We initially set $r_{v,0} = \varphi \circ \rho_v$, where $\varphi: \mathrm{SL}_2 \rightarrow \mathrm{Spin}_{2n+1}$ is a principal SL_2 (for instance, with respect to a choice of pinning for our based root datum for GSpin_{2n+1}). Then $r_{v,0}$ is a Steinberg parameter with $N(r_{v,0}) = 1$, and it remains to introduce a twist to get the correct Clifford norm. Since $\theta|_{\Gamma_{K_v^+}}$ is unramified with

$$\theta(\mathrm{Fr}_v) = q_v^s [q_v^{-s}] \equiv 1 \pmod{p},$$

there exists an unramified character $\alpha_v: \Gamma_{K_v^+} \rightarrow \mathbb{Z}_p^\times$ such that $\alpha_v(\mathrm{Fr}_v)^2 = \theta(\mathrm{Fr}_v)$. We then define r_v to be $r_{v,0} \cdot \alpha_v$, so that now $N(r_v) = N(r_{v,0}) \cdot \alpha_v^2 = \theta|_{\Gamma_{K_v^+}}$. A straightforward computation shows that r_v is robustly smooth in the sense of [5] (or “very smooth” in the sense of [3]).

The lifts r_v we have just defined for $v|p$ and $v|v_{\mathrm{St}}$ satisfy the desired local conditions and have Clifford norm equal to the restriction of θ . For other places $v \in S$, recalling that $\bar{r}|_{\Gamma_{K_v^+}}$ is trivial, we can simply let r_v be an unramified twist (as in the Steinberg case, to get the correct central character) of the trivial representation.⁵ Thus, we can apply Theorem 3.4 (and in fact its simpler variant based on Remark 3.2) to deduce the existence of a lift $r: \Gamma_{K^+,S} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ of $\bar{r}|_{\Gamma_{K^+}}$ such that for all $v \in S$, $r|_{\Gamma_{K_v^+}}$ lies on the same irreducible component of $\mathrm{Spec}(R_{\bar{r}|_v}^{\square,\theta} \otimes \overline{\mathbb{Q}}_p)$ as r_v , and for $v|p$ also has the same p -adic Hodge type as r_v . The proof of the proposition is complete. \square

For each $v|p$ and each embedding $\tau: K_v^+ \rightarrow \overline{\mathbb{Q}}_p$, we let $\mu(r|_v, \tau) = \mu(r_v, \tau): \mathbb{G}_m \rightarrow \mathrm{GSpin}_{2n+1}$ be the Hodge-Tate cocharacter (properly, a conjugacy class of cocharacters) of $r|_v$, or equivalently of r_v .

We will now apply the potential automorphy theorem of [3] to the representation $\mathrm{std}(r): \Gamma_{K^+} \rightarrow \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_p)$. We begin with a remark on the normalization of the construction of automorphic Galois representations, which is summarized in [3, Theorem 2.1.1] and proven by work of many people, with the result we need contained in [37, Theorem 1.2, Remark 7.6]. These references imply that for every regular L-algebraic self-dual cuspidal automorphic representation Π of $\mathrm{GL}_{2n+1}(\mathbb{A}_{K^+})$, there is an associated $r_{\Pi,\iota}: \Gamma_{K^+} \rightarrow \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_p)$ such that for all v not above p , the Weil-Deligne representation associated to $r_{\Pi,\iota}|_{\Gamma_{K_v^+}}$ is, up to Frobenius semisimplification, isomorphic to the local Langlands parameter (in the standard unitary normalization, not the “arithmetic” normalization) of Π_v . Moreover, for $v|p$, $r_{\Pi,\iota}|_{\Gamma_{K_v^+}}$ is de Rham (and crystalline if Π_v is unramified) with labeled Hodge-Tate weights read off from Π_∞ according to the usual recipe, which we will now outline. If M is a number field, G is a connected reductive group over M , and π is an automorphic representation of $G(\mathbb{A}_M)$, then for each choice of field embedding $\epsilon: M \rightarrow \mathbb{C}$, inducing an infinite place of M (that we will also denote by ϵ) and an identification $\overline{M}_\epsilon = \mathbb{C}$, we consider the restriction $\phi_{\pi_\epsilon}: W_{\overline{M}_\epsilon} \rightarrow {}^L G(\mathbb{C})$ of the archimedean L-parameter. Precisely when π_ϵ is L-algebraic, this homomorphism is induced by a pair of algebraic cocharacters, $z \mapsto z^{\mu(\pi_\epsilon, \epsilon)} \bar{z}^{\nu(\pi_\epsilon, \epsilon)}$ of a maximal torus inside the dual group G^\vee , and we thus associate to the pair (π, ϵ) the G^\vee -conjugacy class of cocharacters $\mu(\pi_\epsilon, \epsilon)$. A conjectural Galois representation $r_{\pi,\iota}$ associated to π should

⁵One could certainly refine this argument to achieve more control over our eventual lifts at places in S away from p and v_{St} , but we have no need to do so.

then have the property that for any field embedding $\tau: M \rightarrow \overline{\mathbb{Q}}_p$, which induces $M_v \rightarrow \overline{\mathbb{Q}}_p$ for some $v|p$, and which via ι induces an embedding $\epsilon = \iota^{-1}\tau: M \rightarrow \mathbb{C}$, the Hodge-Tate cocharacter $\mu(r_{\pi, \iota}|_v, \tau)$ and $\mu(\pi_\epsilon, \epsilon)$ define the same G^\vee -conjugacy class of cocharacters. We will use these notions and notation in what follows in the cases $G = \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{GL}_{2n+1}$.

Proposition 4.3. *There exist a totally real extension L^+/K^+ and a regular L -algebraic, cuspidal automorphic representation Π of $\mathrm{GL}_{2n+1}(\mathbb{A}_{L^+})$ such that $\mathrm{std}(r)|_{\Gamma_{L^+}} \cong r_{\Pi, \iota}$. Moreover, the automorphic representation Π satisfies the following properties:*

- (1) *For each embedding $\epsilon: L^+ \rightarrow \mathbb{C}$, inducing an embedding $\tau := \iota\epsilon: L^+ \rightarrow \overline{\mathbb{Q}}_p$ together with a place $v|p$ of L^+ , the cocharacter $\mu(\Pi_\epsilon, \epsilon)$ coming from the archimedean L -parameter ϕ_{Π_ϵ} is conjugate to the Hodge-Tate cocharacter $\mu(\mathrm{std}(r)|_v, \tau)$.*
- (2) *Π_w is isomorphic to a twist of the Steinberg representation of $\mathrm{GL}_{2n+1}(L_w^+)$ for any finite L^+ -place $w|v_{\mathrm{St}}$.*

Proof. The existence of the field L^+ and the cuspidal automorphic representation Π follow from [3, Theorem C], where the assumptions are clearly satisfied by $\mathrm{std}(r)$. The first enumerated item is part of [37, Theorem 1.2]. For the second item, since Π is cuspidal, Π_w is generic for all places w , and therefore local-global compatibility ([37, Theorem 1.2(i)]) and [3, Lemma 1.3.2] imply that $\mathrm{std}(r)|_{\Gamma_{L_w^+}}$ is a smooth point on its (generic fiber) local lifting ring. Since $r|_{\Gamma_{L_w^+}}$ lies on the same irreducible component as the restriction $r_v|_{\Gamma_{L_w^+}}$ of the Steinberg-type lift constructed in Proposition 4.2, so do $\mathrm{std}(r)|_{\Gamma_{L_w^+}}$ and $\mathrm{std}(r_v)|_{\Gamma_{L_w^+}}$ (since the canonical map from the GSpin_{2n+1} to the GL_{2n+1} deformation ring of course preserves irreducible components). The latter is visibly a smooth point, so by [3, Lemma 1.3.4 (2)] (due to Choi) the inertial restrictions $\mathrm{std}(r)|_{I_{L_w^+}}$ and $\mathrm{std}(r_v)|_{I_{L_w^+}}$ are isomorphic. It follows from the local Langlands correspondence for $\mathrm{GL}_{2n+1}(L_w^+)$ that Π_w is an unramified twist of the Steinberg representation of $\mathrm{GL}_{2n+1}(L_w)$. \square

Remark 4.4. The extension L^+/K^+ may be chosen with the following properties: for any fixed finite Galois extension K^{avoid}/K^+ , and any finite set U of places of K^+ , the extension L^+ can be taken to be linearly disjoint from K^{avoid} over K^+ , and such that the primes in U all split completely in L^+/K^+ . The former assertion is part of [3, Theorem C], whereas the latter requires a small additional argument and is explained in the last paragraph of the proof of [11, Theorem 2.1.16].

In the next result we use Arthur's endoscopic classification, which for our purposes proves the local Langlands correspondence for $\mathrm{Sp}_{2n}(L_w^+)$ (for all places w ; of course, for infinite places this is due to Langlands) and describes the discrete automorphic spectrum of $\mathrm{Sp}_{2n}(\mathbb{A}_{L^+})$ in terms of self-dual isobaric automorphic representations of $\mathrm{GL}_{2n+1}(\mathbb{A}_{L^+})$. We will try to phrase the following argument in a way that will help the uninitiated reader to parse the main results of [2]. In what follows, for an admissible smooth representation (or admissible (\mathfrak{g}, K) -module) τ of either $\mathrm{GL}_{2n+1}(L_v^+)$ or $\mathrm{Sp}_{2n}(L_v^+)$, we will write ϕ_τ for the local Langlands parameter associated to τ in, respectively, [22] or [2].

Proposition 4.5. *There is a cuspidal automorphic representation π of $\mathrm{Sp}_{2n}(\mathbb{A}_{L^+})$ with the following properties:*

(1) For every place v of L^+ ,

$$\phi_{\Pi_v} \cong \text{std} \circ \phi_{\pi_v}.$$

(2) For each embedding $\epsilon : L^+ \rightarrow \mathbb{C}$ inducing as before an embedding $\tau : L^+ \rightarrow \overline{\mathbb{Q}}_p$ together with a place $v|p$ of L^+ , the cocharacters $\mu(\pi_\epsilon, \epsilon)$ and $\mu(P(r)|_v, \tau)$ are conjugate in SO_{2n+1} . Moreover, π_ϵ is a discrete series representation of $\text{Sp}_{2n}(L_\epsilon^+)$, with infinitesimal character $\mu(\pi_\epsilon, \epsilon)$.

(3) For any finite L^+ -place $w|v_{\text{St}}$, the local component π_w is isomorphic to the Steinberg representation of $\text{Sp}_{2n}(L_w^+)$.

Proof. Since Π is self-dual cuspidal automorphic, it corresponds to a simple generic parameter in the sense of [2]. By [2, Theorem 1.4.1] there is a unique elliptic (and in fact simple) endoscopic group G with a discrete automorphic representation π of G whose system of unramified Hecke eigenvalues gives rise to those of Π ; and since $2n+1$ is odd, G is necessarily the split group Sp_{2n}/L^+ . Simply by definition, Π thus gives rise to a parameter $\psi \in \tilde{\Psi}(G)$, in the sense of [2, §1.4]. By [2, Theorem 1.4.2], the local Langlands parameters ϕ_{Π_v} factor through $\text{SO}_{2n+1} \subset \text{GL}_{2n+1}$, and then the central results [2, Theorem 1.5.1, Theorem 1.5.2] establish the local Langlands correspondence for G and show that the descent π of Π (the multiplicity here is one, in fact) is everywhere locally compatible with this local Langlands correspondence, which yields part (1) of our Proposition.

The relation between $\phi_{\pi_\epsilon}|_{W_{L_\epsilon^+}}$ and $\mu(P(r)|_v, \tau)$ in (2) follows from (1), part (1) of Proposition 4.3, and the fact that SO_{2n+1} -valued cocharacters are SO_{2n+1} -conjugate if they are GL_{2n+1} -conjugate. The fact that π_ϵ is in the discrete series with the claimed infinitesimal character is then a standard consequence, since there are no other possibilities for the extension from $W_{\mathbb{C}}$ to $W_{\mathbb{R}}$ of this (regular and tempered) parameter.

Now we consider a place $w|v_{\text{St}}$ and show (3). As Π_w is a twist of a Steinberg representation, and the parameter ϕ_{Π_w} takes values in SO_{2n+1} , Π_w must itself be the Steinberg representation: self-duality forces it to be at most a quadratic twist of Steinberg, and then trivial determinant forces the twist to be trivial. Thus the parameter ϕ_{π_w} has $\text{std}(\phi_{\pi_w})$ equal to the Steinberg parameter for GL_{2n+1} . It was shown in [32, Proposition 8.2] that the Steinberg representation St_w of $\text{Sp}_{2n}(L_w^+)$ has this same parameter. Thus π_w and St_w lie in the same local L-packet $\tilde{\Pi}_{\phi_{\pi_w}}$ (in the notation of [2, Theorem 1.5.1]). This parameter is bounded, and Arthur shows ([2, Theorem 1.5.1(b)]) that the packet is thus in bijection with the group denoted $\mathcal{S}_{\phi_{\pi_w}}$ (the component group of the centralizer of the parameter). This group is visibly trivial, so the L-packet contains a single element, and therefore $\pi_w \cong \text{St}_w$. \square

The following argument is complicated slightly by the fact that we do not know at the outset the expectation ([7, III.10.3]) of how to compute the central character ω_π from the local L-parameters of π (except at archimedean and unramified primes).

Proposition 4.6. *There is a cuspidal automorphic representation $\tilde{\pi}$ of $\text{GSp}_{2n}(\mathbb{A}_L^+)$ satisfying the following properties:*

(1) For any place v of L^+ that is either archimedean or for which $\tilde{\pi}_v$ is unramified (in these two cases the corresponding L-parameters $\phi_{\tilde{\pi}_v}$ are defined), $P \circ \phi_{\tilde{\pi}_v} = \phi_{\pi_v}$.

- (2) The representation $\tilde{\pi}$ is L -cohomological in the sense of [26]. More precisely, for each embedding $\epsilon : L^+ \rightarrow \mathbb{C}$, which identifies the algebraic closure of L_ϵ^+ with \mathbb{C} and induces (via ι) an embedding $\tau : L^+ \rightarrow \overline{\mathbb{Q}_p}$ together with a place $v|p$ of L^+ , $\mu(\tilde{\pi}_\epsilon, \epsilon)$ equals $\mu(r|_v, \tau)$.
- (3) For any place $w|v_{\mathrm{St}}$ of L^+ , the local component $\tilde{\pi}_w$ is a twist of the Steinberg representation of $\mathrm{GSp}_{2n}(L_w^+)$.

Proof. Let $\tilde{\omega}_0$ be a finite-order Hecke character of L^+ extending the central character ω_π , and set $\tilde{\omega} = \tilde{\omega}_0 \cdot |\cdot|^s$, where $s = \sum_{i=1}^n s_i$ is the integer appearing in the definition of $\theta = N(r)$ in Proposition 4.2. Then $\tilde{\omega}$ clearly also extends ω_π . By the argument of Langlands-Labesse (for SL_2 in [28] and generalized in [34, Proposition 3.1.4] using an argument of Flicker from [18]), there is a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ with central character $\omega_{\tilde{\pi}} = \tilde{\omega}$ and extending π (i.e., π is a constituent of the restriction $\tilde{\pi}|_{\mathrm{Sp}_{2n}(\mathbb{A}_{L^+})}$). Part (1) follows from [34, Corollary 3.1.6]. We claim that for all $\epsilon : L_\epsilon^+ \rightarrow \mathbb{C}$, $\tilde{\pi}_\epsilon$ is L -cohomological with the claimed infinitesimal character. Indeed, the only quasi-cocharacter of $T \subset \mathrm{GSpin}_{2n+1}$ (notation as in Proposition 4.2) lifting $\mu(\pi_\epsilon, \epsilon)$ and having norm equal to $z \mapsto z^s$, $s = \sum_{i=1}^n s_i$, is the (genuine) cocharacter denoted (in the root datum notation introduced above) $\sum_{i=1}^n s_i(\lambda_i + \frac{\lambda_0}{2})$, and equal to $\mu(r|_v, \tau)$. This must be the infinitesimal character $\mu(\tilde{\pi}_\epsilon, \epsilon)$, so we see that $\tilde{\pi}_\epsilon$ is L -algebraic and (because it extends π_ϵ) essentially discrete series. We will check that $\tilde{\pi}_\epsilon$ is moreover “ L -cohomological” in the sense of [26], i.e. that $\tilde{\pi}_\epsilon \cdot |\lambda_0|^{\frac{n(n+1)}{4}}$ is cohomological in the sense of [26, Definition 1.12] (λ_0 corresponds to the similitude character of GSp_{2n}). Since π_ϵ is discrete series with infinitesimal character $\mu(\pi_\epsilon, \epsilon)$, it is $\xi_{\mu(\pi_\epsilon, \epsilon) - \rho}^\vee$ -cohomological ([8, Theorem V.3.3]), where $\xi_{\mu(\pi_\epsilon, \epsilon) - \rho}$ is the highest-weight $\mu(\pi_\epsilon, \epsilon) - \rho$ representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, and ρ is the half-sum of positive roots for the choice of root basis used also to parametrize highest weights. We need an extension $\tilde{\xi}$ of $\xi_{\mu(\pi_\epsilon, \epsilon) - \rho}$ to $\mathrm{GSp}_{2n}(\mathbb{C})$ such that (letting K be a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$)

$$H^*(\mathfrak{gsp}_{2n}(\mathbb{C}), K \cdot Z_{\mathrm{GSp}_{2n}}(\mathbb{R}); \tilde{\pi}_\epsilon \cdot |\lambda_0|^{\frac{n(n+1)}{4}} \otimes \tilde{\xi}^\vee) \neq 0.$$

From the definition of this cohomology group, and the fact that $\pi_\epsilon \otimes \xi^\vee$ has non-zero $(\mathfrak{sp}_{2n}(\mathbb{C}), K)$ -cohomology, this reduces to checking that we can find an extension $\tilde{\xi}^\vee$ of $\xi_{\mu(\pi_\epsilon, \epsilon) - \rho}^\vee$ such that $\tilde{\xi}$ and $\tilde{\pi}_\epsilon \cdot |\lambda_0|^{\frac{n(n+1)}{4}}$ have the same central character; note that we will only have to check this on $\mathbb{R}_{>0}^\times \subset Z_{\mathrm{GSp}_{2n}}(\mathbb{R})$ since $-1 \in K$.⁶ We choose $\tilde{\xi}$ to be the representation with highest weight $\mu(\tilde{\pi}_\epsilon, \epsilon) - \rho + \frac{n(n+1)}{4}\lambda_0 \in X_\bullet(T)$ (the λ_0 term corrects the failure of ρ to be integral for GSp_{2n}). The claim is now proven, since this highest weight representation has the same infinitesimal character as $\tilde{\pi}_\epsilon \cdot |\lambda_0|^{\frac{n(n+1)}{4}}$.

Now we prove (3). By Proposition 4.5, part (3), π_w is Steinberg for any finite L^+ -place $w|v_{\mathrm{St}}$. Consider the representation $\tau_w := \pi_w \times \tilde{\omega}_w$ of the subgroup $H_w := \mathrm{Sp}_{2n}(L_w^+) \cdot \tilde{Z}(L_w^+) \subset \mathrm{GSp}_{2n}(L_w^+)$, where \tilde{Z} denotes the center of GSp_{2n} (such a representation makes sense because $\tilde{\omega}$ extends ω_π). We also let $\tilde{\mathrm{St}}_w$ denote the Steinberg representation of $\mathrm{GSp}_{2n}(L_w^+)$. We want, for a suitable twist $\tilde{\mathrm{St}}_w \cdot \alpha$ (i.e., twisting by $\mathrm{GSp}_{2n}(L_w^+) \xrightarrow{\lambda_0} L_w^{+, \times} \xrightarrow{\alpha} \mathbb{C}^\times$), to compare $\tilde{\mathrm{St}}_w \cdot \alpha$ to

⁶The definition of “cohomological” used in [26] for a general real reductive group H uses $(\mathrm{Lie}(H(\mathbb{C})), K_H^0 Z_H(\mathbb{R}))$ -cohomology, where K_H^0 is the identity component of a maximal compact subgroup K_H of $H(\mathbb{R})$. For $H = \mathrm{GSp}_{2n}$, K_H^0 is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$.

$\text{Ind}_{H_w}^{\text{GSp}_{2n}(L_w^+)}(\tau_w)$, for which we will simply write $\text{Ind}(\tau_w)$ in what follows. First we note that the restriction $\tilde{\text{St}}_w|_{H_w}$ is isomorphic to St_w extended trivially along $\tilde{Z}(L_w^+)$, as is clear from the definition of the Steinberg representation. Next, since $\tilde{\omega}_w|_{\pm 1}$ is trivial (as $\tilde{\omega}_w$ extends ω_{π_w}), there exists a character α , in fact $L_w^{+,\times}/(L_w^{+,\times})^2 = [\text{GSp}_{2n}(L_w^+) : H_w]$ distinct characters, satisfying $\alpha^2 = \tilde{\omega}_w$. Then, for each such α , Frobenius reciprocity and Schur's lemma imply that

$$\text{Hom}_{\text{GSp}_{2n}(L_w^+)}(\tilde{\text{St}}_w \cdot \alpha, \text{Ind}(\tau_w)) = \text{Hom}_{H_w}(\tau_w, \tau_w),$$

is one-dimensional. Again by Frobenius reciprocity $\tilde{\pi}_w$ is a Jordan-Hölder constituent of the (easily seen to be semisimple) representation $\text{Ind}(\tau_w)$, and since (by definition of the Steinberg representation) all $\text{GSp}_{2n}(L_w^+)$ -conjugates of τ_w are isomorphic, another application of Frobenius reciprocity shows that $\text{Hom}_{\text{GSp}_{2n}(L_w^+)}(\text{Ind}(\tau_w), \text{Ind}(\tau_w))$ has dimension $[\text{GSp}_{2n}(L_w^+) : H_w]$. It follows that

$$\text{Ind}(\tau_w) \cong \bigoplus_{\alpha^2 = \tilde{\omega}_w} \tilde{\text{St}}_w \cdot \alpha,$$

and therefore for some such α , $\tilde{\pi}_w$ is isomorphic to $\tilde{\text{St}}_w \cdot \alpha$. \square

The main result of [26] now implies the existence of a Galois representation

$$r_{\tilde{\pi}} = r_{\tilde{\pi},\iota} : \Gamma_L^+ \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$$

satisfying most of the expected properties, including that $P(r_{\tilde{\pi}}) = P(r)$, $N(r_{\tilde{\pi}}) = \tilde{\omega}$ (abusively writing $\tilde{\omega}$ for the Galois character associated to $\tilde{\omega}$ by global class field theory), and $r_{\tilde{\pi}}$ is locally compatible with $\tilde{\pi}$ at almost all (unramified) places. This $r_{\tilde{\pi}}$ is nearly the Galois representation we want, but first we need to relate $N(r)$ to the Hecke character $\tilde{\omega}$:

Corollary 4.7. *At all places v of L^+ , the central character ω_{π_v} is computed in terms of ϕ_{π_v} according to the expected description in [7, III.10.3]. The representation $r_{\tilde{\pi}}$ is conjugate to $r \otimes \delta$ for a continuous Galois character $\delta : \Gamma_L^+ \rightarrow Z(\overline{\mathbb{Q}}_p)$, and we can choose $\tilde{\omega}$ in Proposition 4.6 to be the Hecke character corresponding to $N(r)$ under global class field theory, so that $\delta^2 = 1$.*

Proof. Recall the local Langlands desideratum that if ϕ_{π_v} is the L-parameter of π_v , then ω_{π_v} should be computed as follows: lift ϕ_{π_v} to a $\text{GSpin}_{2n+1}(\mathbb{C})$ -valued parameter $\tilde{\phi}_{\pi_v}$, and then restrict

$$N(\tilde{\phi}_{\pi_v}) \circ \text{rec}_v^{-1} : (L_v^+)^{\times} \rightarrow \mathbb{C}^{\times}$$

to $\mu_2(L_v^+)$. The Weil-Deligne representation associated to $r_{\tilde{\pi}}|_{\Gamma_{L_v^+}}$ (for v above p this must be taken in the sense of Fontaine: see, e.g., [5] for a description for representations valued in general groups) provides such a lift $\tilde{\phi}_{\pi_v}$,⁷ and then (since, by [26, Theorem A], $N(r_{\tilde{\pi}})$ corresponds to $\tilde{\omega}$) $N(\tilde{\phi}_{\pi_v})$ corresponds to $\tilde{\omega}_v$ under local class field theory; but by construction $\tilde{\omega}_v$ extends ω_{π_v} , so the first claim is proven. Since r and $r_{\tilde{\pi}}$ both lift $P(r)$, they are twists: $r_{\tilde{\pi}} \cong r \otimes \delta$ for some character $\delta : \Gamma_{L^+} \rightarrow Z(\overline{\mathbb{Q}}_p)$. As they both have Clifford norm with the same Hodge-Tate weights, δ is finite-order. It follows, since $N(r_{\tilde{\pi}}) = \delta^2 \cdot N(r)$, that the Hecke character corresponding to $N(r)$ also extends ω_{π} , hence that we can choose $\tilde{\omega}$, and thus $\tilde{\pi}$, in Proposition 4.6 such that $\tilde{\omega}$ corresponds to $N(r)$ under global class field theory. \square

⁷At v above p , this claim makes use of local-global compatibility of $r_{\Pi,\iota}|_{\Gamma_{L_v^+}}$. Since $2n+1$ is odd, this is proven in [4, Theorem A].

Choosing $\tilde{\omega}$ corresponding to $N(r)$, constructing $\tilde{\pi}$ as in Proposition 4.6, and then if necessary using Corollary 4.7 to twist $\tilde{\pi}$ by the Hecke character corresponding to δ^{-1} , we obtain our first main result:

Theorem 4.8. *Let \bar{r} be as in the beginning of this section, satisfying Hypothesis 4.1. Then \bar{r} is potentially automorphic, i.e., there exists a totally real extension L^+/F^+ and a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ that is L -cohomological and twist of Steinberg at some finite prime, such that a suitable GSpin_{2n+1} -conjugate of the Galois representation $r_{\tilde{\pi}} : \Gamma_{L^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ constructed in [26] lifts $\bar{r}|_{L^+}$.*

4.2. The p -adic case. Note that Theorem 4.8 makes no local hypothesis on \bar{r} at finite primes. We can prove an analogue of Theorem 4.8 for p -adic representations if we assume they satisfy a Steinberg-type local condition at some finite prime. The argument given above implies this with little modification:

Theorem 4.9. *Let F^+ be a totally real field, and let $r : \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ be a continuous representation, unramified outside a finite set S of primes containing all $v|p$, having geometric Clifford norm $N(r)$, and satisfying:*

- $p > 2(2n + 1)$.
- \bar{r} is odd.
- For some finite place v_{St} of F^+ , the Frobenius semisimple Weil-Deligne representation associated to $r|_{\Gamma_{F^+_{v_{\mathrm{St}}}}}$ is equivalent to a twist of the Steinberg-type Langlands parameter (equivalently, $P(r|_{\Gamma_{F^+_{v_{\mathrm{St}}}}})$ is a Steinberg parameter for $\mathrm{Sp}_{2n}(F^+_{v_{\mathrm{St}}})$).
- There exist a quadratic CM extension F/F^+ and a character $\mu : \Gamma_{F^+} \rightarrow \overline{\mathbb{Z}}_p^\times$ such that
 - F does not contain ζ_p , and $\mathrm{std}(\bar{r})|_{\Gamma_{F(\zeta_p)}}$ is irreducible.
 - $(\mathrm{std}(r)|_{\Gamma_F}, \mu)$ is polarized, and for some (any) choice of prolongation

$$\rho(\mathrm{std}(\bar{r})|_{\Gamma_F}, \mu) : \Gamma_{F^+} \rightarrow \mathcal{G}_{2n+1}(\overline{\mathbb{Z}}_p),$$

$\rho(\mathrm{std}(r), \mu)|_{\Gamma_{F^+_v}}$ is globally realizable for each $v|p$.

Then there exist a totally real extension L^+/F^+ and a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ that is L -cohomological and locally at primes above v_{St} isomorphic to twists of the Steinberg representation, such that $r_{\tilde{\pi}, \iota} \cong r|_{\Gamma_{L^+}}$. The extension L^+ may be chosen to avoid any fixed finite Galois extension F^{avoid}/F^+ , and to be split at all places above S (and in particular above v_{St}).

Remark 4.10. As previously noted, the local hypothesis at $v|p$ is strictly weaker than assuming $\mathrm{std}(r|_{\Gamma_{F^+_v}})$ is regular and potentially diagonalizable. We note that the definition of globally realizable in [11] has already fixed a CM extension, and we have tried to phrase the statement of the theorem in order to allow flexibility in this choice of extension. It is possible, for instance, to replace an initially given F with one that is split at places above $S \setminus \{v|p\}$.

Proof. The polarization condition implies $\text{std}(r)|_{\Gamma_F} \cong \text{std}(r)|_{\Gamma_F} \otimes \mu|_{\Gamma_F}$, since of course $\text{std}(r)|_{\Gamma_F}$ arises from $P(r)$. Under the hypothesis on $r|_{\Gamma_{F^+}^{\text{vSt}}}$, this implies $\mu|_{\Gamma_F} = 1$ (else $\text{std}(r)|_{\Gamma_F}$ would be induced from a finite extension). By oddness (which by [11, Lemma 1.4.4] is automatic), we conclude that $\mu = \delta_{F/F^+}$, and then we may ([11, Remark 2.1.9]) take our prolongation that is globally realizable at places above p simply to be the familiar composite

$$\Gamma_{F^+} \xrightarrow{P(r) \times \text{res}_F} \text{SO}_{2n+1}(\overline{\mathbb{Z}}_p) \times \text{Gal}(F/F^+) \rightarrow \mathcal{G}_{2n+1}(\overline{\mathbb{Z}}_p),$$

the standard prolongation $\rho(r)$ defined in §2.

The prolongation $\rho(r)$ satisfies the hypotheses of [11, Corollary 4.2.12], so we find a CM extension L/F (with totally real subfield L^+) and a cuspidal automorphic representation Π_L of $\text{GL}_{2n+1}(\mathbb{A}_L)$ that is polarized and regular L-algebraic such that $r_{\Pi_L, \iota} \cong \text{std}(r)|_{\Gamma_L}$. Moreover, we may assume that L^+/F^+ is split at the primes in S and linearly disjoint from the fixed F^{avoid} : indeed, once we know $\text{std}(r)|_{\Gamma_F}$ is potentially automorphic, we deduce from [3, Theorem 5.5.1] (the potentially diagonalizable hypothesis there is only to invoke the potential automorphy theorem of *loc. cit.*) that it belongs to a compatible system of odd, regular, weakly irreducible (as in [11]) polarized representations of Γ_F , and then [11, Theorem 2.1.16] shows that we can indeed choose L/F (and Π_L) such that L^+/F^+ is split at all primes above S .⁸ Next, we note that $\Pi_L^c \cong \Pi_L$ (where c is complex conjugation), so Π_L descends to a (regular L-algebraic) cuspidal automorphic representation Π of $\text{GL}_{2n+1}(\mathbb{A}_{L^+})$; it has an associated Galois representation that restricts to the irreducible $\text{std}(r)|_{\Gamma_L}$, so twisting if necessary we may in fact assume $r_{\Pi, \iota} \cong \text{std}(r)|_{\Gamma_{L^+}}$. We take this Π in Proposition 4.3, and from this point on the proof of our theorem is identical to the proof of Theorem 4.8; note that both $N(r)$ and $\text{std}(r)$ being geometric suffices to imply that r is geometric: this follows from work of Wintenberger and Conrad (see [15, Theorem 6.2]). \square

5. SOLVABLE DESCENT AND COMPATIBLE SYSTEMS

Potential automorphy theorems for the group GL_N imply, by Taylor's Brauer induction argument ([38, §5.3.3]), that one can often put a single p -adic representation $\Gamma_{F^+} \rightarrow \text{GL}_N(\overline{\mathbb{Q}}_p)$ into a compatible system (over this exact field F^+ , not over the extension L^+ where it is shown to be automorphic). In this section we will give a variant of Taylor's argument that applies to a single $\text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ -valued representation. In order to do this, we will have to establish some cases of solvable descent for GSp_{2n} . As for GL_N , this occurs in two steps: we establish cyclic prime degree descent directly by a trace formula argument, which is only a minor supplement to the cyclic base-change argument in [26, Proposition 6.6]. The second step, passing from cyclic to solvable descent in contexts where automorphic Galois representations have been constructed, is considerably more subtle for GSp_{2n} , essentially because of the failure of strong multiplicity one and the presence of non-singleton global L-packets. In addition, since Lefschetz/Kottwitz functions detect unramified but not ramified twists of the Steinberg representation, the argument is complicated by not knowing local-global compatibility of the Kret-Shin construction at the Steinberg prime (taking the extension L^+/F^+ in Theorem 4.9 to be split at the primes of ramification allows us to circumvent this difficulty).

⁸Here we are feeding the conclusion of [11, Corollary 4.2.12] back into one of its essential ingredients in order to obtain a slightly stronger result.

We postpone the proofs of these two descent results to Proposition 5.2 and Theorem 5.3, and we first explain how to apply them and our potential automorphy theorem to put certain $\mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ -valued Galois representations in compatible systems:

Theorem 5.1. *Let $r: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Z}}_p)$ satisfy the hypotheses of Theorem 4.9. Then for all primes ℓ and choices of isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ there is a continuous representation $r_{\iota_\ell}: \Gamma_{F^+} \rightarrow \mathrm{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ such that:*

- For all primes v at which r is unramified, the semisimple conjugacy class of $\iota^{-1}r(\mathrm{Fr}_v)^{\mathrm{ss}}$ agrees with that of $\iota_\ell^{-1}r_{\iota_\ell}(\mathrm{Fr}_v)^{\mathrm{ss}}$.
- For all primes $v|\ell$, r_{ι_ℓ} is de Rham, and its Hodge-Tate cocharacters are determined up to conjugacy by those of r : for all embeddings $\tau: F^+ \rightarrow \overline{\mathbb{Q}}_\ell$, determining a place $v|\ell$ of F^+ , and inducing $\mu_\ell^{-1}\tau: F^+ \rightarrow \overline{\mathbb{Q}}_p$ and a corresponding place $\iota_\ell^*(v)|p$ of F^+ , the conjugacy classes of $\iota_\ell^{-1}\mu(r_{\iota_\ell}|_{\Gamma_{F^+}}, \tau)$ and $\iota^{-1}\mu(r|_{\Gamma_{\iota_\ell^*(v)}}, \mu_\ell^{-1}\tau)$ coincide.

Proof. We write G_r for the Zariski closure of the image of r , and we write G_r^0 for the identity component of G_r (and we use analogous notation for other Galois representations). Let $F^{+,0}$ be the fixed field of the preimage $r^{-1}(G_r^0(\overline{\mathbb{Q}}_p))$. By Theorem 4.9, there exist

- a totally real field L^+ , which we may assume linearly disjoint from the composite of $F^{+,0}(\mu_p)$ and the fixed field of \bar{r} , and split at all places above v_{St} ;
- and a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ that is L-cohomological, and at all places v above v_{St} is isomorphic to a twist of the Steinberg representation, such that $r_{\tilde{\pi}, \iota} \cong r|_{\Gamma_{L^+}}$.

We then have an associated compatible system $\{r_{\tilde{\pi}, \iota_\ell}\}_{\iota_\ell}$ indexed by field isomorphisms $\iota_\ell: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Moreover, $r_{\tilde{\pi}, \iota_\ell}$ lifts r_{π, ι_ℓ} , where as previously we denote by π a (cuspidal automorphic) constituent of $\tilde{\pi}|_{\mathrm{Sp}_{2n}(\mathbb{A}_{L^+})}$ and $r_{\pi, \iota_\ell}: \Gamma_{L^+} \rightarrow \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ the associated Galois representation. The algebraic monodromy groups $G_{r_{\pi, \iota_\ell}}$ are independent of ι_ℓ : each is reductive with a regular unipotent element in the image, hence is irreducible with monodromy group equal to one of the following: all of SO_{2n+1} ; the image of a principal SL_2 ; or G_2 when $n = 3$. Independence of ℓ of the formal character (or even just the rank) of maximal tori in the algebraic monodromy groups (a result of Serre; see, e.g., [31, Proposition 6.12]) implies the claim. It follows then that $G_{r_{\tilde{\pi}, \iota_\ell}}$ is also independent of ι_ℓ , since the $r_{\tilde{\pi}, \iota_\ell}$ moreover have compatible Clifford norms, and component groups are again by Serre's work independent of ℓ ([31, Proposition 6.14]).

By Proposition 5.2 and Theorem 5.3 below, for any intermediate extension $L^+ \supset K^+ \supset F^+$ with L^+/K^+ solvable, $r|_{\Gamma_{K^+}}$ is automorphic; more precisely, there exist cuspidal automorphic representations $\tilde{\pi}_{K^+}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{K^+})$ (L-cohomological and twist of Steinberg at all places above v_{St}) such that $r_{\tilde{\pi}_{K^+}, \iota} = r|_{\Gamma_{K^+}}$ (we are justified in writing “=” since the representation constructed by Kret-Shin is unique up to GSpin_{2n+1} -conjugacy).

Consider the composite $\text{spin}(r): \Gamma_{F^+} \rightarrow \text{GL}_{2^n}(\overline{\mathbb{Q}}_p)$. Writing the trivial representation of $\text{Gal}(L^+/F^+)$ as a linear combination

$$1 = \sum_j n_j \text{Ind}_{\text{Gal}(L^+/L_j^+)}^{\text{Gal}(L^+/F^+)}(\psi_j)$$

for some intermediate extensions L_j^+ such that $\text{Gal}(L^+/L_j^+)$ is nilpotent for each j , some characters $\psi_j: \text{Gal}(L^+/L_j^+) \rightarrow \mathbb{C}^\times$, and some (possibly negative) integers n_j , we see that as virtual representations

$$\text{spin}(r) = \sum_j n_j \text{Ind}_{\Gamma_{L_j^+}}^{\Gamma_{F^+}}(\text{spin}(r|_{\Gamma_{L_j^+}}) \otimes \psi_j).$$

We now define, for every prime ℓ and every choice of isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, the virtual ℓ -adic representation

$$R_{\iota_\ell} = \sum_j n_j \text{Ind}_{\Gamma_{L_j^+}}^{\Gamma_{F^+}}(\text{spin}(r_{\tilde{\pi}_{L_j^+, \iota_\ell}}) \otimes \psi_j).$$

We have to check three things:

- The $\{R_{\iota_\ell}\}_{\iota_\ell}$ form a (weakly) compatible system of actual representations.
- With the previous point established, each $R_{\iota_\ell}: \Gamma_{F^+} \rightarrow \text{GL}_{2^n}(\overline{\mathbb{Q}}_\ell)$ will, up to conjugacy, factor as $\text{spin}(r_{\iota_\ell})$ for some $r_{\iota_\ell}: \Gamma_{F^+} \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$.
- The r_{ι_ℓ} form a compatible system of GSpin_{2n+1} -valued representations, containing $r = r_\iota$ (when $\iota_\ell = \iota$).

Since $\text{spin}(r)$ is not in general irreducible, it seems easiest to give the following *ad hoc* argument, treating the different possible algebraic monodromy groups of $\text{spin}(r)$ separately. If $n = 3$ and $G_{P(r)} = \text{G}_2$, then the result is obvious, since the spin representation restricted to G_r^{der} is the direct sum of the trivial representation and $\text{std}(r)$; thus it suffices to put $\text{std}(r)$ (and a character) in a compatible system, which can be done by [3, Theorem 5.5.1]. If $G_{P(r)} = \text{PGL}_2$, then $\text{SL}_2 \subset G_r \subset \text{GL}_2$, and we can simply regard r as the composite $\varphi \circ r_0$ of a principal homomorphism $\varphi: \text{GL}_2 \rightarrow \text{GSpin}_{2n+1}$ (a principal SL_2 extended to the identity map between the centers) and some $r_0: \Gamma_{F^+} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$. There is a compatible system r_{0, ι_ℓ} containing r_0 , and then we can construct the desired GSpin_{2n+1} -valued compatible system as $\varphi \circ r_{0, \iota_\ell}$.

We now consider the remaining case, where G_r contains Spin_{2n+1} . Then $\text{spin}(r)$ is irreducible, as is $\text{spin}(r|_{\Gamma_{K^+}})$ for any finite extension K^+/F^+ . By the independence-of- ℓ observation above, it also follows that $\text{spin}(r_{\tilde{\pi}_{K^+, \iota_\ell}})$ is irreducible for any ι_ℓ and any $L^+ \supset K^+ \supset F^+$ with L^+/K^+ solvable. The argument of [3, Theorem 5.5.1] then immediately implies that the virtual representation R_{ι_ℓ} is an actual representation and is moreover irreducible. It is then an easy consequence of [30, Theorem 4] that each R_{ι_ℓ} (up to conjugacy) factors through GSpin_{2n+1} ; indeed, the only ‘‘basic similarity classes’’ in the sense of *loc. cit.* relevant to the type $(G_r^{\text{der}} \subset \text{GL}_{2^n}) = (\text{Spin}_{2n+1}, \text{spin})$ are products $(\prod_{i=1}^k \text{Spin}_{2n_i+1}, \boxtimes_i \text{spin}_{2n_i+1})$ for some decomposition $n_1 + \dots + n_k = n$, where spin_{2n_i+1} denotes the appropriate spin representation. This external product is simply the restriction of the 2^n -dimensional spin representation to $\prod \text{Spin}_{2n_i+1} \subset \text{Spin}_{2n+1}$, so in any case $G_{R_{\iota_\ell}}^{\text{der}} \subset \text{GL}_{2^n}$ factors (up to conjugacy) through $\text{Spin}_{2n+1} \subset \text{GL}_{2^n}$. It follows that R_{ι_ℓ} factors as $\text{spin}(r_{\iota_\ell})$ for some $r_{\iota_\ell}: \Gamma_{F^+} \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$.

Moreover, since $R_{\iota_\ell}|_{\Gamma_{L^+}}$ is isomorphic to $\mathrm{spin}(r_{\tilde{\pi}, \iota_\ell})$, each member of the compatible system $\{R_{\iota_\ell}\}_{\iota_\ell}$ of Γ_{F^+} -representations contains the image under the spin representation of a regular unipotent element of Spin_{2n+1} . By [9, Lemma 3.5], each r_{ι_ℓ} then has image containing a regular unipotent element. We deduce from Dynkin's theorem that each algebraic monodromy group $G_{r_{\iota_\ell}}$ contains Spin_{2n+1} .

That the collection $\{r_{\iota_\ell}\}_{\iota_\ell}$ forms a compatible system in the sense of GSpin_{2n+1} -valued representations will now follow from [26, Lemma 1.3] and the fact that $N, \mathrm{std}, \mathrm{spin}$ form a fundamental set of representations of GSpin_{2n+1} . Indeed, by construction the $\mathrm{spin}(r_{\iota_\ell})$ are compatible; $N(r_{\iota_\ell})$ can be read off (see [26, Lemma 0.1]) as the similitude character of the essentially self-dual representation R_{ι_ℓ} , and these are also clearly compatible. To see compatibility of the representations $\mathrm{std}(r_{\iota_\ell})$, note that by a standard plethysm

$$R_{\iota_\ell}^{\otimes 2} \otimes N(r_{\iota_\ell})^{-1} \cong \bigoplus_{i=0}^n \wedge^i \mathrm{std}(r_{\iota_\ell}).$$

The representations on the left-hand side of this isomorphism, for varying ι_ℓ , form a compatible system, so the direct sum on the right-hand side does as well. At the same time, $\mathrm{std}(r)$ is already known to belong a compatible system by [3, Theorem 5.5.1], and consequently so does each $\wedge^i \mathrm{std}(r)$. Let us denote these compatible systems by $\{\wedge^i \tau_{\iota_\ell}\}_{\iota_\ell}$. By uniqueness of the compatible system containing a given representation, we deduce an isomorphism

$$\bigoplus_{i=0}^n \wedge^i \mathrm{std}(r_{\iota_\ell}) \cong \bigoplus_{i=0}^n \wedge^i \tau_{\iota_\ell}.$$

Since $\mathrm{std}(r_{\iota_\ell})$ and τ_{ι_ℓ} both have algebraic monodromy group SO_{2n+1} , all of these wedge powers (as i varies) are irreducible of different dimensions, and we conclude that $\mathrm{std}(r_{\iota_\ell}) \cong \tau_{\iota_\ell}$ for all ι_ℓ . In other words, the representations $\mathrm{std}(r_{\iota_\ell})$ do indeed form a compatible system.

This concludes the proof of the theorem, modulo Proposition 5.2 and Theorem 5.3. \square

We next establish the cases we need of cyclic descent of prime degree. We simply follow [26, §6], and in particular we first recall some of the setup preceding their Proposition 6.6 (the reader should have [26] at hand to follow the discussion), recast in a way more consistent with our notation. Let L^+/K^+ be a cyclic Galois extension of totally real fields of prime degree ℓ , with $\mathrm{Gal}(L^+/K^+)$ generated by an automorphism σ . Let G be the split reductive group $\mathrm{GSp}_{2n}/\mathcal{O}_{L^+}$, let S be a finite set of places containing those that ramify in L^+/K^+ , and let \tilde{G} be the restriction of scalars $\mathrm{Res}_{\mathcal{O}_{L^+}/\mathcal{O}_{K^+}}(\mathrm{GSp}_{2n})$. Thus \tilde{G} is a reductive group scheme over $\mathcal{O}_{K^+}[\frac{1}{S}]$. The reductive integral models of G and \tilde{G} determine hyperspecial maximal compact subgroups, and thus unramified Hecke algebras and unramified representations, away from S .

Proposition 5.2. *Let $\tilde{\pi}$ be a cuspidal automorphic representation of $\tilde{G}(\mathbb{A}_{K^+}) = G(\mathbb{A}_{L^+})$ such that*

- (1) $\tilde{\pi}$ is unramified outside (all places above) a finite set S of places of K^+ , containing all those that are ramified in L^+/K^+ .
- (2) $\tilde{\pi}^\sigma \cong \tilde{\pi}$.

- (3) For some finite place v_{St} of K^+ , which for simplicity we assume splits in L^+/K^+ , $\tilde{\pi}_{v_{\text{St}}} = \otimes_{v|v_{\text{St}}} \tilde{\pi}_v$ is an unramified twist of the Steinberg representation of $\tilde{G}(K_{v_{\text{St}}}^+) = \prod_{v|v_{\text{St}}} G(L_v^+)$.
- (4) For some irreducible algebraic representation $\xi = \otimes_{v|\infty} \xi_v$ of $G(K_\infty^+) = \prod_{v|\infty} G(K_v^+)$, with base-change ξ_{L^+} to $G(L_\infty^+)$, $\tilde{\pi}_\infty$ is ξ_{L^+} -cohomological.

Then there exists a cuspidal automorphic representation $\tilde{\pi}_{K^+}$ of $G(\mathbb{A}_{K^+})$ such that

- (1) For all finite places $v \notin S$, $\tilde{\pi}_v$ is unramified at v , and moreover $\tilde{\pi}_v \cong \text{BC}_{L^+/K^+}(\tilde{\pi}_{K^+,v})$.
- (2) $\tilde{\pi}_{K^+,v_{\text{St}}}$ is isomorphic to an unramified twist of the Steinberg representation.
- (3) $\tilde{\pi}_{K^+,\infty}$ is ξ -cohomological.

Proof. We just explain the modifications to the base-change argument of [26, Proposition 6.6]. The central character $\omega_{\tilde{\pi}}$ is σ -invariant, so it descends (L^+/K^+ is cyclic) to a Hecke character $\omega_{\tilde{\pi},K^+}$ of K^+ . Let \tilde{U} be a sufficiently small compact open subgroup of $G(\mathbb{A}_{L^+}^\infty)$, satisfying

- $(\tilde{\pi}^\infty)^{\tilde{U}} \neq 0$.
- $\tilde{U} \cap Z_G(\mathbb{A}_{L^+}^\infty)$ is contained in the kernel of $\omega_{\tilde{\pi}}$.

Set $\tilde{\mathfrak{X}} = Z_G(L_\infty^+) \cdot (Z_G(\mathbb{A}_{L^+}^\infty) \cap \tilde{U})$, a closed subgroup of $Z_G(\mathbb{A}_{L^+})$. Let $\tilde{\chi}$ denote the restriction of $\omega_{\tilde{\pi}}$ to $\tilde{\mathfrak{X}}$, inducing $\tilde{\chi}: (\tilde{\mathfrak{X}} \cap Z_G(L^+)) \backslash \tilde{\mathfrak{X}} \rightarrow \mathbb{C}^\times$. Then $(\tilde{\mathfrak{X}}, \tilde{\chi})$ is a central character datum (for \tilde{G}) satisfying the assumptions in [26, §6]. and we can define a central character datum (for G) by taking $\mathfrak{X} = N_{L^+/K^+}(\tilde{\mathfrak{X}})$ and $\chi: (Z_G(K^+) \cap \mathfrak{X}) \backslash \mathfrak{X} \rightarrow \mathbb{C}^\times$ equal to the restriction of $\omega_{\tilde{\pi},K^+}$. Following [26, §6], we can then define the adelic Hecke algebras $\mathcal{H}(G(\mathbb{A}_{L^+}), \tilde{\chi}^{-1})$ and $\mathcal{H}(G(\mathbb{A}_{K^+}), \chi^{-1})$ (note that for all finite places v , the components at v of these Hecke algebras are just the usual Hecke algebras, since $\tilde{\chi}|_{\tilde{\mathfrak{X}}^\infty} = 1$, and $\chi|_{\mathfrak{X}^\infty} = 1$).

Continuing to follow [26], we choose test functions $\tilde{f} = \prod_w \tilde{f}_w \in \mathcal{H}(G(\mathbb{A}_{L^+}), \tilde{\chi}^{-1})$ and $f = \prod_v f_v \in \mathcal{H}(G(\mathbb{A}_{K^+}), \chi^{-1})$ as follows: for w not above S , we let \tilde{f}_w be an arbitrary element of the unramified Hecke algebra at w , and we let f_v (for v lying below w) be the image (in the unramified Hecke algebra at v) of $\prod_{w|v} \tilde{f}_w$ under the base-change map. For w above v_{St} , we let $\tilde{f}_{v_{\text{St}}} = \prod_{w|v_{\text{St}}} \tilde{f}_w$ and $f_{v_{\text{St}}}$ be the Kottwitz and generalized Kottwitz functions of [27, §3.9]; then [27, Proposition 3.9.2] shows that up to non-zero scalar $\tilde{f}_{v_{\text{St}}}$ and $f_{v_{\text{St}}}$ are associated. Because of our simplifying assumption that L^+/K^+ splits at v_{St} , the ‘‘generalized Kottwitz function’’ of [27, §3.9] is simply the (non-generalized) Kottwitz function of *loc. cit.*⁹ At infinite places, we let $\tilde{f}_\infty = \prod_{w|\infty} \tilde{f}_w$ be the twisted Lefschetz function corresponding to ξ_{L^+} , and likewise we let $f_\infty = \prod_{v|\infty} f_v$ be the usual Lefschetz function corresponding to ξ , as in [26, Appendix A]; then [26, Lemma A.11] recalls that these are associated up to non-zero scalar. Finally, for other places w above S , we let \tilde{f}_w be arbitrary, and we let f_v be

⁹The Kottwitz functions of [27] are the functions denoted f_{Lef}^G and $f_{\text{Lef}}^{\tilde{G}}$ in [26]. Strictly speaking, if we did not make our simplifying hypothesis that v_{St} splits, we could not directly invoke [26, Lemma A.7] at the end of the present proof. In general, we would also not be able to conclude that $\tilde{\pi}_{K^+,v_{\text{St}}}$ is an unramified twist of Steinberg—it could be some other twist factoring through $K_{v_{\text{St}}}^{+,\times}/N(L_{v_{\text{St}}}^{+,\times})$.

the transfer of $\prod_{w|v} f_{\tilde{w}}$ under the base-change map, which exists by [27, Theorem 3.3.1]. Then [26, Proposition 6.6] explains (with reference to [27]) that with any such choice of test functions we have the identity

$$T_{\mathrm{cusp}, \tilde{\chi}}^{\tilde{G} \cdot \sigma}(\tilde{f}) = c T_{\mathrm{cusp}, \chi}^G(f)$$

for some $c \in \mathbb{C}^\times$ (depending on \tilde{f}, f); we refer to [26] for the definition of these distributions. Varying the choice of \tilde{f}^S (i.e. the unramified test function), linear independence of characters then separates systems of Hecke eigenvalues, and as in [26, Proposition 6.6] we find

$$\sum_{\substack{\tilde{\tau} \in \mathcal{A}_{\tilde{\chi}}(G(\mathbb{A}_{L^+})) \\ \tilde{\tau}^\sigma \cong \tilde{\tau}, \tilde{\tau}^S \cong \tilde{\tau}^S}} \tilde{m}(\tilde{\tau}) \mathrm{Tr}^\sigma(\tilde{f}_S | \tilde{\tau}_S) = c \cdot \sum_{\substack{\tau \in \mathcal{A}_\chi(G(\mathbb{A}_{K^+})) \\ \mathrm{BC}_{L^+/K^+}(\tau^S) \cong \tilde{\tau}^S}} m(\tau) \mathrm{Tr}(f_S | \tau_S).$$

Here $m(\tau)$ is the multiplicity of τ in the cuspidal spectrum of $G(\mathbb{A}_{K^+})$, and $\tilde{m}(\tilde{\tau})$ is the relative multiplicity in the sense of [27, §4.4]. $\mathcal{A}_{\tilde{\chi}}(G(\mathbb{A}_{L^+}))$ is the set of isomorphism classes of cuspidal automorphic representations of $G(\mathbb{A}_{L^+})$ whose central character restricts to $\tilde{\chi}$ on $\tilde{\mathfrak{X}}$, and similarly for $\mathcal{A}_\chi(G(\mathbb{A}_{K^+}))$. We claim the left-hand side of this equality is non-zero. We first claim that any $\tilde{\tau}$ contributing a non-zero term to the left-hand side must have $\tilde{\tau}_{v_{\mathrm{St}}}$ isomorphic to an unramified twist of the Steinberg representation of $G(L_{v_{\mathrm{St}}}^+)$. If $\mathrm{Tr}^\sigma(\tilde{f}_{v_{\mathrm{St}}} | \tilde{\tau}_{v_{\mathrm{St}}})$ is non-zero, then $\tilde{\tau}_{v_{\mathrm{St}}} = \prod_{w|v_{\mathrm{St}}} \tilde{\tau}_w$ is in each component (recall that v_{St} splits in L^+) isomorphic to an unramified twist of either the trivial or the Steinberg representation.¹⁰ If one of these components $\tilde{\tau}_w$ is abelian, a standard strong approximation argument (e.g. [26, Lemma A.12]), using that $G^{\mathrm{der}}(L_w^+)$ is non-compact and G^{der} is simply-connected, implies that $\tilde{\tau}$ itself would have to be one-dimensional. This contradicts cuspidality. We conclude that for all $\tilde{\tau}$ contributing non-trivially, $\tilde{\tau}_{v_{\mathrm{St}}}$ is an unramified twist of the Steinberg representation, and moreover the (twisted) trace $\mathrm{Tr}^\sigma(\tilde{f}_{v_{\mathrm{St}}} | \tilde{\tau}_{v_{\mathrm{St}}})$ is independent of the particular $\tilde{\tau}$.

Knowing that $\tilde{\tau}_{v_{\mathrm{St}}}$ is a twist of the Steinberg representation, [26, Lemma 2.7] shows that $\tilde{\tau}$ is essentially tempered at all places. Since $\mathrm{Tr}^\sigma(\tilde{f}_\infty | \tilde{\tau}_\infty)$ is non-zero, $\tilde{\tau}_\infty$ is ξ_{L^+} -cohomological (see the discussion preceding [26, Lemma A.11]), and then as in [26, Corollary 2.8] we conclude that $\tilde{\tau}_\infty$ belongs to the discrete series L-packet associated to ξ_{L^+} . In particular, as in the v_{St} case, for all such $\tilde{\tau}$, the values $\mathrm{Tr}^\sigma(\tilde{f}_\infty | \tilde{\tau}_\infty)$ do not depend on $\tilde{\tau}$. Moreover, we conclude from [26, Theorem 12.1] that the usual automorphic multiplicity $m(\tilde{\tau})$ is 1 for all such $\tilde{\tau}$. The argument of [27, §4.4] then shows the relative multiplicity $\tilde{m}(\tilde{\tau})$ is also 1: indeed, $\tilde{\tau}$ then has a unique automorphic extension to $\tilde{G}(\mathbb{A}_{K^+}) \cdot \tilde{G}^+(K^+)$, where, as in [27], \tilde{G}^+ denotes the (not connected) reductive group $\tilde{G} \rtimes \langle \sigma \rangle$. The formulae in [27, §4.4], which are elementary calculations, show that this extension occurs with multiplicity one (for \tilde{G}^+), and that the corresponding relative multiplicity $\tilde{m}(\tilde{\tau})$ is also 1 (the polynomial denoted $C(\pi^G, \pi_0^L, X)$ in

¹⁰This is essentially noted in the proof of [27, Proposition 3.9.1], once we take into account that the only unitary subquotients of the principal series representation with the trivial representation as Langlands quotient are the Steinberg and the trivial representations. Alternatively, a twisted version of the argument in [26, Lemma A.7] shows that up to an unramified twist we may assume $\tilde{\tau}_{v_{\mathrm{St}}}$ factors through the quotient of \tilde{G} by its maximal split central torus, as a representation we will call $\tilde{\tau}_{v_{\mathrm{St}}}^{\mathrm{ad}}$. Since v_{St} splits in L^+ , this quotient is just a product of ℓ copies of the adjoint group $\mathrm{GSp}_{2n}(F_{v_{\mathrm{St}}}^+)/F_{v_{\mathrm{St}}}^{+, \times}$. In that case, [12, Proposition 3.8] shows that these are twisted Lefschetz functions, so in particular only representations with non-zero cohomology can have $\mathrm{Tr}^\sigma(\tilde{f}_{v_{\mathrm{St}}} | \tilde{\tau}_{v_{\mathrm{St}}}) \neq 0$. By [8, Theorem XI.3.9], for each $w|v_{\mathrm{St}}$ the w -component of $\tilde{\tau}_{v_{\mathrm{St}}}^{\mathrm{ad}}$ must be the trivial representation or the Steinberg representation.

Labesse's notation is identically 1). Thus, the left-hand side of our trace formula expression reduces to some non-zero constant times $\sum_{\tilde{\tau}} \text{Tr}^\sigma(\tilde{f}_{S \setminus v_{\text{St}} \cup \infty} | \tilde{\tau}_{S \setminus v_{\text{St}} \cup \infty})$, where the sum is indexed by the same collection of $\tilde{\tau}$ as before. Collecting terms with $\tilde{\tau}_{S \setminus \{v_{\text{St}} \cup \infty\}}$ in a given isomorphism class, we obtain a sum indexed over irreducible admissible representations V of $\tilde{G}(K_{S \setminus \{v_{\text{St}} \cup \infty\}}^+)$ of the form $\sum_V B_V \text{Tr}^\sigma(\tilde{f}_{S \setminus v_{\text{St}} \cup \infty} | V)$, with at least one non-zero coefficient B_V . We conclude by linear independence of characters that the test function $\tilde{f}_{S \setminus v_{\text{St}} \cup \infty}$ can be chosen to make the left-hand side of our trace formula identity non-zero.

Thus there exists $\tau \in \mathcal{A}_\chi(G(\mathbb{A}_{K^+}))$ such that $\text{BC}_{L^+/K^+}(\tau^S) \cong \tilde{\pi}^S$, and the traces $\text{Tr}(f_v | \tau_v)$ are non-zero for $v | \infty$ and $v = v_{\text{St}}$. The conclusion of the Proposition follows then from the choice of Kottwitz/Lefschetz functions at these places, by [26, Lemma A.7] (and the same strong approximation argument as above to rule out the trivial representation) and, as before, [26, Lemma 2.8]. \square

The following theorem is, for GL_N , a standard consequence of cyclic prime degree descent and the existence of automorphic Galois representations. In the absence of a strong multiplicity one theorem, however, the argument is more complicated. Before beginning the proof, we remark on some of the setup in [2] that will show up in the argument: to normalize transfer factors, Arthur uses the Whittaker normalization based on fixing a non-trivial character $\mathbb{A}_F/F \rightarrow \mathbb{C}^\times$ (for any number field F) and the standard splitting of various endoscopic groups for GL_{2n+1} . We will fix $\psi_{F^+}: \mathbb{A}_{F^+}/F^+ \rightarrow \mathbb{C}^\times$, and for extensions K^+/F^+ define $\psi_{K^+} = \psi_{F^+} \circ \text{tr}_{K^+/F^+}$, and we will let $B \subset \text{Sp}_{2n}$ denote the Borel subgroup associated with Arthur's standard splitting. Letting N denote the unipotent radical of B , ψ_{K^+} then (using the fixed splitting) induces a character $\psi_{K^+,N}: N(\mathbb{A}_{K^+})/N(K^+) \rightarrow \mathbb{C}^\times$, with local restrictions $\psi_{K_v^+,N}$. With respect to these characters we can speak of representations, local or global, being $(B, \psi_{K^+,N})$ -generic or $(B, \psi_{K_v^+,N})$ -generic.

Theorem 5.3. *Let F^+ be a totally real field, and let $r: \Gamma_{F^+} \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_p)$ be a continuous representation such that for some finite Galois extension L^+/F^+ , $r|_{\Gamma_{L^+}}$ is equivalent to $r_{\tilde{\pi},\iota}$ for some cuspidal automorphic representation $\tilde{\pi}$ of $\text{GSp}_{2n}(\mathbb{A}_{L^+})$ such that*

- (1) $\tilde{\pi}$ is L -cohomological with respect to the base-change ξ_{L^+} of an irreducible algebraic representation ξ of $\text{GSp}_{2n}(F_\infty^+)$.
- (2) At some finite place v_{St} of F^+ that splits completely in L^+/F^+ , $\tilde{\pi}_{v_{\text{St}}}$ is isomorphic to a twist of the Steinberg representation of $\text{GSp}_{2n}(L_{v_{\text{St}}}^+) = \prod_{v|v_{\text{St}}} \text{GSp}_{2n}(L_v^+)$.

Then for any intermediate field $L^+ \supset K^+ \supset F^+$ such that L^+/K^+ is a solvable Galois extension, there exists a cuspidal automorphic representation $\tilde{\pi}_{K^+}$ of $\text{GSp}_{2n}(\mathbb{A}_{K^+})$, with $\tilde{\pi}_{K^+}$ L -cohomological with respect to ξ_{K^+} and isomorphic to a twist of the Steinberg representation above v_{St} , such that $r_{\tilde{\pi}_{K^+},\iota} \cong r|_{\Gamma_{K^+}}$.

Proof. By induction, we may assume $\text{Gal}(L^+/K^+)$ is cyclic of prime degree, generated by an automorphism σ . The conjugate r^σ is obviously equivalent to r , and we would like to use this to choose the $\tilde{\pi}$ to be σ -invariant. We begin by taking some $\tilde{\pi}^1$ as in the theorem statement. Fix an automorphic constituent π^1 of $\tilde{\pi}^1|_{\text{Sp}_{2n}(\mathbb{A}_{L^+})}$. Then there are SO_{2n+1} -valued automorphic Galois representations $r_{\pi^1,\iota}$ and $r_{\pi^1,\sigma,\iota}$, and these are clearly isomorphic. Since full local-global compatibility has been established for these representations (see [26,

Theorem 2.4] for a concise summary of the known properties), we see that for all v , π_v^1 and $\pi_v^{1,\sigma}$ belong to the same (tempered, since the Ramanujan conjecture is known in these cases) local L-packet Π_{ϕ_v} , and so π^1 and $\pi^{1,\sigma}$ belong to the same global L-packet, which we will denote Π_ϕ . This does not suffice to imply that π^1 and $\pi^{1,\sigma}$ are isomorphic, but it will allow us to replace π^1 with another member of Π_ϕ that does satisfy $\pi^\sigma \cong \pi$. Namely, let π be the unique member of Π_ϕ with the property that all of the characters $\langle \cdot, \pi_v \rangle$ on \mathcal{S}_{ϕ_v} are trivial (notation as in [2]). Existence of such a π follows from on the one hand Arthur's main local theorem in the non-archimedean case, which includes the statement that Π_{ϕ_v} and $\widehat{\mathcal{S}}_{\phi_v}$ are in bijection via $\pi_v \mapsto \langle \cdot, \pi_v \rangle$; and on the other hand Shelstad's local theory in the archimedean case, which shows ([36, Theorem 11.5]) that the unique (by results of [40], [25]) $(B, \psi_{K_v^+, N})$ -generic representation in the local L-packet gives rise to the trivial character (such a result is needed, since in the archimedean case the map $\Pi_v \rightarrow \widehat{\mathcal{S}}_{\phi_v}$ is not surjective). For this π , we can deduce that $\pi^\sigma \cong \pi$. Indeed, at primes of L^+ split over K^+ this is obvious. At inert primes v we first observe that if π_v is $(B, \psi_{K_v^+, N})$ -generic, then π_v^σ is also $(B, \psi_{K_v^+, N})$ -generic, by an elementary computation and the fact that $\psi_{K^+} = \psi_{F^+} \circ \mathrm{tr}_{K^+/F^+}$ is σ -invariant. Then we invoke (see [2, Remark 2 after Proposition 8.3.2]) Arthur's deduction of the strong form of Shahidi's generic packet conjecture, namely that the tempered L-packet Π_{ϕ_v} has a unique $(B, \psi_{K_v^+, N})$ -generic member, and it corresponds to the trivial character of \mathcal{S}_{ϕ_v} under Arthur's map $\Pi_{\phi_v} \rightarrow \widehat{\mathcal{S}}_{\phi_v}$. Moreover, since the multiplicity of the packet Π_ϕ is 1, we in fact have $\pi^\sigma = \pi$.

Next we show that this invariance may be lifted to $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$. The central character $\omega_{\tilde{\pi}^1}$ of $\tilde{\pi}^1$ extends ω_{π^1} , which by Corollary 4.7 equals ω_π , so we may as before choose a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{L^+})$ containing π and with central character equal to $\omega_{\tilde{\pi}^1}$ (which we note is σ -invariant, since it corresponds to $N(r)|_{\Gamma_{L^+}}$). Moreover, we claim that we may choose $\tilde{\pi}$ to satisfy $\tilde{\pi}^\sigma \cong \tilde{\pi}$. Indeed, by [1], the local restrictions $\tilde{\pi}_v|_{\mathrm{Sp}_{2n}(L_v^+)}$ have multiplicity-free restriction, and it follows ([20, Lemma 2.4]) that for all places v , $\tilde{\pi}_v^\sigma \cong \tilde{\pi}_v \otimes \chi_v$ for some character χ_v . Then we may apply [42, Proposition 1.7(2)] to deduce that $\tilde{\pi}^\sigma \cong \tilde{\pi} \otimes \chi$ for a (finite-order) Hecke character

$$\chi: \mathrm{GSp}_{2n}(\mathbb{A}_{L^+})/\mathrm{GSp}_{2n}(L^+)Z(\mathbb{A}_{L^+})\mathrm{Sp}_{2n}(\mathbb{A}_{L^+}) \rightarrow \mathbb{C}^\times.$$

The construction of Kret-Shin applies to $\tilde{\pi}$, so we obtain $r_{\tilde{\pi}, \iota}^\sigma \cong r_{\tilde{\pi}, \iota} \otimes \chi$; we claim that $\chi = \psi^{1-\sigma}$ for some finite-order character ψ of Γ_{L^+} . Granted the claim, we have that $\tilde{\pi} \otimes \psi$ is σ -invariant. We would now like to apply Proposition 5.2 to $\tilde{\pi} \otimes \psi$, but we do not know it is an *unramified* twist of the Steinberg representation at places above v_{St} . We can get around this difficulty as follows. Suppose that for some (all, by invariance) places $v|v_{\mathrm{St}}$, $\tilde{\pi}_v \otimes \psi_v$ is isomorphic to $\mathrm{St}_v \otimes \delta_{\mathrm{un}, v} \otimes \delta_v$, where $\delta_{\mathrm{un}, v}$ is an unramified character, and $\delta_v: \Gamma_{L_v^+} \rightarrow \mathbb{C}^\times$ is a non-trivial ramified character. Since L^+/F^+ is split at v_{St} , we may regard this as a character $\delta_{v_{\mathrm{St}}}: \Gamma_{F_{v_{\mathrm{St}}}^+} \rightarrow \mathbb{C}^\times$ that restricts to δ_v for all $v|v_{\mathrm{St}}$, and we can then find a global (finite-order) character $\delta: \Gamma_{F^+} \rightarrow \mathbb{C}^\times$ such that $\delta|_{\Gamma_{F_{v_{\mathrm{St}}}^+}} = \delta_{v_{\mathrm{St}}}$. Now, $\tilde{\pi} \otimes (\psi \cdot \mathrm{BC}_{L^+/F^+}(\delta^{-1}))$ is clearly still σ -invariant and is at places above v_{St} an unramified twist of Steinberg, so we may apply Proposition 5.2 to deduce the existence of a cuspidal automorphic representation $\tilde{\pi}_{K^+}$ of $\mathrm{GSp}_{2n}(\mathbb{A}_{K^+})$ whose weak base-change is isomorphic to $\tilde{\pi} \otimes (\psi \cdot \mathrm{BC}_{L^+/F^+}(\delta^{-1}))$, and which is locally at places above v_{St} an unramified twist of Steinberg, and which is suitably cohomological at infinity. Both $r_{\tilde{\pi}^1, \iota}$ and $r_{\tilde{\pi} \otimes (\psi \cdot \mathrm{BC}_{L^+/F^+}(\delta)), \iota}$ lift $r_{\pi, \iota}$, so they are isomorphic

up to a twist by a (finite-order) character α , so we deduce that

$$r|_{\Gamma_{L^+}} \cong r_{\tilde{\pi}_{K^+, \iota}}|_{\Gamma_{L^+}} \otimes \alpha.$$

Conjugating by σ and applying [26, Proposition 5.2], we deduce that $\alpha^\sigma = \alpha$, and since L^+/K^+ is cyclic, there exists a descent α_{K^+} of α to a (finite-order) Hecke character of K^+ . Thus $r|_{\Gamma_{L^+}} \cong r_{\tilde{\pi}_{K^+} \otimes \alpha_{K^+, \iota}}|_{\Gamma_{L^+}}$, so (since the centralizer of the image of $r|_{\Gamma_{L^+}}$ is the center of GSpin_{2n_1}) $r|_{\Gamma_{K^+}} \cong r|_{\tilde{\pi}_{K^+} \otimes \alpha_{K^+, \iota}} \otimes \beta$ for some (finite-order) character $\beta: \Gamma_{K^+} \rightarrow \overline{\mathbb{Q}}_p^\times$. At last we can conclude that $r|_{\Gamma_{K^+}}$ is automorphic, isomorphic to the Galois representation associated to $\tilde{\pi}_{K^+} \otimes (\alpha_{K^+} \cdot \beta)$, and the induction can proceed.

Finally, we prove the above claim, that the character χ can in fact be written in the form $\chi = \psi^{1-\sigma}$ for some finite-order character ψ of Γ_{L^+} . When the Zariski closure of the image of $r_{\tilde{\pi}, \iota}$ (equivalently, of r) contains Spin_{2n+1} , $\mathrm{spin}(r_{\tilde{\pi}, \iota})$ is still irreducible, and the fact that there is some Hecke character $\psi: C_{L^+} \rightarrow \mathbb{C}^\times$ such that $\psi^{1-\sigma} = \chi$ follows from the argument of [29, Remark after Statement A] (that discussion is for complex representations of the Weil group; see [34, Lemma 3.3.3, Lemma 3.3.5] for the p -adic analogue, noting that the first paragraph of the proof of [34, Lemma 3.3.5] is unnecessary in our totally real context, since all of our twisting characters are finite-order). We need only supplement this conclusion with the assertion that ψ can in fact be chosen to have finite-order. By the classification of Hecke characters of the totally real field L^+ , ψ has the form $|\cdot|^w \cdot \psi_{\mathrm{fin}} \cdot \psi_{\mathrm{Maass}}$, where ψ_{fin} is finite-order, $w \in \mathbb{R}$, and ψ_{Maass} is a unitary character whose restrictions at places $v|\infty$ have the form $x_v \mapsto |x_v|^{t_v}$ for some real numbers t_v . We may of course take $w = 0$ without changing $\psi^{1-\sigma}$, so it suffices to show that we may replace ψ_{Maass} with a finite-order character. This follows from Weil's classification ([41]; see [34, §2.3.1] for a discussion) of the possible ∞ -types of Hecke characters. Namely, the existence of $\psi_{\mathrm{Maass}}: C_{L^+} \rightarrow S^1$ with the given archimedean data is equivalent to the statement that there is an integer M such that for all units $u \in \mathcal{O}_{L^+}^\times$, $\left(\prod_{v|\infty} |t_v(u)|^{t_v}\right)^M = 1$, where $t_v: L^+ \rightarrow \mathbb{R}$ is the embedding corresponding to v . Since $\psi_{\mathrm{Maass}}^{1-\sigma}$ has finite-order, for all real places v_0 of K^+ , the real numbers t_v must be independent of $v|v_0$, and we set t_{v_0} to be this common value. Then there must be a Hecke character $\psi_{\mathrm{Maass}, K^+}: C_{K^+} \rightarrow S^1$ with ∞ -type given by the $\{t_{v_0}\}_{v_0|\infty}$: it suffices to check that for all $u \in \mathcal{O}_{K^+}^\times$,

$$\left(\prod_{v_0|\infty} |t_{v_0}(u)|^{t_{v_0}}\right)^{M \cdot [L^+: F^+]} = \left(\prod_{v_0|\infty} \prod_{v|v_0} |t_{v_0}(u)|^{t_{v_0}}\right)^M = \left(\prod_{v|\infty} |t_v(u)|^{t_v}\right)^M = 1.$$

Thus we can rewrite

$$\psi^{1-\sigma} = \psi_{\mathrm{fin}}^{1-\sigma} \cdot \psi_{\mathrm{Maass}}^{1-\sigma} = \psi_{\mathrm{fin}}^{1-\sigma} \cdot (\psi_{\mathrm{Maass}} \cdot \mathrm{BC}_{L^+/K^+}(\psi_{\mathrm{Maass}, K^+}^{-1}))^{1-\sigma},$$

where now $\psi_{\mathrm{Maass}} \cdot (\psi_{\mathrm{Maass}, K^+}^{-1} \circ N_{L^+/K^+})$ is itself a finite-order character, and so we may indeed take ψ to have finite order.

Finally, we treat the other two cases (of the existence of ψ claim), in which the Zariski-closure of the image of $r_{\tilde{\pi}, \iota}$ satisfies $H^{\mathrm{der}} \subset G_{r_{\tilde{\pi}, \iota}} \subset H$, where H is either the inclusion $\mathrm{GL}_2 \hookrightarrow \mathrm{GSpin}_{2n+1}$ via the principal homomorphism or the inclusion $G_2 \times \mathbb{G}_m \hookrightarrow \mathrm{GSpin}_{2n+1}$. Then we note that since GL_2 and $G_2 \times \mathbb{G}_m$ are their own normalizers in GSpin_{2n+1} , the equivalence $r_{\tilde{\pi}, \iota}^\sigma \cong r_{\tilde{\pi}, \iota} \otimes \chi$ can be taken to be given by H -conjugation, and since in either case the center of H maps by the identity to the center of GSpin_{2n+1} , we can take the equivalence entirely as a statement about H -valued representations. Then instead of using

the spin representation, we use the standard representation (when $H = \mathrm{GL}_2$) or the faithful 7-dimensional representation (when $H = G_2 \times \mathbb{G}_m$), and the exact same argument applies to show that $\chi = \psi^{1-\sigma}$ for a finite-order character ψ .

□

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