RELATIVE DEFORMATION THEORY AND LIFTING IRREDUCIBLE GALOIS REPRESENTATIONS

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Abstract. We study irreducible odd mod p Galois representations $\bar{\rho}\colon \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{F}}_p)$, for F a totally real number field and G a general reductive group. For $p \gg_{G,F} 0$, we show that any $\bar{\rho}$ that lifts locally, and at places above p to de Rham and Hodge–Tate regular representations, has a geometric p-adic lift. We also prove non-geometric lifting results without any oddness assumption.

In memory of Jean-Pierre Wintenberger 1954–2019

1. Introduction

Let $\overline{\mathbb{Z}}_p$ be the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$, and let G be a smooth group scheme over $\overline{\mathbb{Z}}_p$ such that G^0 is a split connected reductive group.

1.1. **The lifting problem for odd representations.** The starting point of this paper is the following basic question:

Question 1.1. Let F be a number field with algebraic closure $\overline{F}$ and absolute Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, and let $\bar{\rho}\colon \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{F}}_p)$ be a continuous homomorphism. Does there exist a lift ρ

$$\begin{tikzcd} & G(\overline{\mathbb{Z}}_p) \arrow[d] \\ \Gamma_F \arrow[ur, dashed, "\rho"] \arrow[r, "\bar{\rho}"'] & G(\overline{\mathbb{F}}_p) \end{tikzcd}$$

that is geometric in the sense of Fontaine–Mazur?

This question has attracted a great deal of attention, at least since Serre proposed his modularity conjecture ([Ser87]). We begin by recalling a few instances of this general problem, beginning with Serre's conjecture. Serre proposed that every irreducible representation

$$\bar{\rho}\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

that was moreover *odd* in the sense that $\det \bar{\rho}(c) = -1$ for any complex conjugation $c \in \Gamma_{\mathbb{Q}}$ should be isomorphic to the mod p reduction of a p-adic Galois representation attached to a classical modular eigenform. In particular, such a $\bar{\rho}$ should a admit a geometric p-adic lift. Around the time Serre first made his conjecture, as recounted in a letter of Serre to Tate on 12th July, 1974 ([ST15]), Deligne raised the objection that the conjecture implied the existence of geometric lifts of $\bar{\rho}$ which were moreover minimally ramified (for example unramified outside p if $\bar{\rho}$ is unramified outside p).

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The papers [KW09a], [Kha06], [KW09b], [KW09c] proved Serre's modularity conjecture, and as a key step lifted $\bar{\rho}$ to a geometric representation with prescribed local properties. The proof of this key step in loc. cit. uses the modularity lifting results of Wiles and Taylor ([Wil95], [TW95]). In contrast, prior to the resolution of Serre's conjecture, Ramakrishna ([Ram99], [Ram02]) developed a beautiful, purely Galois-theoretic, method that in most cases settled Question 1.1 in the setting of Serre's conjecture ($F = \mathbb{Q}$, $G = \mathrm{GL}_2$, $\bar{\rho}$ odd and irreducible). Ramakrishna's lifts cannot be ensured to be minimally ramified.

We might then turn to asking Question 1.1 for $\bar{\rho}\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that are *even*, in the sense that $\det(\bar{\rho}(c)) = 1$. For instance, suppose that the image of $\bar{\rho}$ is $\mathrm{SL}_2(\mathbb{F}_p)$. Any geometric lift would (for $p \neq 2$) itself be even, and so conjecturally would be the p-adic representation ρ attached to an algebraic Maass form. Such a ρ should, up to twist, have finite image (because up to twist the associated motive should have Hodge realization of type $(0,0)$); but for $p > 5$, Dickson's classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ rules out the possibility of such a lift. Thus one expects that $\bar{\rho}$ has no geometric lift. We have no general means of translating this conjectural heuristic into a proof, but Calegari ([Cal12, Theorem 5.1]) has given an ingenious argument that proves unconditionally that certain such even $\bar{\rho}$ have no geometric lift.

In other settings, Question 1.1 is even more mysterious. For instance, if $G = \mathrm{GL}_2$ and $F/\mathbb{Q}$ is quadratic imaginary, we do not even have a reliable heuristic for predicting whether $\bar{\rho}\colon \Gamma_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ should have a geometric lift! It is a remarkable and widely-tested phenomenon that torsion cohomology (Hecke eigen-) classes for the locally symmetric spaces associated to congruence subgroups of GL_2/F need not lift to characteristic zero; one can ask whether after raising the level (passing to a finite covering space of the arithmetic 3-manifold) their Hecke eigensystems lift, and that the corresponding Galois-theoretic statement holds as well. However, we have little evidence to support this.

This paper addresses Question 1.1 for general G, but for $\bar{\rho}$ that are *odd* in a sense generalizing Serre's formulation for GL_2. The following definition is essentially due to Gross ([Gro]), who suggested parallels between this class of Galois representations and the "odd" representations of Serre's original conjecture:

Definition 1.2. We say $\bar{\rho}\colon \Gamma_F \to G(\overline{\mathbb{F}}_p)$ is odd if for all $v \mid \infty$,

$$h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = \dim(\mathrm{Flag}_{G^0}),$$

where $\mathfrak{g}^{\mathrm{der}}$ is the Lie algebra of the derived group G^{der} of G^0, and Flag_{G^0} is the flag variety of G^0.

Note that for any involution of $\mathfrak{g}^{\mathrm{der}}$, the dimension of the space of invariants must be at least $\dim(\mathrm{Flag}_{G^0})$. An adjoint group contains an order 2 element whose invariants have dimension $\dim(\mathrm{Flag}_{G^0})$ if and only if -1 belongs to the Weyl group of G. When -1 does not belong to the Weyl group, we can (after choosing a pinning) find such an order two element in $G \rtimes \mathrm{Out}(G)$; for more details, see [Pat16, §4.5, §10.1]. Also note that the definition implies that F is totally real. That said, the "odd" case does have implications in certain CM settings. For example, let F be quadratic imaginary, and let $\bar{\rho}\colon \Gamma_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible representation such that

$$\bar{\rho}^c \cong \bar{\rho}^\vee \otimes \mu|_{\Gamma_F},$$

where $\mu\colon \Gamma_{\mathbb{Q}} \to \overline{\mathbb{F}}_p^\times$ is a character. Moreover assume that when we realize this essential conjugate self-duality as a relation

$$\bar{\rho}(cgc^{-1}) = A^t\bar{\rho}(g)^{-1}A^{-1}\mu(g)$$

for some $A \in \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ (and all $g \in \Gamma_F$), the scalar $A \cdot {}^t A^{-1}$ (which is easily seen to be ± 1) actually equals $+1$. Then the pair $(\bar{\rho}, \mu)$ can be extended to a homomorphism

$$\bar{r} \colon \Gamma_{\mathbb{Q}} \to (\mathrm{GL}_n \times \mathrm{GL}_1)(\overline{\mathbb{F}}_p) \rtimes \{1, j\},$$

where $j^2 = 1$ and $j(g, a)j^{-1} = (a \cdot {}^t g^{-1}, a)$, and this $\bar{r}$ is odd in the sense of Definition 1.2.

There are essentially two techniques for approaching cases of Question 1.1. For classical groups, automorphy lifting and potential automorphy theorems, via a technique introduced in [KW09a], yield the most robust results. For instance, the strongest lifting results in the previous example ($\bar{\rho}$ essentially conjugate self-dual over a quadratic imaginary field) follow from the work of Barnet-Lamb, Gee, Geraghty, and Taylor ([BLGGT14]). For general G, however, we do not have a good understanding of automorphic Galois representations, and we must rely on purely Galois-theoretic methods. Ramakrishna developed the first such method in the papers [Ram99] and [Ram02], which, as noted above, resolved Question 1.1 in the setting of Serre's original modularity conjecture ($F = \mathbb{Q}$, $G = \mathrm{GL}_2$, $\bar{\rho}$ odd and irreducible). Our work relies on the methods of [Ram99] and [Ram02], particularly as extended in the "doubling method" of [KLR05] and the work of Hamblen and Ramakrishna ([HR08]).

From now on, we will replace $\overline{\mathbb{Z}}_p$ (resp. $\overline{\mathbb{F}}_p$) by the ring of integers $\mathcal{O}$ in some finite extension of $\mathbb{Q}_p$ (resp. the residue field k of $\mathcal{O}$). We let ϖ denote a uniformizer of $\mathcal{O}$ and $\mathfrak{m} = (\varpi)$ the maximal ideal of $\mathcal{O}$. Thus we now take G as before but defined over $\mathcal{O}$, and we study continuous homomorphisms $\bar{\rho} \colon \Gamma_{F,S} \to G(k)$, where S is a finite set of primes containing those above p, and $\Gamma_{F,S}$ denotes $\mathrm{Gal}(F(S)/F)$ for $F(S)$ the maximal (inside $\overline{F}$) Galois extension of F that is unramified away from S.

There are several difficulties in extending the method of [Ram02] to lifting odd irreducible representations to $G(k)$ for general groups:

- In the arguments of [Ram02] one must construct at all primes v at which $\bar{\rho}$ is ramified a formally smooth irreducible component of the local lifting ring $R^{\square}_{\bar{\rho}|_{\Gamma_{F_v}}}$ (for v not above p) or a formally smooth component of the lifting ring that parametrizes lifts of a fixed (Hodge–Tate regular) p-adic Hodge type (for v above p). Such components do not always exist in the level of generality in which we work.
- The lack of smooth components as above necessitates working with more general (typically ramified) coefficients $\mathcal{O}$, while in loc. cit. one can work with the ring of Witt vectors $W(k)$. This causes complications related to the fact that if $\mathcal{O}$ is ramified, $\mathcal{O}/\mathfrak{m}^2$ is of characteristic p and hence isomorphic to the dual numbers $k \oplus k[\varepsilon]$.
- The auxiliary prime arguments of [Ram02] break down as the image of $\bar{\rho}$ gets smaller. For general G, where many possible images can still lead to "irreducible" $\bar{\rho}$, this is a basic difficulty.

These difficulties are not as serious an impediment for $G = \mathrm{GL}_2$ as compared to the case of general G. In [Ram02], under mild hypotheses on $\bar{\rho}|_{\Gamma_{\mathbb{Q}_p}}$, the necessary local theory is worked out (we should note, however, that particularly at the prime p the situation is here considerably simplified by working over $\mathbb{Q}_p$ rather than a ramified extension). As for the global hypotheses, by a theorem of Dickson any irreducible subgroup of $\mathrm{GL}_2(k)$ (for $p \geq 7$) either has order prime to p, in which case one can take the "Teichmüller" lift, or has projective image conjugate to a subgroup of the form $\mathrm{PSL}_2(k')$ or $\mathrm{PGL}_2(k')$ for some subfield k' of k. This allows Ramakrishna to restrict to the case where the adjoint representation $\mathrm{ad}^0(\bar{\rho})$ is absolutely irreducible.

For higher-rank G, the global arguments of [Ram02] work with little change under the corresponding assumption that the adjoint representation $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ (this will be our notation for the Galois module $\mathfrak{g}^{\mathrm{der}}$, equipped with the action of Γ_F via $\mathrm{Ad} \circ \bar{\rho}$) is absolutely irreducible. Such a generalization is carried out in [Pat16]. The same paper also proves a variant with somewhat smaller image, in which $\mathrm{im}(\bar{\rho})$ contains (approximately) $\varphi(\mathrm{SL}_2(k))$, where $\varphi\colon \mathrm{SL}_2 \to G$ is a principal SL_2. In this case $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ decomposes into r irreducible factors, where r is the semisimple rank of G, and the final result depended on an explicit analysis of this decomposition, requiring case-by-case calculations depending on the Dynkin type, with the result only verified for the exceptional groups via a computer calculation. More seriously, the method did not apply to groups of type D_{2m}, for which $\mathfrak{g}^{\mathrm{der}}$ is not multiplicity-free as an SL_2-module (one factor occurs with multiplicity two). Some other instructive examples of how variants of the familiar Ramakrishna arguments still fail to treat relatively simple images can be found in [Tan18].

1.2. **Main theorem.** Before explaining how we overcome the difficulties mentioned above, we will state the main theorem. From now on we will require of G that the component group $\pi_0(G)$ is finite étale of order prime to p.

Theorem A (See Theorem 6.9)**.** *Let $p \gg_G 0$ be a prime. Let F be a totally real field, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a continuous representation unramified outside a finite set of finite places S containing the places above p. Let $\widetilde{F}$ denote the smallest extension of F such that $\bar{\rho}(\Gamma_{\widetilde{F}})$ is contained in $G^0(k)$, and assume that $[\widetilde{F}(\zeta_p) : \widetilde{F}]$ is strictly greater than the integer a_G arising in Lemma A.6 (which depends only on the root datum of G). Fix a geometric lift $\mu\colon \Gamma_{F,S} \to G/G^{\mathrm{der}}(\mathcal{O})$ of $\bar{\mu} := \bar{\rho} \pmod{G^{\mathrm{der}}}$, and assume that $\bar{\rho}$ satisfies the following:*

- *$\bar{\rho}$ is odd, i.e. for all infinite places v of F, $h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = \dim(\mathrm{Flag}_{G^{\mathrm{der}}})$.*
- *$\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible.*
- *For all $v \in S$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift $\rho_v\colon \Gamma_{F_v} \to G(\mathcal{O})$ of type $\mu|_{\Gamma_{F_v}}$; and that for $v \mid p$ this lift may be chosen to be de Rham and regular in the sense that the associated Hodge–Tate cocharacters are regular.*

Then there exist a finite extension K' of $K = \mathrm{Frac}(\mathcal{O})$ (whose ring of integers and residue field we denote by $\mathcal{O}'$ and k') depending only on the set $\{\rho_v\}_{v \in S}$; a finite set of places $\widetilde{S}$ containing S; and a geometric lift

$$\begin{tikzcd} & G(\mathcal{O}') \arrow[d] \\ \Gamma_{F,\widetilde{S}} \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & G(k') \end{tikzcd}$$

of $\bar{\rho}$ such that $\rho(\Gamma_{F,\widetilde{S}})$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}')$. Moreover, if we fix an integer t and for each $v \in S$ an irreducible component defined over $\mathcal{O}$ and containing ρ_v of:

- *for $v \in S \setminus \{v \mid p\}$, the generic fiber of the local lifting ring, $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}[\frac{1}{\varpi}]$ (where $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$ pro-represents $\mathrm{Lift}_{\bar{\rho}|_{\Gamma_{F_v}}}$); and*
- *for $v \mid p$, the lifting ring $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ whose $\overline{K}$-points parametrize lifts of $\bar{\rho}|_{\Gamma_{F_v}}$ with specified Hodge type $\mathbf{v}$ (see* [Bal12, Prop. 3.0.12] *for the construction of this ring);*

then the global lift ρ may be constructed such that, for all $v \in S$, $\rho|_{\Gamma_{F_v}}$ is congruent modulo ϖ^t to some $\widehat{G}(\mathcal{O}')$-conjugate of ρ_v, and $\rho|_{\Gamma_{F_v}}$ belongs to the specified irreducible component for every $v \in S$.

Thus we prove an essentially complete "local-to-global" principle for finding geometric lifts of irreducible odd Galois representations. We make a few remarks:

Remark 1.3.

- The arguments proceed from a somewhat different global image assumption—see Assumption 5.1—but for $p \gg_{G,\widetilde{F}} 0$ the absolute irreducibility hypothesis implies the other conditions in Assumption 5.1 (see Corollary A.7).
- The method also allows us to lift a representation $\rho_n \colon \Gamma_{F,S} \to G(\mathcal{O}/\varpi^n)$ to a geometric representation, provided that (in addition to the global hypotheses on $\bar{\rho}$) the restrictions $\rho_n|_{\Gamma_{F_v}}$ for $v \in S$ have $G(\mathcal{O})$-lifts ρ_v as in the theorem statement.
- The bound on p can be made effective: for detailed remarks, see Remark 6.15.
- In §7 we give some examples of the theorem.
- All the lifts ρ produced by the theorem have image containing $\widehat{G^{\mathrm{der}}}(\mathcal{O}')$; that is, we find lifts whose image is "as large as possible" subject to the given $\mathrm{im}(\bar{\rho})$. Thus, even in a setting where an obvious "Teichmüller" lift exists (i.e., when $|\mathrm{im}(\bar{\rho})|$ is coprime to p), we produce very different sorts of lifts, producing congruences between finite-image and "full" image Galois representations.
- For more detailed remarks on how, for $G^0 = \mathrm{GL}_n$, this theorem compares to results coming from potential automorphy theorems, see Remark 6.17.

We mention two variants of the theorem that are straightforward given our techniques. The first (see Theorem 6.19) is a non-geometric but finitely-ramified lifting theorem for $\bar{\rho}$ without any constraints on the action of Γ_{F_v} for $v \mid \infty$ (and in particular allowing F to be any number field); this holds under the same image hypotheses as Theorem 1.2. The second produces possibly de Rham but infinitely-ramified lifts, generalizing the main theorem of [KLR05] from the case $G = \mathrm{GL}_2$ and $\mathrm{SL}_2(\mathbb{F}_p) \subset \mathrm{im}(\bar{\rho})$.

Theorem B (See Corollary 5.13)**.** *Let F be any number field. Assume $p \gg_G 0$, and let $\bar{\rho} \colon \Gamma_{F,S} \to G(k)$ be a representation such that $\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible and $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$ (see Lemma A.6). Fix a lift μ of $\bar{\rho}$ $\pmod{G^{\mathrm{der}}}$ as in Theorem 1.2. Assume that for all $v \in S$, there are lifts $\rho_v \colon \Gamma_{F_v} \to G(\mathcal{O})$ of $\bar{\rho}|_{\Gamma_{F_v}}$ with multiplier μ. Then there exists an infinitely ramified lift*

$$\begin{tikzcd} & G(\mathcal{O}) \arrow[d] \\ \Gamma_F \arrow[r, "\bar{\rho}"'] \arrow[ur, dashed, "\rho"] & G(k) \end{tikzcd}$$

such that $\rho|_{\Gamma_{F_v}}$ is $\widehat{G^{\mathrm{der}}}(\mathcal{O})$-conjugate to ρ_v for all $v \in S$, and $\rho(\Gamma_F)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O})$.

1.3. **Strategy of proof, the doubling method and relative deformation theory.**

The method of Hamblen-Ramakrishna. We first briefly recall the original technique of Ramakrishna (see [Ram02] and [Tay03]). Under the oddness hypothesis, one defines a global Galois deformation problem by imposing formally smooth local deformation conditions on the restriction of $\bar{\rho}$ to primes in S, and whose associated (mod p) Selmer and dual Selmer groups have the same dimension (we will informally say that Selmer and dual Selmer are "balanced"). In this setting an application of the Selmer group variant of the Poitou–Tate sequence (see [Tay03, Lemma 1.1]) implies that if the dual Selmer group vanishes, then the corresponding universal deformation ring is

$\mathcal{O}$ and therefore gives rise to a geometric deformation of $\bar{\rho}$. The task, then, is to allow ramification at a set Q of auxiliary primes such that:

- the allowed ramification at each $q \in Q$ is a formally smooth local condition;
- the conditions at $q \in Q$ have large enough tangent space that the resulting new Selmer and dual Selmer groups remain "balanced" as we add each $q \in Q$; and
- when we have allowed the entire auxiliary set Q of ramification, the dual Selmer group, hence also the Selmer group, vanishes.

In the case $F = \mathbb{Q}$, Ramakrishna takes a Steinberg local condition at primes $q \not\equiv 1 \pmod p$ at which $\bar{\rho}$ is unramified with distinct Frobenius eigenvalues with ratio q. By comparing splitting conditions on Selmer and dual Selmer classes, he shows (when the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(k)$) that such q can be chosen that inductively decrease the size of Selmer and dual Selmer.

We refer to the process of starting with a Selmer group $H^1_{\mathcal{L}}(\Gamma_{F,S}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ (resp. dual Selmer group $H^1_{\mathcal{L}^{\perp}}(\Gamma_{F,S}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$), with local conditions defined by subspaces $L_v \subset H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, $v \in S$, and finding a finite set of places Q with subspaces $L_v \subset H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, $v \in Q$, such that $H^1_{\mathcal{L}}(\Gamma_{F,S\cup Q}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ (resp. $H^1_{\mathcal{L}^{\perp}}(\Gamma_{F,S\cup Q}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$) is 0, as killing mod p Selmer (resp. dual Selmer). We call these groups *intrinsic* mod p (dual) Selmer groups as opposed to *relative* (extrinsic) versions of them that we define later.

In higher rank, it is better to think of the GL_2 "Steinberg" condition as allowing unipotent ramification in the direction of a fixed root space and constraining Frobenius to act by the cyclotomic character on this root space, since a key point in controlling the Selmer and dual Selmer groups simultaneously is that the Ramakrishna deformation condition should intersect the unramified condition in a codimension one subspace. At each step of the inductive argument that decreases the size of the Selmer groups, one has to, given non-zero Selmer and dual Selmer classes, be able to choose this root space in a suitably general position with respect to the images of cocycles representing these classes. This is not always possible when (as in [Pat16]) the auxiliary primes are chosen so that Frobenius acts by a regular semisimple element. To overcome this, we follow the path taken in [HR08], and generalize the notion of *trivial primes* from the work of Hamblen and Ramakrishna ([HR08]) to the present context. Thus, we use auxiliary primes q having the one behavior we are guaranteed to find in the image of any representation, namely, that $\bar{\rho}|_{\Gamma_{F_q}}$ is trivial; note that as $\bar{\rho}(\mathrm{Frob}_q)$ is then contained in *every* maximal torus of G, we win a great deal of flexibility in the choice of root space in which to allow ramification (contrast the condition (6) in [Pat16, §5] with our Proposition 6.7).

Hamblen and Ramakrishna show how to deform a *reducible* but indecomposable representation $\Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ to an irreducible representation over $W(k)$ by allowing Steinberg-type ramification at primes q such that $q \equiv 1 \pmod p$, $q \not\equiv 1 \pmod{p^2}$, and $\bar{\rho}|_{\Gamma_{\mathbb{Q}_q}}$ is trivial. The resulting local condition on lifts of $\bar{\rho}|_{\Gamma_{\mathbb{Q}_q}}$ is liftable but is not representable, the latter point being reflected in the fact that the local condition behaves very differently modulo different powers of p: while its tangent space is "too small" for the global applications, certain lifts mod p^m for $m \geq 3$ do indeed witness that the condition is coming from a sufficiently large characteristic zero condition (see Lemma 3.5 for a precise formulation of this distinction). The consequence of this distinction is that the global argument must treat separately the problems of lifting $\bar{\rho}$ to a mod p^3 representation and lifting it modulo higher powers of p. Because of the generality in which we work, and the demands of the relative deformation theory argument we have to adopt, we need a more elaborate version that separates the two problems of lifting mod ϖ^N for some $N \gg 0$ and lifting beyond mod ϖ^N.

The lifting method of this paper. Now we come to the main technical innovations in the paper. We do not spell out here the local conditions at the places where we allow our representations to ramify (for which see §3), but instead concentrate on the shape of the global arguments that are novel to our work. (The notation used here is lighter, less accurate, and not identical to what is used in the main text.)

We find it convenient first to make a couple of definitions. Let v be a finite place of F and $\rho_M : \Gamma_{F_v} \to G(\mathcal{O}/\varpi^M)$ be a lifting of a residual representation $\bar{\rho} : \Gamma_{F_v} \to G(k)$. For any $G(\mathcal{O}/\varpi^r)$-valued homomorphism ρ_r of a local or global Galois group, $\rho_r(\mathfrak{g}^{\mathrm{der}})$ will denote $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r$ equipped with the $\mathrm{Ad} \circ \rho_r$ action.

Definition 1.4. We say that an $\mathcal{O}$-submodule $L_{M,v} \subset H^1(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ is *balanced* if

$$|L_{M,v}| = \begin{cases} |\rho_M(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F,v}}| & \text{if } v \nmid p; \\ |\rho_M(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F,v}}| \cdot |\mathcal{O}/\mathfrak{m}^M|^{\dim_k(\mathfrak{n})[F:\mathbb{Q}_p]} & \text{if } v \mid p. \end{cases}$$

Here $\mathfrak{n}$ is the Lie algebra of the unipotent radical of a Borel subgroup of G.

Definition 1.5. Given positive integers $M \leq m$, and a place v of F, we say that, for $n \geq m$, a pair of representations (ρ_n, ρ_{n+M}), with $\rho_n : \Gamma_{F_v} \to G(\mathcal{O}/\varpi^n)$, $\rho_{n+M} : \Gamma_{F_v} \to G(\mathcal{O}/\varpi^{n+M})$, and ρ_{n+M} reducing to ρ_n modulo ϖ^n, is in *relative good position* with respect to the data $(\{\mathrm{Lift}_v(\mathcal{O}/\varpi^n)\}_{n\geq m}, L_{M,v})$ if:

- For all $n \geq m$, $\mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ is a set of lifts of $\bar{\rho}$ such that reduction induces a surjective map $\mathrm{Lift}_v(\mathcal{O}/\varpi^{n+M}) \to \mathrm{Lift}_v(\mathcal{O}/\varpi^n)$;
- ρ_n and ρ_{n+M} belong to $\mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ and $\mathrm{Lift}_v(\mathcal{O}/\varpi^{n+M})$;
- $L_{M,v} \subset H^1(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ is an $\mathcal{O}$-submodule, and the fibers of the map $\mathrm{Lift}_v(\mathcal{O}/\varpi^{n+M}) \to \mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ are stable under the natural action of the preimage of $L_{M,v}$ in the space of one-cocycles;
- $L_{M,v}$ is balanced.

If the data $(\{\mathrm{Lift}_v(\mathcal{O}/\varpi^n)\}_{n\geq m}, L_{M,v})$ is understood, then we simply say that the pair of representations (ρ_n, ρ_{n+M}) is in relative good position.

Our results as in Theorem 1.2 have a local-global flavor, and for simplicity in this section we assume that the $\mathcal{O}$-valued lifts ρ_v for $v \in S$ that we interpolate (mod ϖ^t) are smooth points of the corresponding local framed deformation ring. (In this case our methods prove the existence of geometric lifts of $\bar{\rho}$ that are themseleves $\mathcal{O}$-valued, without the need to make a finite extension.) By a lemma that we deduce from a result of Serre (cf. Lemma 4.3), given a positive integer M (specified in advance, and in practice for us coming from the global setup), there exists an $m \gg 0$, in particular $m \geq \max(t, M)$, and an open neighborhood of the lift ρ_v (viewed as a point in the generic fibre of a suitable framed deformation ring), enabling one for all $n \geq m$ to specify sets of lifts $\mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ that reduce to $\rho_v \pmod{\varpi^m}$, and such that the fibers of the surjective maps $\mathrm{Lift}_v(\mathcal{O}/\varpi^{n+r}) \to \mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ for all $r \leq M$ are stable under an $\mathcal{O}$-submodule of cocycles in $Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ that contains all coboundaries, and whose image in $H^1(\Gamma_v, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is balanced (cf. Definition 1.4). This purely local condition implies that the corresponding Selmer groups $H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ have the same cardinality as $H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, (\rho_r(\mathfrak{g}^{\mathrm{der}}))^*)$. Furthermore the lifts in (the non-empty set) $\varprojlim \mathrm{Lift}_v(\mathcal{O}/\varpi^m)$ lie in the same component of the local framed deformation ring as ρ_v (for $v \in S$).

As in [HR08], our work takes place in two steps with different flavors.

Step 1: Mod ϖ^N liftings using the doubling method of [KLR05]**.** Let N be a positive integer, which we will choose sufficiently large compared to an integer M coming from the global situation (namely, $\mathrm{im}(\bar{\rho})$) and compared to local information (the fixed lifts ρ_v, $v \in S$, and the local geometry of local lifting rings around the ρ_v, as just discussed). Using a refinement of the doubling method which was introduced in [KLR05], we lift $\bar{\rho}$ to a $\rho_N \colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$ that at places $v \in S$ equals (up to strict equivalence) the given liftings $\rho_v \pmod{\varpi^t}$. In doing so we have to enlarge the set S to a finite set of places $S' \supset S$ by allowing ramification at an auxiliary set of primes (cf. Definition 3.4 and Lemma 3.5), which are again a generalization of the trivial primes of Hamblen–Ramakrishna, and we have to specify a class of liftings $\mathrm{Lift}_v(\mathcal{O}/\varpi^n)$, for $n \geq N - M$, $v \in S'$, such that $\rho_N|_{\Gamma_{F_v}}$ is such a lifting, and such that for all $v \in S'$ and $1 \leq r \leq M$, the fibers of the surjective map $\mathrm{Lift}_v(\mathcal{O}/\varpi^{n+r}) \to \mathrm{Lift}_v(\mathcal{O}/\varpi^n)$ are stable under a set of cocycles in $Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ whose image $L_{r,v}$ in $H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is balanced.

The construction of the lift $\rho_N \colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$ is achieved by a generalization of the methods of [KLR05] and [HR08], new arguments being needed to handle general $\mathrm{im}(\bar{\rho})$. In §5.1, we start with any mod ϖ^2 lift ρ_2 of $\bar{\rho}$ (easily seen to exist after enlarging S by a set of trivial primes); to further lift it we modify ρ_2 so that its local restrictions at primes of ramification match certain specified local lifts (namely, those coming from the reductions of the $G(\mathcal{O})$-lifts ρ_v assumed to exist in Theorem 1.2). This leads to the following question: given local cohomology classes $z_T = (z_w)_{w \in T} \in \bigoplus_{w \in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ (here T will be a finite set of primes containing the original set S of ramification), can we find a global class $h \in H^1(\Gamma_{F,T}, (\mathfrak{g}^{\mathrm{der}}))$ such that $h|_{\Gamma_{F_w}} = z_w$ for all $w \in T$?

The answer is no, so we aim for the next best thing: to enlarge T to a finite set $T \cup U$, and to find a class $h^U \in H^1(\Gamma_{F,T\cup U}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $h^U|_T = z_T$. This would allow us to modify ρ_2 to some $(1 + \varpi h^U)\rho_2$ that is well-behaved at primes in T. The problem here is that we sacrifice control at the primes in U, and this necessitates the use of an idea from [KLR05] (as exploited in a simpler setting by [HR08]), which we will refer to as the "doubling method": roughly speaking, we consider two such sets U and U', with corresponding cocycles h^U and $h^{U'}$. By considering all possibilities $(1 + \varpi(2h^U - h^{U'}))\rho_2$ as U and U' vary (each through Čebotarev multi-sets of primes), we show by a limiting argument that there is a pair (not, in fact, a Čebotarev set!) of U and U' such that $\rho_2' = (1 + \varpi(2h^U - h^{U'}))\rho_2$ both has the desired behavior at T and is under enough control at U and U' (the detailed desiderata come out of Lemma 3.5) for subsequent steps in the lifting argument. In §5.2 we run a more complicated version of this argument, iterating it to produce the desired lift modulo ϖ^N. In both of these arguments, handling the case of general $\mathrm{im}(\bar{\rho})$ poses a significant challenge beyond the GL_2 arguments of [KLR05] and [HR08]; in particular, handling multiplicities in the $\mathbb{F}_p[\Gamma_F]$-decomposition of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ requires new arguments.

Step 2: Relative deformation theory. Having "risen above" the singularities of the local deformation rings by lifting $\bar{\rho}$ to $\rho_N \colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$ for $N \gg 0$, we would like to lift ρ_N to a geometric lift following the methods of Hamblen–Ramakrishna [HR08] (who only needed to consider $N = 2$). We run into the problem that we may not be able to kill the dual Selmer group $H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S'}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ for any $1 \leq r \leq M$: the classes coming from inflation from $H^1(\mathrm{im}(\rho_N), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ cannot be killed using "trivial primes" that are good for ρ_N (as in Definition 3.4). Note that when $\mathcal{O}$ is ramified, the group $H^1(\mathrm{im}(\rho_N), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ does not vanish even for $N = 2$ (as $\mathcal{O}/\varpi^2$ is isomorphic to the dual numbers $k \oplus k[\varepsilon]$ as alluded to earlier). It also does not vanish for certain choices of $\mathrm{im}(\bar{\rho})$ even when $\mathcal{O}$ is unramified (cf. [FKP18, Example 5.5]).

One of our main observations at this stage is that we can still kill, for an appropriate choice of M and for a choice of a finite set of "trivial primes" Q, the relative mod p dual Selmer group

$\overline{H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)}$, namely the image of the map $H^1_{\mathcal{L}_M{}^\perp}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_1^\perp}(\Gamma_{F,S'\cup Q}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$ (cf. Definition 6.2). Here the choice of M is dictated by having to ensure that the problematic classes vanish relatively, namely the image of $H^1(\mathrm{im}(\rho_N), \rho_M(\mathfrak{g}^{\mathrm{der}})) \to H^1(\mathrm{im}(\rho_N), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ is 0 (cf. Lemma 6.4) for $N \geq M$. This is deduced from well-known results on vanishing of cohomology of semi-simple Lie algebras and some results of Lazard on cohomology of p-adic Lie groups (cf. Lemma B.2). After this, a crucial ingredient of the proof of the killing of relative (dual) Selmer (cf. Proposition 6.7 and Theorem 6.8), beyond the versatility of trivial primes for killing cohomology classes, is that the relative mod p Selmer and dual Selmer are balanced under the assumption that $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ have no global invariants (cf. Lemma 6.3).

Because of our inability to kill intrinsic mod p dual Selmer, and only being able to kill a relative version of it, after allowing ramification at a finite set of trivial primes Q which have preseribed properties in ρ_N, we have to consider a more elaborate deformation problem. Thus instead of directly lifting ρ_N, we now exploit the fact that the lifting ρ_N produced in Step 1, and the set Q used to kill relative dual Selmer, is such that the pair (ρ_{N-M}, ρ_N) is in "relative good position" (cf. Definition 1.5) at all places in $S' \cup Q$, including the places in S (this requires having chosen $N \gg 0$ relative both to M and to the bounds coming from our local analysis). We construct pairs of representations (τ_n, ρ_{n+M}) in relative good position at all places $v \in S' \cup Q$, starting for $n = N - M$ with the pair (ρ_{N-M}, ρ_N). Namely, for each $n \geq N - M$, we inductively construct pairs of (fixed multiplier μ) liftings (τ_n, ρ_{n+M}), where $\tau_n \colon \Gamma_{F,S'\cup Q} \to G(\mathcal{O}/\varpi^n)$ and $\rho_{n+M} \colon \Gamma_{F,S'\cup Q} \to G(\mathcal{O}/\varpi^{n+M})$, with the following properties:

(1) For each $v \in S' \cup Q$, $\tau_n|_{\Gamma_{F_v}}$ belongs to $\mathrm{Lift}_v(\mathcal{O}/\varpi^n)$, and $\rho_{n+M}|_{\Gamma_{F_v}}$ belongs to $\mathrm{Lift}_v(\mathcal{O}/\varpi^{M+n})$
(2) $\tau_{n+1} = \rho_{n+M} \pmod{\varpi^{n+1}}$.
(3) $\tau_n = \tau_{n+1} \pmod{\varpi^n}$.
(4) (τ_n, ρ_{n+M}) are in relatively good position at all places $v \in S' \cup Q$.

$$\begin{tikzcd}
(\tau_{N-M}, & \rho_N) \arrow[dl, Rightarrow] \\
(\tau_{N+1-M}, & \rho_{N+1}) \\
\cdots & \\
(\tau_n, & \rho_{n+M}) \arrow[dl, Rightarrow] \\
(\tau_{n+1} & \rho_{n+1+M}) \\
\cdots &
\end{tikzcd}$$

The ρ_{n+M} may not be compatible as we increase n, but as the τ_n's are we get our desired geometric lift (of ρ_{N-M} and hence of $\bar{\rho}$) by setting $\rho = \varprojlim \tau_n \colon \Gamma_{F,S'\cup Q} \to G(\mathcal{O})$.

To carry out the inductive step we use the vanishing of the relative dual Selmer and the "diagram of relative deformation theory" which arises by comparing the Poitou–Tate sequence for $\rho_M(\mathfrak{g}^{\mathrm{der}})$ and $\rho_{M-1}(\mathfrak{g}^{\mathrm{der}})$ coefficients:

$$\begin{array}{ccccc} H^1(\Gamma_{S'\cup Q},\rho_M(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & \bigoplus_{v\in S'\cup Q}\frac{H^1(\Gamma_{F_v},\,\rho_M(\mathfrak{g}^{\mathrm{der}}))}{L_{M,v}} & \longrightarrow & H^1_{\mathcal{L}_M^{\perp}\cup\{L_{M,v}^{\perp}\}_{v\in Q}}(\Gamma_{S'\cup Q},\rho_M(\mathfrak{g}^{\mathrm{der}})^*)^{\vee} \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\Gamma_{S'\cup Q},\rho_{M-1}(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & \bigoplus_{v\in S'\cup Q}\frac{H^1(\Gamma_{F_v},\,\rho_{M-1}(\mathfrak{g}^{\mathrm{der}}))}{L_{M-1,v}} & \longrightarrow & H^1_{\mathcal{L}_{M-1}^{\perp}\cup\{L_{M-1,v}^{\perp}\}_{v\in Q}}(\Gamma_{S'\cup Q},\rho_{M-1}(\mathfrak{g}^{\mathrm{der}})^*)^{\vee} \end{array}$$

in which the rows come from (a part) of the Poitou–Tate exact sequence, and the vertical arrows are induced by the reduction map $\rho_M(\mathfrak{g}^{\mathrm{der}}) \to \rho_{M-1}(\mathfrak{g}^{\mathrm{der}})$ (with the third vertical arrow arising from dualizing twice), cf. §6 and Theorem 6.9 (especially Claim 6.12).

Remark 1.6.

- Thus the arguments are relative in two different, albeit related, aspects and apply once we have lifted $\bar{\rho}$ to ρ_N for $N \gg 0$.
  - The lifting arguments are relative to ρ_N. In fact we lift ρ_{N-M} for suitable $0 \ll M \ll N$. This allows us to use only generic smoothness of generic fibres of local deformation rings.
  - The lifting problem is relative in that we lift pairs of representations. This allows us to get away with killing relative mod p dual Selmer groups rather than intrinsic mod p dual Selmer groups.
- The obstacle that relative deformation theory helps overcome is that given $\rho_N : \Gamma_{F,S} \to G(\mathcal{O}/\varpi^N)$, imposing on a place v that $\rho_N(\mathrm{Frob}_v)$ has a prescribed shape and that a given class $\phi \in H^1_{\mathcal{L}_M}(\Gamma_{F,S}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ does not lie locally in a prescribed subspace $L_v \subset H^1(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ might be incompatible. The incompatibility, essentially a lack of linear disjointness issue, arises from the classes that are inflated from $H^1(\mathrm{im}(\rho_N), \rho_M(\mathfrak{g}^{\mathrm{der}}))$. This issue has been encountered before, for example in [Kha04] and [KR15].

  In the paper [KR15] this issue is overcome by a different route which works under an ordinarity assumption (and certain smoothness assumptions). In [KR15], ρ_N is first lifted to an ordinary, possibly non-geometric, representation by making the global, ordinary at places above p, deformation ring unobstructed. As a second step, [KR15] uses the smoothness of the global ordinary deformation ring to massage the non-geometric lift to a geometric one (using the ideas of [KLR05]). The relative deformation argument of this paper deals with the lack of linear disjointness alluded to above by a more direct route without relaxing the Selmer conditions at places above p.
- G. Boeckle has lifted odd irreducible representations $\bar{\rho} : \Gamma_{\mathbb{Q}} \to GL_2(k)$ to geometric representations that are ramified at a finite auxiliary set in a manuscript that is of a few years' vintage. In this work he anticipates using generic smoothness of generic fibers of local deformation rings, rather than finer information like their formal smoothness, when lifting Galois representations using the techniques of [HR08].
- In this work we use the vanishing of relative mod p dual Selmer to prove lifting results for residual Galois representations. In forthcoming work with Jack Thorne, one of us (C.K.) will use the vanishing of relative mod p Selmer to prove automorphy of geometric representations ρ relative to automorphy of their mod ϖ^N cutoffs for $N \gg 0$. The vanishing of

certain relative mod p Selmer groups $\overline{H^1_{\mathcal{L}_M}(\Gamma_{F,S\cup Q_n}, \rho_M(\mathfrak{g}^{\mathrm{der}}))}$ ensures infinitesimal uniqueness of certain lifts of ρ_n for $n \geq N$ with specified local properties at a finite set of places $S \cup Q_n$.

- In an earlier version of this paper ([FKP18], available at `arXiv:1810.05803v1`), we could deal with the non-vanishing of $H^1(\mathrm{im}(\rho_2), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ (i.e., the lack of linear disjointness alluded to above) for ρ_2 a $W_2(k)$-valued lift of $\bar{\rho}$ (essentially) only when the adjoint representation $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ was multiplicity free as $\mathbb{F}_p[\Gamma_F]$-module, with the help of a group-theoretic argument we owe to Larsen. This has been obviated in the present approach. The present paper supersedes the older version [FKP18]. The older version is no longer intended for publication in a journal, but will be available on the arXiv. Some of the arguments of [FKP18] might prove useful, for example in killing mod p intrinsic (rather than relative) Selmer groups in certain situations.

Here is a summary of the contents of the paper. In §3 and §4 we carry out the necessary preliminaries in local deformation theory: §3 studies the local theory at our auxiliary primes, and §4 explains the consequences of results on generic fibers of local deformation rings. We explain the relative deformation theory argument and prove the main result, Theorem 6.9, in §6, using as input the technical mod ϖ^N lifting results of §5.1 and §5.2. In §7 we gather a few examples of the main theorem. In Appendix A we present the group theory arguments needed to streamline some of the global hypotheses on $\bar{\rho}$ in §5.1 and §5.2 to an irreducibility hypothesis. In Appendix B we deduce results on cohomology of p-adic Lie groups that are necessary for the relative deformation theory argument.

Finally, we remark that a number of arguments are made technically more intricate by the fact that we have worked with groups G having rather general (finite étale, order prime to p) component groups. To get the essence of the argument nothing is lost if the reader focuses on the case of connected adjoint groups G.

1.4. **Notation and conventions.** We embed local Galois groups into global Galois groups by fixing embeddings $\overline{F} \hookrightarrow \overline{F}_v$. We write κ for the p-adic cyclotomic character and $\bar{\kappa}$ for its mod p reduction, and we write $\mathbb{Q}_p/\mathbb{Z}_p(1)$ for the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ equipped with Galois module structure via κ. We once and for all fix an isomorphism (i.e. a compatible collection of p-power roots of unity) $\zeta\colon \mathbb{Q}_p/\mathbb{Z}_p(1) \xrightarrow{\sim} \mu_{p^\infty}(\overline{F}) \xrightarrow{\sim} \mu_{p^\infty}(\overline{F}_v)$, and use this to identify the Tate dual $V^* = \mathrm{Hom}(V, \mu_{p^\infty}(\overline{F}))$ of a Γ_F-module V with $\mathrm{Hom}(V, \mathbb{Q}_p/\mathbb{Z}_p(1))$. The reader should always assume we are doing this; only in the proof of Lemma 3.7 will we make the identifications explicit. At some places v, we will need to analyze certain kinds of tamely-ramified deformations, and we will routinely write σ_v and τ_v (or σ and τ if there is no risk of confusion) for elements of $\mathrm{Gal}(F_v^{\mathrm{tame},p}/F_v)$, $F_v^{\mathrm{tame},p}$ denoting the maximal tamely-ramified with p-power ramification quotient of Γ_{F_v}, that lift the (arithmetic) Frobenius and a generator of the p-part of the tame inertia group. See the beginning of §3 for a discussion of normalization.

For any finite set of primes S of F, we let $\Gamma_{F,S}$ denote $\mathrm{Gal}(F(S)/F)$, where $F(S)$ is the maximal extension of F inside $\overline{F}$ that is unramified outside the primes in S; here we impose no constraint on the "ramification" at ∞, but for notational convenience we do not want the set S to contain the archimedean places (as would often be the convention for what we are referring to as $\Gamma_{F,S}$).

Given a homomorphism $\rho\colon \Gamma \to H$ for some groups Γ and H, and an H-module V, we will write $\rho(V)$ for the associated Γ-module (we typically apply this with V the adjoint representation of an algebraic group).

Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, let ϖ be a uniformizer of $\mathcal{O}$, let $\mathfrak{m} = (\varpi)$ be the maximal ideal and let $k = \mathcal{O}/\mathfrak{m}$ be the residue field. Let e be the ramification index of $\mathcal{O}/\mathbb{Z}_p$. For integers $0 < r \leq s$, we use the choice of ϖ to view $\mathcal{O}/\mathfrak{m}^r$ as a submodule of $\mathcal{O}/\mathfrak{m}^s$ via the map induced by $1 \mapsto \varpi^{s-r}$.

Throughout this paper, except in Appendix §B, G will be a smooth group scheme over $\mathcal{O}$ with Lie algebra $\mathfrak{g}$ such that G^0 is split connected reductive, and G/G^0 is finite étale of order prime to p. We will sometimes write $\pi_0(G)$ for this quotient G/G^0. We let G^{der} denote the derived group of G^0, and we denote by $\mathfrak{g}^{\mathrm{der}}$ the Lie algebra of G^{der}; in the sequel, when there is no chance of confusion, we will sometimes abuse notation and use $\mathfrak{g}^{\mathrm{der}}$ also for $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} k$. We also let Z_{G^0} and $\mathfrak{z}_G$ be the center of G^0 and its Lie algebra.

For a continuous representation $\rho_s : \Gamma \to G(\mathcal{O}/\varpi^s)$ of a (topological) group Γ, we shall denote by $\rho_s(\mathfrak{g}^{\mathrm{der}})$ the $\mathcal{O}$-module $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^s$ with the action of Γ given by $\mathrm{Ad} \circ \rho_s$. Given moreover r such that $0 < r \leq s$, we set $\rho_r := \rho_s \pmod{\varpi^r}$, and the choice of uniformizer gives us inclusions $\rho_r(\mathfrak{g}^{\mathrm{der}}) \to \rho_s(\mathfrak{g}^{\mathrm{der}})$ which we shall use without further mention.

2. Deformation theory preliminaries

The following assumptions on p will implicitly be in effect for the remainder of the paper:

Assumption 2.1. We assume that $p \neq 2$ is very good ([Car85, §1.14]) for G^{der}, which in particular holds if $p \geq 7$ and $p \nmid n+1$ whenever G^{der} has a simple factor of type A_n. We also assume that the canonical central isogeny $G^{\mathrm{der}} \times Z_G^0 \to G^0$ has kernel of order prime to p (and in particular is étale).

Then in particular we have a G-equivariant direct sum decompositions $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \oplus \mathfrak{z}_G$, $(\mathfrak{g}^{\mathrm{der}})^{G^0} = 0$, and there is a non-degenerate G-invariant trace form $\mathfrak{g}^{\mathrm{der}} \times \mathfrak{g}^{\mathrm{der}} \to k$ ([Car85, 1.16]). The isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$ to the adjoint group of G^{der} also induces an isomorphism on Lie algebras.

Let Γ be a profinite group, and let $\bar{\rho}\colon \Gamma \to G(k)$ be a continuous homomorphism. Set

$$\bar{\mu} := \bar{\rho} \pmod{G^{\mathrm{der}}}\colon \Gamma \to G/G^{\mathrm{der}}(k),$$

and fix a lift $\mu\colon \Gamma \to G/G^{\mathrm{der}}(\mathcal{O})$ of $\bar{\mu}$; we can always choose the "Teichmüller" lift, since $G/G^{\mathrm{der}}(k)$ has order prime to p. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local noetherian algebras R with $\mathcal{O} \to R$ inducing an isomorphism of residue fields (and morphisms the local homomorphisms), and let $\mathcal{C}_{\mathcal{O}}^f$ be the full subcategory of those algebras that are artinian. Note that for any $R \in \mathcal{C}_{\mathcal{O}}$, $\pi_0(G)(R) \xrightarrow{\sim} \pi_0(G)(k)$, so we will just identify any $\pi_0(G)(R)$ to this fixed finite group $\pi_0(G)$.

Define the lifting and deformation functors

$$\mathrm{Lift}_{\bar{\rho}}, \mathrm{Def}_{\bar{\rho}}, \mathrm{Lift}_{\bar{\rho}}^{\mu}, \mathrm{Def}_{\bar{\rho}}^{\mu}\colon \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$$

by letting $\mathrm{Lift}_{\bar{\rho}}(R)$ be the set of lifts of $\bar{\rho}$ to $G(R)$, and by letting $\mathrm{Lift}_{\bar{\rho}}^{\mu}(R) \subset \mathrm{Lift}_{\bar{\rho}}(R)$ be the subset of lifts ρ such that $\mu \circ \rho = \mu$; and then letting the corresponding deformation functors be the quotients by the equivalence relation

$$\rho \sim \rho' \iff \rho = g\rho' g^{-1} \text{ for some } g \in \widehat{G}(R) = \ker(G(R) \to G(k)).$$

Note that $\widehat{G}(R) = \widehat{G^0}(R)$, and conjugation by $\widehat{G^{\mathrm{der}}}(R)$ induces the same equivalence relation since $G^{\mathrm{der}} \cap Z_{G^0}$ has order prime to p. The tangent spaces of the lifting functors are canonically isomorphic to $Z^1(\Gamma, \bar{\rho}(\mathfrak{g}))$ (recall that $\bar{\rho}(\mathfrak{g})$ is the Γ-module obtained by composing $\bar{\rho}$ with the adjoint representation of G) and $Z^1(\Gamma, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$; the tangent spaces of the corresponding deformation functors are canonically isomorphic to $H^1(\Gamma, \bar{\rho}(\mathfrak{g}))$ and $H^1(\Gamma, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, and (by our running assumptions) the latter is a direct summand of the former. In some cases we will have a global Galois representation valued in a non-connected group G, but it will be convenient to develop certain local deformation conditions only for the group G^0: since $\widehat{G}$ is contained in G^0 (and as above $\pi_0(G)$ has order prime to p), a G^0-deformation of a $G^0(k)$-valued $\bar{\rho}$ is exactly the same thing as a G-deformation of a G^0-valued $\bar{\rho}$.

As usual, when $R \to R/I$ is a small extension the obstruction to lifting a $\rho \in \mathrm{Lift}_{\bar{\rho}}(R/I)$ to a $\tilde{\rho} \in \mathrm{Lift}_{\bar{\rho}}(R)$ is a class in $H^2(\Gamma, \bar{\rho}(\mathfrak{g}) \otimes_k I)$ (the two-cocycle one defines by choosing a topological lift of ρ to $G(R)$ takes values in $\ker(G(R) \to G(R/I)) = \ker(G^0(R) \to G^0(R/I)) = \exp(\mathfrak{g} \otimes_k I)$), and similarly for deforming lifts of type μ. Note also that when $\rho \in \mathrm{Lift}^{\mu}_{\bar{\rho}}(R/I)$ has a lift $\tilde{\rho} \in \mathrm{Lift}_{\bar{\rho}}(R)$, then it also has a lift in $\mathrm{Lift}^{\mu}_{\bar{\rho}}(R)$: the discrepancy between $\tilde{\rho} \pmod{G^{\mathrm{der}}}$ and μ is measured by a class in $H^1(\Gamma, \bar{\mu}(\mathfrak{g}/\mathfrak{g}^{\mathrm{der}})) \otimes_k I$, and the canonical map $H^1(\Gamma, \bar{\rho}(\mathfrak{z}_G)) \to H^1(\Gamma, \bar{\mu}(\mathfrak{g}/\mathfrak{g}^{\mathrm{der}}))$ is an isomorphism (as $G^{\mathrm{der}} \cap Z_{G^0}$ has order prime to p). Thus we can modify the lift $\tilde{\rho}$ to one of type μ.

3. Local deformation theory: trivial primes

Let $F/\mathbb{Q}_\ell$ be a finite extension with residue field of order q. Assume q is congruent to 1 mod p, and let $\bar{\rho}\colon \Gamma_F \to G(k)$ be the trivial homomorphism; in particular, all lifts of $\bar{\rho}$ land in G^0. Moreover, all lifts of $\bar{\rho}$ factor through the quotient of Γ_F topologically generated by a lift σ of (arithmetic) Frobenius and a generator τ of the p-part of the tame inertia group. At one point we will invoke a calculation (Lemma 3.7) that depends on the normalization of τ. Suppose that F_v in fact contains μ_{p^b} for some integer b, and that we have a fixed $(p^b)^{th}$ root of unity $\zeta \in \mu_{p^b}(F_v)$, which we may regard (by evaluation at $\frac{1}{p^b}$) as a choice of isomorphism $\zeta\colon \mathbb{Z}/p^b(1) \xrightarrow{\sim} \mu_{p^b}(F_v)$ (in the global setting, this will come from a global choice, as in §1.4). We then choose τ such that for any uniformizer ϖ_F of F, $\frac{\tau(\varpi_F^{1/p^b})}{\varpi_F^{1/p^b}} = \zeta$.

3.1. **The local conditions at trivial primes.** We will now define the kinds of local lifts of $\bar{\rho}$ that we will make use of at auxiliary primes. First, we introduce some notation. For a split maximal torus T of G^0 (over $\mathcal{O}$) and an $\alpha \in \Phi(G^0, T)$, we let $U_\alpha \subset G^0$ denote the root subgroup that is the image of the root homomorphism ("exponential mapping") $u_\alpha\colon \mathfrak{g}_\alpha \to G$. The homomorphism u_α is a T-equivariant isomorphism $\mathfrak{g}_\alpha \to U_\alpha$ (see [Con14, Theorem 4.1.4]), and its characterizing properties (*loc. cit.*) imply that $Z_{G^0}(\mathfrak{g}_\alpha) = Z_{G^0}(U_\alpha)$.

Definition 3.1. Fix a split maximal torus T of G^0 (over $\mathcal{O}$) and an $\alpha \in \Phi(G^0, T)$. Define $\mathrm{Lift}^{\alpha}_{\bar{\rho}}(R)$ to be the set of lifts $\widehat{G}(R)$-conjugate to one satisfying

- $\rho(\sigma) \in T \cdot Z_{G^0}(\mathfrak{g}_\alpha)(R)$
- Under the composite (note that T normalizes the centralizer)

$$T \cdot Z_{G^0}(\mathfrak{g}_\alpha)(R) \to T(R)/(T(R) \cap Z_{G^0}(\mathfrak{g}_\alpha)(R)) \xrightarrow{\alpha} R^\times,$$

  $\rho(\sigma)$ maps to q.
- $\rho(\tau) \in U_\alpha(R)$.

Lemma 3.2. *For any pair (T, α) consisting of a split maximal torus T of G^0 and an $\alpha \in \Phi(G^0, T)$, the functor $\mathrm{Lift}^{\alpha}_{\bar{\rho}}$ is formally smooth, i.e. for all maps $R \to R/I$ in $\mathcal{C}^f_{\mathcal{O}}$ with $I \cdot \mathfrak{m}_R = 0$, $\mathrm{Lift}^{\alpha}_{\bar{\rho}}(R) \to \mathrm{Lift}^{\alpha}_{\bar{\rho}}(R/I)$ is surjective.*

Similarly, if we fix a lift $\mu\colon \Gamma_F \to G/G^{\mathrm{der}}(\mathcal{O})$ of the multiplier character $\bar{\mu} := \bar{\rho} \pmod{G^{\mathrm{der}}}$, then the sub-functor of lifts with multiplier μ, $\mathrm{Lift}^{\mu,\alpha}_{\bar{\rho}}$, is formally smooth.

Proof. It is convenient to begin with a slightly different description of $\mathrm{Lift}_{\bar\rho}^{\alpha}$ that will circumvent the need to know that $Z_{G^0}(\mathfrak{g}_\alpha)$ is smooth over $\mathcal{O}$. To that end, let Z_α be the open subscheme of $Z_{G^0}(\mathfrak{g}_\alpha)$ obtained by removing all non-identity components of the special fiber $Z_{G^0_k}(\mathfrak{g}_\alpha \otimes_{\mathcal{O}} k)$. Set $\mathfrak{g}_{\alpha,k} = \mathfrak{g}_\alpha \otimes_{\mathcal{O}} k$. We first claim the special fiber $Z_{\alpha,k} \to \operatorname{Spec} k$ is smooth. By our assumptions on p, $Z_{G^0_k}(\mathfrak{g}_{\alpha,k})$ is smooth if and only if $Z_{G_k^{\mathrm{der}}}(\mathfrak{g}_{\alpha,k})$ is smooth, and then the assumption that p is very good for G^{der} implies, by a criterion of Richardson ([Jan04, Theorem 2.5]), that $Z_{G_k^{\mathrm{der}}}(\mathfrak{g}_{\alpha,k})$ is smooth (recall that $\mathfrak{g}^{\mathrm{der}}$ has a non-degenerate trace form). In particular, $Z_{\alpha,k}$ is smooth. Since $Z_{\alpha,k}$ has a single irreducible component, we can now apply [Boo18, Remark 4.3, Lemma 4.4] to deduce that $Z_\alpha \to \operatorname{Spec} \mathcal{O}$ is smooth.

We next claim that $\mathrm{Lift}_{\bar\rho}^{\alpha}$ is equivalently defined by replacing $Z_{G^0}(\mathfrak{g}_\alpha)$ with Z_α in Definition 3.1. First note that for any object R of $\mathcal{C}_{\mathcal{O}}^f$, the fiber over the identity of $Z_{G^0}(\mathfrak{g}_\alpha)(R) \to Z_{G^0}(\mathfrak{g}_\alpha)(k)$ is contained in $Z_\alpha(R)$, and that T normalizes Z_α (as functors of Artin rings). Now let $x \in T(R)Z_{G^0}(\mathfrak{g}_\alpha)(R)$ be an element in the fiber over $1 \in G(k)$, and correspondingly write $x = t \cdot c$. Writing $\bar c$ for the image of c in $G(k)$, we have $\bar c \in \ker(\alpha|_T)$. This kernel is smooth (our assumptions on p imply that $X^\bullet(T)/\mathbb{Z}\alpha$ has no p-torsion), so we can lift $\bar c$ to an element $t' \in \ker(\alpha|_T)(R)$. Then writing $x = (tt')(t'^{-1}c)$ we have exhibited x as an element of $T(R)Z_\alpha(R)$. Since $\bar\rho$ is trivial, we conclude that $\mathrm{Lift}_{\bar\rho}^{\alpha}$ can equivalently be defined with Z_α in place of $Z_{G^0}(\mathfrak{g}_\alpha)$.

With this reinterpretation, we can now check formal smoothness of $\mathrm{Lift}_{\bar\rho}^{\alpha}$. Let ρ be any element of $\mathrm{Lift}_{\bar\rho}^{\alpha}(R/I)$. Since $\widehat{G}$ is formally smooth, we may assume ρ satisfies the three bulleted items of Definition 3.1. Write $\rho(\sigma) = t_\sigma c_\sigma$ and $\rho(\tau) = u_\alpha(x)$ for some $t_\sigma \in T(R/I)$ satisfying $\alpha(t_\sigma) = q$, $c_\sigma \in Z_\alpha(R/I)$, and $x \in R/I$. Since T and Z_α are formally smooth, we can choose lifts $\widetilde{t_\sigma} \in T(R)$, $\widetilde{c_\sigma} \in Z_\alpha(R)$, and $\widetilde{x} \in R$. We can write $\alpha(\widetilde{t_\sigma}) = q + i$ for some $i \in I$, and then we replace $\widetilde{t_\sigma}$ by $\widetilde{t_\sigma}\alpha^\vee(1 - \frac{i}{2})$ (recall that p is odd). Since $I \cdot \mathfrak{m}_R = 0$, we then find that $\tilde\rho(\sigma) = \widetilde{t_\sigma}\widetilde{c_\sigma}$, $\tilde\rho(\tau) = u_\alpha(\widetilde{x})$ defines a lift $\tilde\rho \in \mathrm{Lift}_{\bar\rho}^{\alpha}(R)$ of ρ. The fixed multiplier analogue is clear from the remarks in §2. □

Remark 3.3. We could have argued directly with the original definition using $Z_{G^0}(\mathfrak{g}_\alpha)$, but lacking a generalization to all groups of [Boo18, §4.4]—namely, sections of $Z_{G^0}(\mathfrak{g}_\alpha)$ hitting any irreducible component in the special fiber—we would only have obtained the smoothness for $p \gg_G 0$ but non-effective (resorting to a spreading-out argument).

We now carry out the local calculation needed for Theorem 5.15. In the application, we will only need to make use of the behavior of a local deformation functor beyond a certain fixed lift modulo ϖ^n (for some n), so we begin by introducing a relative analogue of the functor $\mathrm{Lift}_{\bar\rho}^{\alpha}$:

Definition 3.4. Suppose $F/\mathbb{Q}_\ell$ is a finite extension with residue field of order $q \equiv 1 \pmod p$, let $M > 1$ be a fixed integer, and suppose $\rho_M \colon \Gamma_F \to G(\mathcal{O}/\varpi^M)$ is a homomorphism whose mod p reduction $\bar\rho$ is trivial and that belongs to $\mathrm{Lift}_{\bar\rho}^{\alpha}(\mathcal{O}/\varpi^M)$ for some pair (T, α) of a split maximal torus and a root. Then we define the following functor on the over-category $(\mathcal{C}_{\mathcal{O}}^f)_{/(\mathcal{O}/\varpi^M)}$ of objects of $\mathcal{C}_{\mathcal{O}}^f$ equipped with an augmentation to $\mathcal{O}/\varpi^M$: given an $R \to \mathcal{O}/\varpi^M$ (we will for notational ease not write the augmentation in what follows), define $\mathrm{Lift}_{\rho_M}^{\alpha}(R)$ to be the set of lifts

$$\begin{tikzcd} & G(R) \arrow[d] \\ \Gamma_F \arrow[ur, "\rho"] \arrow[r, "\rho_M"'] & G(\mathcal{O}/\varpi^M) \end{tikzcd}$$

of ρ_M that are $\widehat{G}^{(M)}(R) := \ker\left(G(R) \to G(\mathcal{O}/\varpi^M)\right)$-conjugate to one satisfying

- $\rho(\sigma) \in T \cdot Z_{G^0}(\mathfrak{g}_\alpha)(R)$
- Under the composite

$$T \cdot Z_{G^0}(\mathfrak{g}_\alpha)(R) \to T(R)/(T(R) \cap Z_{G^0}(\mathfrak{g}_\alpha)(R)) \xrightarrow{\alpha} R^\times,$$

  $\rho(\sigma)$ maps to q.
- $\rho(\tau) \in U_\alpha(R)$.

Similarly define $\mathrm{Lift}^{\mu,\alpha}_{\rho_M}$ to be the sub-functor of lifts with prescribed multiplier μ.

Note that $\widehat{G}^{(M)}$ is formally smooth, so $\mathrm{Lift}^{\alpha}_{\rho_M}$ and $\mathrm{Lift}^{\mu,\alpha}_{\rho_M}$ are formally smooth, just as in Lemma 3.2.

Recall that for any $G(\mathcal{O}/\varpi^r)$-valued homomorphism ρ_r, $\rho_r(\mathfrak{g}^{\mathrm{der}})$ will denote $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r$ equipped with the $\mathrm{Ad} \circ \rho_r$ action.

Lemma 3.5. *Let $F/\mathbb{Q}_\ell$ be a finite extension with residue field of order q, let $M > 1$ be a fixed integer, and let $1 \le s \le M$ be another fixed integer. Suppose $\rho_{M+s} \colon \Gamma_F \to G(\mathcal{O}/\varpi^{M+s})$ is a homomorphism with multiplier μ satisfying:*

- *The reduction $\rho_s := \rho_{M+s} \pmod{\varpi^s}$ is trivial (mod center), and $q \equiv 1 \pmod{\varpi^s}$; but $q \not\equiv 1 \pmod{\varpi^{s+1}}$.*
- *There is a suitable choice of split maximal torus T and root $\alpha \in \Phi(G^0, T)$ such that $\rho_{M+s}(\sigma) \in T(\mathcal{O}/\varpi^{M+s})$, $\alpha(\rho_{M+s}(\sigma)) = q$, and $\rho_{M+s}(\tau) \in U_\alpha(\mathcal{O}/\varpi^{M+s})$. In particular, $\rho_{M+s} \in \mathrm{Lift}^{\mu,\alpha}_{\rho_M}(\mathcal{O}/\varpi^{M+s})$.*
- *For any root $\beta \in \Phi(G^0, T)$, $\beta(\rho_{M+s}(\sigma)) \not\equiv 1 \pmod{\varpi^{s+1}}$.*

Then for all $1 \le r \le M$ there are spaces of cocycles $Z^\alpha_r \subset Z^1(\Gamma_F, \rho_r(\mathfrak{g}^{\mathrm{der}}))$, with images $L_r \subset H^1(\Gamma_F, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ such that

- *Z_r contains all coboundaries and is free over $\mathcal{O}/\varpi^r$ of rank $\dim(\mathfrak{g}^{\mathrm{der}})$.*
- *For any integers $a, b > 0$ such that $a+b = r$, the natural maps induce short exact sequences*

$$0 \to Z^\alpha_a \to Z^\alpha_r \to Z^\alpha_b \to 0.$$

- *For all $m \ge 2s + M$ (in particular, for all all $m \ge 3M$), and any lift $\rho_m \in \mathrm{Lift}^{\mu,\alpha}_{\rho_M}(\mathcal{O}/\varpi^m)$ of ρ_{M+s}, the fiber of $\mathrm{Lift}^{\mu,\alpha}_{\rho_M}(\mathcal{O}/\varpi^{m+r}) \to \mathrm{Lift}^{\mu,\alpha}_{\rho_M}(\mathcal{O}/\varpi^m)$ over ρ_m is non-empty and Z^α_r-stable.*

Proof. Note that $2m - 2s \ge m + r$, so for any $X \in \mathfrak{g}$ we have the following computation in $G(\mathcal{O}/\varpi^{m+r})$:

$$\begin{aligned}(1 + \varpi^{m-s}X)\rho_{m+r}(\gamma)(1 + \varpi^{m-s}X)^{-1} &= (1 + \varpi^{m-s}X)\rho_{m+r}(\gamma)(1 - \varpi^{m-s}X) \\ &= (1 + \varpi^{m-s}(X - \mathrm{Ad}(\rho_{m+r}(\gamma))X))\rho_{m+r}(\gamma) \\ &= \left(1 + \varpi^m \left(\frac{X - \mathrm{Ad}(\rho_{m+r}(\gamma))X}{\varpi^s}\right)\right)\rho_{m+r}(\gamma),\end{aligned}$$

where this last expression makes sense since ρ is trivial modulo ϖ^s. Note also that the expression $\frac{X - \mathrm{Ad}(\rho_{m+r}(\gamma))X}{\varpi^s}$ only depends on ρ_{r+s}, and in particular only on ρ_{M+s}. Thus, for each $X \in \mathfrak{g}^{\mathrm{der}}$, we have a cocycle $\phi^r_X \in Z^1(\Gamma_F, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ given by

$$\phi^r_X(\gamma) = \frac{X - \mathrm{Ad}(\rho_{m+r}(\gamma))X}{\varpi^s},$$

and the action of any such ϕ^r_X preserves the fiber of $\mathrm{Lift}^{\alpha}_{\rho_M}(\mathcal{O}/\varpi^{m+r}) \to \mathrm{Lift}^{\alpha}_{\rho_M}(\mathcal{O}/\varpi^m)$ over ρ_m (as $m - s \ge M$). In particular, for any root $\beta \in \Phi(G^0, T)$, taking $X = X_\beta$ a generator (over $\mathcal{O}$) of $\mathfrak{g}_\beta$, the cocycle $\phi^r_{X_\beta}$ has the property that $\phi^r_{X_\beta}(\sigma)$ is an $\mathcal{O}^\times$-multiple of X_β. Our space Z^α_r will be the span of the following three kinds of cocycles:

- $\phi^r_{X_\beta}$ for all $\beta \in \Phi(G^0, T)$.
- For all X in an $\mathcal{O}$-basis of $\ker(\alpha|_{\mathfrak{t}^{\mathrm{der}}})$, the unramified cocycles $\phi_X^{\mathrm{un},r}$ given by $\phi_X^{\mathrm{un},r}(\sigma) = X$, $\phi_X^{\mathrm{un},r}(\tau) = 0$. To see that these are indeed cocycles, we just note that $X \pmod{\varpi^r}$ belongs to $\rho_r(\mathfrak{g})^{\Gamma_F}$, which reduces to the two assumptions that $\rho_r(\sigma) \in T(\mathcal{O}/\varpi^r)$ and $[X, X_\alpha] = 0$.
- The ramified cocycle ϕ^r_α given by $\phi^r_\alpha(\sigma) = 0$ and $\phi^r_\alpha(\tau) = X_\alpha$. The cocycle condition for ϕ^r_α reduces to the fact that $\alpha(\rho_r(\sigma)) = q$.

That ϕ^r_X and ϕ^r_α preserve the fiber is clear when acting on a lift ρ_{m+r} that satisfies $\rho_{m+r}(\sigma) \in T \cdot Z_{G^0}(\mathfrak{g}_\alpha)(\mathcal{O}/\varpi^{m+r})$ and $\rho_{m+r}(\tau) \in U_\alpha(\mathcal{O}/\varpi^{m+r})$; in general it follows since for $g \in \widehat{G}^{(M)}(\mathcal{O}/\varpi^{m+r})$

$$g(1 + \varpi^m \phi)\rho_{m+r} g^{-1} = (1 + \varpi^m \phi) g \rho_{m+r} g^{-1},$$

as $r \leq M$.

We claim that the $\mathcal{O}/\varpi^r$-span Z^α_r of the collection $\{\phi^r_{X_\beta}, \phi^r_X, \phi^r_\alpha\}$ satisfies all the properties in the Lemma's conclusion. To see that Z^α_r is free of rank $\dim(\mathfrak{g}^{\mathrm{der}})$, note that a linear combination of the $\phi^r_{X_\beta}, \phi^{\mathrm{un},r}_X, \phi^r_\alpha$ is—by evaluating at σ and then at τ—seen to be a multiple of ϖ if and only if each coefficient is a multiple of ϖ. The claimed exact sequences induced by reduction modulo ϖ^b are clear from the construction.

Finally, to see that Z^α_r contains $B^1(\Gamma_F, \rho_r(\mathfrak{g}^{\mathrm{der}}))$, note that the latter module is spanned by $\gamma \mapsto X - \mathrm{Ad}(\rho_r(\gamma))X$ for X in a basis of $\rho_r(\mathfrak{g}^{\mathrm{der}})$. The coboundaries thus generated by the $\{X_\beta\}_{\beta \in \Phi(G^0, T)}$ are clearly in the span of the $\phi^r_{X_\beta}$. For $X \in \mathfrak{t}^{\mathrm{der}}$, the corresponding coboundary vanishes on σ and maps τ to $X - \mathrm{Ad}(\rho_r(\tau))X$. Since $\rho_r(\tau) = u_\alpha(y)$ for some y, $X - \mathrm{Ad}(\rho_r(\tau))X$ is a multiple of X_α, and so this coboundary is in the span of ϕ^r_α. □

Remark 3.6.

- Theorem 5.15 will use Lemma 3.5 in the cases $s \in \{1, 2, e\}$.
- Proposition 6.7 and Theorem 6.8 will use Lemma 3.5 in the cases $s = M$.

3.2. **Local duality pairing at trivial primes.** We will later require the following calculation of the local duality pairing at trivial primes: for $r = 1$ we will use this Lemma throughout §4, and for general r we will use it in Lemma 6.6. When $r = 1$ (or for general r when $e = 1$), the trace pairing and choice of generator of μ_p induces an isomorphism of $k[\Gamma_F]$-modules

$$\mathrm{tr}_{k/\mathbb{F}_p} \colon \mathrm{Hom}_k(W, k(1)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p}(W, \mathbb{F}_p(1)) \underset{\sim}{\xrightarrow{\zeta}} \mathrm{Hom}_{\mathbb{F}_p}(W, \mu_p) = W^*,$$

where the k-structure on the target is just induced by the k-multiplication on W. In general, we simply fix a generator of the $\mathcal{O}$-module $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}/\varpi^r, \mathbb{Q}_p/\mathbb{Z}_p)$; then for a finite free $\mathcal{O}/\varpi^r$-module W, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(W, \mathcal{O}/\varpi^r) \xrightarrow{\sim} \mathrm{Hom}(W, \mathbb{Q}_p/\mathbb{Z}_p) \tag{1}$$

by composing with our fixed generator. If now W is moreover an $\mathcal{O}/\varpi^r[\Gamma_F]$-module, then having fixed a choice $\zeta \colon \mathbb{Q}_p/\mathbb{Z}_p(1) \xrightarrow{\sim} \mu_{p^\infty}$ of p-power roots of unity, we can identify the Tate dual W^* with $\mathrm{Hom}_{\mathcal{O}}(W, \mathcal{O}/\varpi^r(1))$, and we then define the $\mathcal{O}$-linear local duality by

$$\mathrm{inv}_F(\cdot \cup \cdot) \colon H^1(\Gamma_F, W) \times H^1(\Gamma_F, W^*) \to H^2(\Gamma_F, W \otimes_{\mathcal{O}} W^*) \to H^2(\Gamma_F, \mathcal{O}/\varpi^r(1)) \xrightarrow{\sim} \mathcal{O}/\varpi^r,$$

where the last isomorphism is induced by the composite

$$H^2(\Gamma_F, \mathcal{O}/\varpi^r(1)) \to H^2(\Gamma_F, K/\mathcal{O}(1)) \cong H^2(\Gamma_F, \mathbb{Q}_p/\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow[\mathrm{inv}_F \circ \zeta^{-1}]{\sim} \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O} \cong K/\mathcal{O}.$$

In all, this duality pairing depends on the choice in Equation (1), but only up to $\mathcal{O}^\times$-scaling (so, e.g., $\mathcal{O}$-submodules are canonically defined), and it is independent of the choice of ζ.

Lemma 3.7. *Let W be a free $\mathcal{O}/\varpi^r$-module equipped with trivial Γ_F-action, and assume that $q \equiv 1 \pmod{\varpi^r}$, so that W^* is also a trivial Γ_F-module. Identify $W^* \cong \mathrm{Hom}_{\mathcal{O}}(W, \mathcal{O}/\varpi^r(1))$ as above, and write $\langle \cdot, \cdot \rangle \colon W \times W^* \to \mathcal{O}/\varpi^r$ for the $\mathcal{O}/\varpi^r$-linear evaluation pairing. Then the $\mathcal{O}/\varpi^r$-linear duality pairing*

$$\mathrm{inv}_F(\cdot \cup \cdot) \colon H^1(\Gamma_F, W) \times H^1(\Gamma_F, W^*) \to \mathcal{O}/\varpi^r$$

has the following properties: if ϕ is unramified, then

$$\mathrm{inv}_F(\phi \cup \psi) = -\langle \phi(\sigma), \psi(\tau) \rangle,$$

and if ψ is unramified, then

$$\mathrm{inv}_F(\phi \cup \psi) = \langle \phi(\tau), \psi(\sigma) \rangle.$$

Remark 3.8. These identifications of course depend on the choice of τ; see the beginning of this section for the discussion of how we calibrate τ.

Proof. Since W is trivial, the lemma reduces to the case where W is free of rank one over $\mathcal{O}/\varpi^r$, and the above description of the $\mathcal{O}/\varpi^r$-linear duality pairing, which for $W = \mathcal{O}/\varpi^r$ is the $\mathcal{O}/\varpi^r$-linear extension of the $\mathbb{Z}/p^b$-linear duality pairing on the trivial module $\mathbb{Z}/p^b$, where $b = \lfloor \frac{r}{e} \rfloor$, shows we can further reduce to the case $W = \mathbb{Z}/p^b$.

Then the calculation can be performed, for instance, using the identity

$$\mathrm{inv}_F(\phi \cup \delta(a)) = \phi(\mathrm{rec}_F(a))$$

for any $\phi \in H^1(\Gamma_F, W) = \mathrm{Hom}(\Gamma_F^{\mathrm{ab}}, \mathbb{Z}/p^b)$ and $a \in F^\times/(F^\times)^{p^b} \underset{\sim}{\xrightarrow{\delta}} H^1(\Gamma_F, \mu_{p^b}) = H^1(\Gamma_F, (\mathbb{Z}/p^b)^*)$ (the last identification is the canonical one). If ϕ is unramified, then $\phi(\mathrm{rec}_F(a))$ is simply $-v(a)\phi(\sigma)$ (writing v for the normalized valuation, and normalizing rec_F to take uniformizers to geometric frobenii). On the other hand, if $\psi = \delta(a)$, then

$$\psi(\tau) = \delta(a)(\tau) = \frac{\tau(a^{1/p^b})}{a^{1/p^b}} = \left(\frac{\tau(\varpi_F^{1/p^b})}{\varpi_F^{1/p^b}} \right)^{v(a)} = \zeta^{v(a)}.$$

Then $\langle \phi(\sigma), \psi(\tau) \rangle = \zeta^{v(a)\phi(\sigma)}$, and via our isomorphism $\zeta \colon \mathbb{Z}/p^b \to \mu_{p^b}$ we thus identify $\langle \phi(\sigma), \psi(\tau) \rangle = -\mathrm{inv}_F(\phi \cup \psi)$, as desired. Now suppose ψ is unramified. Then we identify $W = W^{**}$ and apply the previous step to find

$$\mathrm{inv}_F(\phi \cup \psi) = -\mathrm{inv}_F(\psi \cup \phi) = \langle \psi(\sigma), \phi(\tau) \rangle = \langle \phi(\tau), \psi(\sigma) \rangle.$$

∎

4. Generic fibers of local Galois deformation rings

In this section we re-interpret known results ([Kis08], [BG17]) on generic fibers of local (and fixed p-adic Hodge type) deformation rings. We will achieve enough control over a part of the corresponding integral deformation rings—as in the previous section, once we have lifted beyond a certain modulus ϖ^n—to use these softer results on generic fibers in place of the usual local demand of Ramakrishna-style arguments, which typically require having at hand a formally smooth irreducible component of the appropriate local deformation ring.

4.1. **Interpretation of a result of Serre.** Let $\mathcal{O}$, ϖ, m, k be as before and let K be the quotient field of $\mathcal{O}$. Let (R, m_R) be a complete local noetherian $\mathcal{O}$-algebra with residue field k. In this section, we are interested in understanding the structure of the sets X_n of $\mathcal{O}/\mathsf{m}^n$-valued points of $\mathrm{Spec}(R)$ together with the reduction maps $\pi_{n,r} : X_{n+r} \to X_n$ for $n, r \geq 0$.

If R is formally smooth over $\mathcal{O}$ then these sets have a very simple structure: all the reduction maps are surjective and the fibers of the maps for $r \leq n$ are principal homogenous spaces over $Hom_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_R \mathcal{O}/\mathsf{m}^n, \mathcal{O}/\mathsf{m}^r)$, a free module over $\mathcal{O}/\mathsf{m}^r$ of rank $\dim(R) - 1$. Here $\Omega_{R/\mathcal{O}}$ denotes the module of continuous derivations and the map from R to $\mathcal{O}/\mathsf{m}^n$ used to form the tensor product corresponds to the chosen point of X_n, but clearly the module $Hom_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_R \mathcal{O}/\mathsf{m}^n, \mathcal{O}/\mathsf{m}^r)$ only depends on the reduction of this point in X_r. For arbitrary R the nonempty fibers of $\pi_{n,r}$ do have the same property, but these maps need not be surjective. However, we show below that under some relatively mild conditions on R there exist nonempty subsets $Y_n \subset X_n$ such that $\pi_{n,r}$ induces surjections $Y_{n+r} \to Y_n$ and, for fixed r and $n \gg 0$, the fibers of these maps are principal homogenous spaces over a suitable submodule of $Hom_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_R \mathcal{O}/\mathsf{m}^n, \mathcal{O}/\mathsf{m}^r)$ which is free over $\mathcal{O}/\mathsf{m}^r$ of rank equal to the dimension of $R[\varpi^{-1}]$.

Let $A = \mathcal{O}[[x_1, x_2, \ldots, x_N]]$, with $N = \dim_k m_R/(m_R^2, \varpi)$, and let $\alpha : A \to R$ be a surjection of $\mathcal{O}$-algebras. The map α induces an inclusion of the set X of $\mathcal{O}$-valued points of $\mathrm{Spec}(R)$ into $(\mathsf{m})^{\mathsf{N}}$ with image a closed subset[1]. We also have an identification of the K-valued points of $\mathrm{Spec}(R[\varpi^{-1}])$ with the $\mathcal{O}$-valued points of $\mathrm{Spec}(R)$: any map of $\mathcal{O}$-algebras from $R[\varpi^{-1}] \to K$ must map all the $\pi(x_i)$ to elements of $m \subset K$.

We assume that there exist $y \in X$ such that $\mathrm{Spec}(R[\varpi^{-1}])$ is formally smooth and of dimension d at y. Since X is defined as a closed subset (in the ϖ-adic topology) of $(\mathsf{m})^{\mathsf{N}}$ by finitely many power series (generators $f_1, f_2, \ldots, f_e$ of $\ker(\alpha)$), it follows by the Jacobian criterion and the implicit function theorem that there is an open set $y \in U \subset X$ such that the Jacobian matrix of $f_1, f_2, \ldots, f_e$ has rank $n - d$ at all points in U. Thus, U is a "ϖ-adic" (analytic) manifold of dimension d (in the naive sense).

Lemma 4.1. *If U is compact, there exists an integer $v \geq 0$ such that the torsion in $\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}$ is annihilated by ϖ^v for all $y \in U$.*

Proof. Let $y \in U$ and consider the Jacobian matrix of $f_1, f_2, \ldots, f_e$. Since X is a manifold of dimension d at y, this matrix evaluated at y has rank $N - d$, so has an invertible $N - d \times N - d$ minor. By continuity, this minor remains invertible and its determinant has constant valuation in an open neighbourhood of y. We conclude (using the compactness of U) by applying Lemma 4.2 below. □

[1]We use brackets to indicate that here m is viewed simply as an open subset of $\mathcal{O}$ with its ϖ-adic topology and the exponent N denotes the N-fold Cartesian product.

Lemma 4.2. *Let $\mathcal{O}$ be any discrete valuation ring with uniformizer ϖ and let K be its quotient field. Let M be an $e \times N$ matrix with entries in $\mathcal{O}$ such that its rank (as a matrix with entries in K) is $N-d$. If M has an $(N-d)\times(N-d)$ minor whose determinant has valuation v, then the torsion submodule of the quotient Q of $\mathcal{O}^n$ by the submodule generated by the rows of M is annihilated by ϖ^v.*

Proof. We may assume that $e = N-d$ and then by performing row operations and permuting the columns, we may assume that the first $N-d$ columns of M form an upper triangular matrix whose diagonal entries have product a unit times ϖ^v. It follows from this that the image in Q of the submodule of $\mathcal{O}^N$ spanned by $\varpi^v e_i$, $i = 1,2,\ldots,N-d$, with e_i the i-th standard basis vector, is contained in the image of the submodule spanned by $e_{N-d+1},\ldots,e_N$. This submodule must be torsion free since the rank of Q is d, so the lemma is proved. □

We now analyze the structure of the reductions of U modulo $\mathfrak{m}^n$ for $n \gg 0$. This has essentially been done by Serre [Ser81], but we explain part of his proof in order to make explicit a couple of points that are crucial for our applications.

Lemma 4.3. *There exists a compact open set Y such that $y \in Y \subset U$ and with the following properties: For any integer $n > 0$ let Y_n be the reduction of Y modulo $\mathfrak{m}^n$, so we have induced maps $\pi^Y_{n,r} : Y_{n+r} \to Y_n$. Given any integer $r_0 > 0$, there exists an integer $n_0 > 0$, such that for any $n \geq n_0$ the fibers of the map $\pi^Y_{n,r}$, for $n \geq n_0$ and $0 \leq r \leq r_0$, are nonempty principal homogenous spaces over a submodule T_r of $Hom_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}, \mathcal{O}/\mathfrak{m}^r)$ that is free over $\mathcal{O}/\mathfrak{m}^r$ of rank d. Here the action of T_r is induced by the natural action of $Hom_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}, \mathcal{O}/\mathfrak{m}^r)$ on the fibers of the map $\pi_{n,r} : X_{n+r} \to X_n$.*

Proof. Following §3.3 of Serre [Ser81], we first consider some subsets Y of $(\mathfrak{m})^{\mathsf{N}}$ and examine the properties of their reductions.

(i) Y is defined by equations of the form

$$\begin{aligned} x_{d+1} &= \phi_{d+1}(x_1,\ldots,x_d)\\ x_{d+2} &= \phi_{d+2}(x_1,\ldots,x_d)\\ &\cdots\\ x_N &= \phi_N(x_1,\ldots,x_d)\end{aligned}$$

where $\phi \in A$, for $i = d+1,\ldots,N$ and $y = 0$. Clearly Y is the set of $\mathcal{O}$-valued points of the ring R', defined as the quotient of A by the ideal generated $\{x_{d+i} - \phi_{d+i}\}_{i=d+1}^N$. The ring R' is formally smooth over $\mathcal{O}$, so the desired statements hold for Y with $n_0 = r_0$ and $T_r = Hom_{\mathcal{O}}(\Omega_{R'/\mathcal{O}} \otimes_{R',y} \mathcal{O}, \mathcal{O}/\mathfrak{m}^r)$ viewed as a submodule of $Hom_{\mathcal{O}}(\Omega_{A/\mathcal{O}} \otimes_A \mathcal{O}, \mathcal{O}/\mathfrak{m}^r)$ using the surjection $\Omega_{A/\mathcal{O}} \to \Omega_{R'/\mathcal{O}}$ induced by the quotient map $\alpha' : A \to R'$.

(ii) Y is as in the previous case except that we permute the coordinates $(x_1,\ldots,x_N)$. It is clear that the statement holds in this case as well.

(iii) Y is of the form

$$Y = y + \varpi^s Y'$$

where $y \in (\mathfrak{m})^{\mathsf{N}}$, Y' is as in the previous case, and $s \geq 0$ is any integer. In this case we may take $n_0 = r_0 + s$ and T_r as above.

By Proposition 11 of [Ser81],[2] there is an open set Y with $y \in Y \subset U$ such that Y is of type (iii) above. This completes the proof, except that the module T_r that we get is, a priori, only a

[2]Serre considers the case $K = \mathbb{Q}_p$ but the proof extends to general K *mutatis mutandis*.

submodule of $Hom_{O}(\Omega_{A/O} \otimes_{A,y} O, O/\mathfrak{m}^r)$. To see that T_r is a submodule of $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$, let V_n be the reduction modulo $\mathfrak{m}^n$ of the O-valued points of $\mathrm{Spec}(A)$ and let $\pi^A_{n,r} : V_{n+r} \to V_n$ be the reduction maps. The inclusions $Y_n \subset X_n \subset V_n$ are compatible with the reduction maps and the inclusions $X_n \subset V_n$ are also compatible with the action of $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$ on the fibers of $\pi_{n,r}$ and of $Hom_{O}(\Omega_{A/O} \otimes_{A,y} O, O/\mathfrak{m}^r)$ on the fibers of $\pi^A_{n,r}$ (for $n \geq r$). Since the nonempty fibers of $\pi_{n,r}$ (resp. $\pi^A_{n,r}$) are principal homogenous spaces over $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$ (resp. $Hom_{O}(\Omega_{A/O} \otimes_{A,y} O, O/\mathfrak{m}^r)$), it follows that T_r must be contained in (the image of) $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$. $\square$

We now give an explicit description of the spaces T_r from Lemma 4.3. By Lemma 4.1, there is an integer v such that the ϖ-torsion of $\Omega_{R/O} \otimes_{R,y} O$ is annihilated by ϖ^v for all $y \in Y$. For each $r > 0$, consider the O-submodule $T'_r \subset Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$ given by all homomorphisms which are trivial on the ϖ-torsion in $\Omega_{R/O} \otimes_{R,y} O$. It is clearly free over $O/\mathfrak{m}^r$ of rank d.

Lemma 4.4. *The submodules T_r and T'_r are equal for all r.*

Proof. Let n be any integer such that $n > n_0$, where n_0 is as in the conclusion of Lemma 4.3 for $r_0 = \max\{2v, 2r\}$. By that lemma, the orbit of the reduction y_{n+r} of y in Y_{n+r} under the action of T_r consists of points which can be lifted to Y, in particular to Y_{n+2r}. If $t \in Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$ is such that t is nontrivial on the torsion in $\Omega_{R/O} \otimes_{R,y} O$ and $r \leq v$, then t cannot be lifted to $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^{2r})$ since composing the map from the torsion to $O/\mathfrak{m}^{2r}$ with the projection to $O/\mathfrak{m}^r$ one gets the zero map. But this implies that $t \cdot y_{n+r}$ cannot be lifted to Y_{n+2r}, which is a contradiction. Thus, $T_r \subset T'_r$ for $v \leq r$ and by the equality of ranks we see that $T_r = T'_r$. The statement for all r then follows from the fact that for any $r > 1$, T_{r-1} is the reduction of T_r modulo $\mathfrak{m}^{r-1}$ (this follows from their defining property) and similarly for T'_{r-1}. $\square$

4.2. **Application to deformation rings.** In our applications R will be a suitable quotient of the universal framed (fixed "determinant") deformation ring of a mod ϖ representation $\bar{\rho} : \Gamma_F \to G(k)$, for $F/\mathbb{Q}_l$ a finite extension, arising from choosing an irreducible component of the generic fiber of the spectrum of the lifting ring $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_F}}[1/\varpi]$ (when $l \neq p$) or of the fixed p-adic Hodge type $\mathbf{v}$ lifting ring $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}}[1/\varpi]$ (when $l = p$: see [Bal12, Prop. 3.0.12] for the construction of this ring), and then letting R be the quotient ring corresponding to the Zariski closure of this component in $\mathrm{Spec}(R^{\square,\mu}_{\bar{\rho}})$.

The points of Y_n are identified with (certain) $O/\mathfrak{m}^n$-valued points of $\mathrm{Spec}(R)$, so correspond to lifts of $\bar{\rho}$ to $O/\mathfrak{m}^n$. A point $y \in Y$ corresponds to a lift $\rho : \Gamma_F \to G(O)$ of $\bar{\rho}$ and $Hom_{O}(\Omega_{R/O} \otimes_{R,y} O, O/\mathfrak{m}^r)$ is naturally isomorphic to an O-submodule of the group of one-cocycles $Z^1(\Gamma_F, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{O} O/\mathfrak{m}^r)$. The group of one-cocyles also contains the group of coboundaries $B^1(\Gamma_F, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{O} O/\mathfrak{m}^r)$.

Lemma 4.5. *For all $r > 0$ we have $B^1(\Gamma_F, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{O} O/\mathfrak{m}^r) \subset T_r$.*

Proof. Choose $n \gg 0$ so that we can apply Lemma 4.3 with $r = r_0$. Let $y \in Y$ and also assume that n is sufficiently large so that the $\widehat{G}^{(n)}(O)$-orbit (recall that $\widehat{G}^{(n)}$ was defined in Lemma 3.4) of y is contained in Y. Let y_n (resp. y_{n+r}) be the reduction of y modulo $\mathfrak{m}^n$ (resp. $\mathfrak{m}^{n+r}$). Clearly y_n is fixed by the $\widehat{G}^{(n)}(O)$ action, so this action maps y_{n+r} into another lift of y_n in Y_{n+r}. These lifts differ from y_{n+r} by elements of T_r. On the other hand, viewing elements of Y_{n+r} as lifts of $\bar{\rho}$ one easily sees that the action of $\widehat{G}^{(n)}(O)$ corresponds precisely to changing these elements by boundaries via the identification of $\widehat{G}^{(n)}(O)/\widehat{G}^{(n+r)}(O)$ with $\mathfrak{g}_r$ given by the exponential map. Thus, all boundaries are in T_r, so the lemma follows. $\square$

In fact, conjugation of lifts by the group $\widehat{G}(\mathcal{O})$ induces an action on $R[1/\varpi]$, and hence on R, by [BG17, Lemma 3.4.1]; see Remark 4.8 below.

Definition 4.6. We define the submodule $L_{\rho,r} \subset H^1(\Gamma_F, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}^r)$ to be the image of T_r.

Putting all of the above together we get:

Proposition 4.7. *Let $\bar{\rho} : \Gamma_F \to G(k)$ be any representation with $F/\mathbb{Q}_l$ a finite extension. Let R be chosen as above, arising from a choice of irreducible component of either $R^{\square,\mu}_{\bar{\rho}}[1/\varpi]$ ($\ell \neq p$) or some $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}}$ ($\ell = p$). Assume that* $\mathrm{Spec}(R[1/\varpi])$ *has a K-valued point y which is contained in the smooth locus and let $\rho : \Gamma_F \to G(\mathcal{O})$ be the corresponding lift of $\bar{\rho}$. Then there exists a nonempty open set $y \in Y \subset$* $\mathrm{Spec}(R[1/\varpi])(K)$ *with the following properties: Let Y_n be the image of Y in* $\mathrm{Spec}(R)(\mathcal{O}/\mathfrak{m}^n)$ *and for $n, r \geq 0$ let $\pi^Y_{n,r} : Y_{n+r} \to Y_n$ be the induced maps.*

1. *Given $r_0 > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$ and $0 \leq r \leq r_0$ the fibers of $\pi^Y_{n,r}$ are nonempty principal homogenous spaces over a submodule $T_r \subset Z^1(\Gamma_F, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}^r)$ which is free over $\mathcal{O}/\mathfrak{m}^r$ of rank d.*
2. *The inclusions $\mathcal{O}/\mathfrak{m}^{r-1} \to \mathcal{O}/\mathfrak{m}^r$ and the surjections $\mathcal{O}/\mathfrak{m}^r \to \mathcal{O}/\mathfrak{m}^{r-1}$ induce inclusions $T_{r-1} \to T_r$ and surjections $T_r \to T_{r-1}$.*
3. $$|L_{\rho,r}| = \begin{cases} |\rho_r(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F}| & \text{if } l \neq p; \\ |\rho_r(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F}| \cdot |\mathcal{O}/\mathfrak{m}^r|^{\dim\left(\mathrm{Res}_{F\otimes_{\mathbb{Q}_p}K/K}(G^0)/P_{\mathbf{v}}\right)} & \text{if } l = p \end{cases}$$
   Here $P_{\mathbf{v}}$ is the parabolic subgroup of the Weil restriction $\mathrm{Res}_{F\otimes_{\mathbb{Q}_p}K/K}(G^0)$ defining the p-adic Hodge type $\mathbf{v}$: see [BG17, Definition 2.8.2] *for a precise definition. In particular, we note that if the*
4. *The groups $L_{\rho,r}$ are compatible with the maps on cohomology induced by the inclusions $\mathcal{O}/\mathfrak{m}^{r-1} \to \mathcal{O}/\mathfrak{m}^r$ and the surjections $\mathcal{O}/\mathfrak{m}^r \to \mathcal{O}/\mathfrak{m}^{r-1}$.*

Proof. Item (1) follows from Lemma 4.3 and the discussion above. Item (2) follows from Lemma 4.4. Item (3) follows from [BG17, Theorem A] (using that for regular Hodge–Tate lifts the associated parabolic is a Borel), the exact sequence

$$0 \to \rho_r(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F} \to \rho_r(\mathfrak{g}^{\mathrm{der}}) \to B^1(\Gamma_F, \rho_r(\mathfrak{g}^{\mathrm{der}}) \to 0$$

and the definition of $L_{\rho,r}$. Item (4) follows directly from (2). □

Remark 4.8. We will later implicitly make use of [BG17, Lemma 3.4.1], that $\widehat{G}(\mathcal{O})$-conjugation preserves the irreducible component $R[1/\varpi]$, as follows. In the global setting we will fix a local lift ρ as here and extract as in Lemma 4.3 an analytic neighborhood of ρ in the $\mathcal{O}$-points of $R[1/\varpi]$; since our global lifts as constructed in §5.1 and §5.2 will only locally interpolate $\rho \pmod{\varpi^N}$ modulo $\widehat{G}(\mathcal{O})$-conjugacy, we will implicitly in §6 use the fact that Proposition 4.7 applies, with the same output n_0, to any $\widehat{G}(\mathcal{O})$-conjugate of ρ. Moreover, by the result just cited in [BG17], we remain on the same irreducible component $R[1/\varpi]$ of the local lifting ring.

To refine the local conclusion of our main theorem, we will use the following:

Lemma 4.9. *Let R be a complete local noetherian $\mathcal{O}$-algebra, and assume that* $\mathrm{Spec}(R[1/\varpi])$ *has an open dense regular subscheme. Fix an $\mathcal{O}$-point $x\colon R \to \mathcal{O}$. Then there exists a finite extension K'/K, with ring of integers $\mathcal{O}'$, such that for every $t \geq 1$ there exists an $x_t\colon R \to \mathcal{O}'$ such that $x_t \equiv x$* $\pmod{\varpi^t}$ *and x_t defines a formally smooth point of* $\mathrm{Spec}(R[1/\varpi])$.

Proof. For ease of reference we use the language of rigid geometry, but this is certainly not essential for the proof.

Let $\mathfrak{X} = \mathrm{Spf}(R)$ be the formal scheme over $\mathrm{Spf}(\mathcal{O})$ associated to R and let $\mathfrak{X}^{\mathrm{rig}}$ be the rigid space over K associated to $\mathfrak{X}$ as in [dJ95, §7.1]. By [dJ95, Lemma 7.1.9], points of $\mathfrak{X}^{\mathrm{rig}}$ are in canonical bijection with closed points of $\mathrm{Spec}(R[1/\varpi])$, and there is a canonical isomorphism of the completion of the local ring of a closed point of $\mathrm{Spec}(R[1/\varpi])$ with the completion of the local ring of the associated point on $\mathfrak{X}^{\mathrm{rig}}$. Thus, formally smooth points on $\mathrm{Spec}(R[1/\varpi])$ correspond to smooth points on $\mathfrak{X}^{\mathrm{rig}}$. It follows that we may replace $\mathrm{Spec}(R[1/\varpi])$ by $\mathrm{Sp}(B)$, where B is an affinoid algebra over K. By assumption, the singular locus of $\mathrm{Sp}(B)$, which is defined by an ideal I, does not contain any of its irreducible components, so it suffices to prove the statement with formally smooth points replaced by points in the complement of the zero set of any such ideal. By replacing $\mathrm{Sp}(B)$ by an irreducible component containing x we may then assume that B is integral.

By the Noether normalization theorem for affinoid algebras ([BGR84, §6.1.2]), there exists a finite homomorphism $\phi : T_d \to B$ with T_d a Tate algebra over K of dimension $d = \dim(B)$. Since K has only finitely many extensions of a fixed degree (in a fixed algebraic closure $\overline{K}$) we are reduced to proving that for a point $x \in \mathcal{O}^d$ and any proper ideal I of T_d, there is a sequence of points $\{x_n\}_{n\geq 0}$ of $\mathcal{O}^d$ converging to x in the ϖ-topology and with none of the x_n in the closed subset defined by I. Since the only element of T_d vanishing on all of $\mathcal{O}^d$ is 0, by choosing $d-1$ general linear polynomials in T_d vanishing on x we are reduced to the case $d = 1$. In this case the result is clear since any element of T_1 has only finitely many zeros. □

5. The doubling method: constructing mod ϖ^N lifts

Let F be a number field, let S be a finite set of places of F containing all places above p, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a continuous homomorphism. In this section we explain a broad generalization of the techniques of [KLR05]. This will allow us to construct (after increasing S) a mod ϖ^N lift of $\bar{\rho}$ with good local properties, including but not limited to interpolating (modulo $\widehat{G}$-conjugation) any fixed set of local mod ϖ^N lifts at places in S. We begin in §5.1 with the case $N = 2$, which contains the essence of the technique; then in §5.2 (see Theorem 5.15) we iterate (a rather more elaborate version of) this argument to produce lifts modulo higher powers of ϖ.

5.1. **Constructing the mod ϖ^2 lift.** We may assume $\bar{\rho}$ surjects onto $\pi_0(G)$ (if not, we replace G by the preimage in G of the image of $\bar{\rho}$ in $\pi_0(G)$; the deformation theory of $\bar{\rho}$ is unchanged by this replacement). There is then a unique finite Galois extension $\widetilde{F}/F$ such that $\bar{\rho}$ induces an isomorphism $\mathrm{Gal}(\widetilde{F}/F) \to \pi_0(G)$. We make the following assumptions on $\bar{\rho}$:

Assumption 5.1. Assume $p \gg_G 0$, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a continuous representation unramified outside a finite set of finite places S; we may and do assume that S contains all places above p. Assume that $\bar{\rho}$ satisfies the following:

- The field $K = \widetilde{F}(\bar{\rho}(\mathfrak{g}^{\mathrm{der}}), \mu_p)$ does not contain μ_{p^2}.
- $H^1(\mathrm{Gal}(K/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$=0.
- $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ are semisimple $\mathbb{F}_p[\Gamma_F]$-modules (equivalently, semisimple $k[\Gamma_F]$-modules) having no common $\mathbb{F}_p[\Gamma_F]$-sub-quotient, and neither contains the trivial representation.

How large p must be given (the root datum of) G can be extracted from the arguments of this section, but we do not make it explicit.

Remark 5.2. Given a global Galois representation $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$, we will refer to places $w \notin S$ of F such that $\bar{\rho}|_{\Gamma_{F_v}} = 1$ and $N(w) \equiv 1 \pmod{p}$ as "trivial primes." All of the auxiliary primes constructed in this paper will satisfy this condition (and frequently some refinement of this condition).

Remark 5.3. We could carry out the analysis of this section without the assumption that K does not contain μ_{p^2}; the difference is that the sets of trivial primes w that we produce would not necessarily satisfy $N(w) \not\equiv 1 \pmod{p^2}$ (equivalently, $N(w) \not\equiv 1 \pmod{\varpi^{e+1}}$). In particular, Corollary 5.13 does not require this assumption.

We decompose

$$\bar{\rho}(\mathfrak{g}^{\mathrm{der}}) = \oplus_{i \in I} W_i^{\oplus m_i}$$

where each W_i is an irreducible $\mathbb{F}_p[\Gamma_F]$-module, and $W_i \not\cong W_j$ for $i \neq j$. Dually we obtain the decomposition $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^* = \oplus_{i \in I}(W_i^*)^{m_i}$, where $W^* = \mathrm{Hom}_{\mathbb{F}_p}(W, \mathbb{F}_p)(1)$ is the $\mathbb{F}_p$-dual. Each W_i is a $k_{W_i} = \mathrm{End}_{\mathbb{F}_p[\Gamma_F]}(W_i)$-module, and since $\mathrm{Br}(\mathbb{F}_p) = 0$, k_{W_i} is a finite field extension of $\mathbb{F}_p$. We may then also regard W_i^* as the k_{W_i}-dual, with the trace identifying the k_{W_i}-vector spaces

$$\mathrm{tr}_{k_{W_i}/\mathbb{F}_p}\colon\ \mathrm{Hom}_{k_{W_i}}(W_i, k_{W_i}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p}(W_i, \mathbb{F}_p).$$

We begin by finding some mod ϖ^2 lift of $\bar{\rho}$. Let $T \supset S$ be a finite set of places with $T \setminus S$ consisting of trivial primes w not split in $K(\mu_{p^2})$ such that $\text{Ш}^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = 0$. That T can be so arranged follows from the first two items of Assumption 5.1: the cocycles in question restrict non-trivially to Γ_K, and then we choose places v that are split in K and non-split in both $K(\mu_{p^2})$ and the fixed field (over K) of the cocycle (the latter two conditions are compatible whether or not the fixed field is disjoint from $K(\mu_{p^2})$, since they are both just the condition of being non-trivial). By global duality, $\text{Ш}^2_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ also vanishes, so to produce some lift $\rho_2\colon \Gamma_{F,T} \to G(\mathcal{O}/\varpi^2)$ of $\bar{\rho}$ with multiplier μ it suffices to check there are no local lifting obstructions:

Lemma 5.4. *Assume $p \gg_G 0$. Then for all finite places v, the set of mod ϖ^2 local lifts $\mathrm{Lift}_{\bar{\rho}|_{\Gamma_{F_v}}}(\mathcal{O}/\varpi^2)$ is non-empty, and similarly for lifts of type μ.*

Proof. For $p \gg_G 0$, there exists a faithful representation $r\colon G \hookrightarrow \mathrm{GL}_N$ such that $\mathfrak{g}$ is a direct summand of $\mathfrak{gl}_n$ (as G-modules). The induced map $H^2(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g})) \to H^2(\Gamma_{F_v}, r \circ \bar{\rho}(\mathfrak{gl}_n))$ is thus injective, and it clearly sends the obstruction to lifting $\bar{\rho}$ (to $G(\mathcal{O}/\varpi^2)$) to the obstruction to lifting $r \circ \bar{\rho}$. But the latter is unobstructed by [BÖ3, Theorem 1.1]. The fixed multiplier character analogue follows from choosing some lift modulo ϖ^2 and then, using the fact that $Z_{G^0} \to G/G^{\mathrm{der}}$ has kernel of order prime to p, modifying it to a lift of type μ. ◻

In fact, in the application we will make a stronger assumption on $\bar{\rho}|_{\Gamma_{F_v}}$ for $v \in S$, obviating the need for this lemma; for now we are trying to proceed without superfluous hypotheses.

In what follows, it will be technically convenient to enlarge the set T by trivial primes non-split in $K(\mu_{p^2})$, beyond what is necessary to annihilate $\text{Ш}^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$. We may and do assume that our T strictly contains whatever initial choice of T was used to annihilate the Shafarevich-Tate groups; more precise enlargements of this set T will follow. We can compute such an enlargement's effect on global Galois cohomology. More generally, if W is any $\mathbb{F}_p[\Gamma_{F,T}]$-module, we have the analogous notion of a trivial prime v for W: $W|_{\Gamma_{F_v}}$ is trivial, and $N(v) \equiv 1 \pmod{p}$. If moreover W satisfies $\text{Ш}^1_T(\Gamma_{F,T}, W) = 0$, then for any trivial prime $v \notin T$ we have an exact sequence

$$0 \to H^1(\Gamma_{F,T}, W) \to H^1(\Gamma_{F,T\cup v}, W) \to H^1(\Gamma_{F_v}, W)/H^1_{\mathrm{unr}}(\Gamma_{F_v}, W) \to 0,$$

where the second map is given by evaluation at τ_v; surjectivity of this map follows from the Greenberg–Wiles Euler-characteristic formula ([DDT94, Theorem 2.19]). In particular, the cokernel of the inflation map has dimension $\dim W$.

We must modify our initial ρ_2 so that its local behavior allows further lifting. We will now fix certain local lifts to $G(\mathcal{O}/\varpi^2)$ that we would like to interpolate into a global mod ϖ^2 representation. In §5.15, we will be more particular about what lifts we choose, and we will make additional assumptions on the local behavior of $\bar{\rho}$; so as to be clear about what assumptions are used at what point in the paper, we delay imposing these additional hypotheses.

Construction 5.5. For the remainder of this section, we fix local lifts $\{\lambda_w\}_{w\in T}$ as follows:

- For $w \in S$, fix any lift $\lambda_w \in \mathrm{Lift}^{\mu}_{\bar{\rho}|_{\Gamma_{F_w}}}(\mathcal{O}/\varpi^2)$.
- For $w \in T \setminus S$, we simply choose any unramified lift (with multiplier μ) λ_w with the property that the elements $\lambda_w(\sigma_w)$ generate $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^2)$. For this to be possible, we may have to enlarge the set T, and we do this implicitly at this step of the argument.

Having fixed the local mod ϖ^2 lifts λ_w as in Construction 5.5, we have that for each $w \in T$ there is a class $z_w \in H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that

$$(1 + \varpi z_w)\rho_2|_w \sim \lambda_w \tag{2}$$

(with $\sim$ denoting strict equivalence). We wish to modify ρ_2 by a global cohomology class so that the resulting lift of $\bar{\rho}$ matches the specified local lifts λ_w.

If there exists a global class $h \in H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ mapping to $z_T := (z_w)_{w\in T}$ under the localization map

$$\Psi_T \colon H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{w\in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})),$$

then we proceed to §5.2. For the remainder of this section, we assume there is no such h. Denote by Ψ_T^* the corresponding localization map for $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$. To construct auxiliary primes, we will need the following lemma, which we will eventually apply to the irreducible constituents of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$:

Lemma 5.6. *Let W be an irreducible $\mathbb{F}_p[\Gamma_{F,T}]$-module such that $H^1(\mathrm{Gal}(F(W)/F), W) = 0$. Set $k_W = \mathrm{End}_{\mathbb{F}_p[\Gamma_{F,T}]}(W)$ (a finite extension of $\mathbb{F}_p$, since $\mathrm{Br}(\mathbb{F}_p) = 0$), and set $K = F(W, \mu_p)$. Let $\psi_1, \dots, \psi_s$ be a k_W-basis of $H^1(\Gamma_{F,T}, W)$. Then the fixed fields $K_{\psi_1}, \dots, K_{\psi_s}$ of the cocycles ψ_i are strongly linearly disjoint over K, and for each i, $\mathrm{Gal}(K_{\psi_i}/K) \xrightarrow[\psi_i]{\sim} W$. If moreover μ_{p^2} is not contained in K, and W is not isomorphic to the trivial representation, then for any $w \in W$ and any non-zero class $\psi \in H^1(\Gamma_{F,T}, W)$, there exists a Čebotarev set of trivial primes v not split in $K(\mu_{p^2})$ such that $\psi(\sigma_v) = w$.*

Proof. We must show that restriction gives an isomorphism

$$\mathrm{Gal}(K_{\psi_1} \cdots K_{\psi_s}/K) \xrightarrow{\sim} \prod_{i=1}^{s} \mathrm{Gal}(K_{\psi_i}/K) \xrightarrow{\sim} \prod_{i=1}^{s} W.$$

To see this, we induct on the number of factors. For $s = 1$, the isomorphism follows from simplicity of the $\mathbb{F}_p[\Gamma_F]$-module W (note that $\psi_i|_{\Gamma_K} \neq 0$). If the linear disjointness is known for $\psi_1, \dots, \psi_i$, and if $K_{\psi_{i+1}}$ is contained in the composite $K_{\psi_1} \cdots K_{\psi_i}$, then we have a map of $\mathbb{F}_p[\Gamma_{F,T}]$-modules

$$W^{\oplus i} \xleftarrow[\psi_1,\dots,\psi_i]{\sim} \mathrm{Gal}(K_{\psi_1} \cdots K_{\psi_i}/K) \twoheadrightarrow \mathrm{Gal}(K_{\psi_{i+1}}/K) \xrightarrow[\psi_{i+1}]{\sim} W.$$

Since W is irreducible, the composite $W^{\oplus i} \to W$ has the form $(a_1, \ldots, a_i)$ for some $a_i \in k_W$, and we deduce that $\psi_{i+1} = \sum_{j=1}^{i} a_j \psi_j$, contradicting linear independence. We conclude that $K_{\psi_{i+1}}$ is not contained in $K_{\psi_1} \cdots K_{\psi_i}$, but again since W is irreducible this forces these fields to be linearly disjoint over K.

The last claim is clear if K_ψ and $K(\mu_{p^2})$ are linearly disjoint over K. Otherwise, $K(\mu_{p^2})$ is contained in K_ψ, and so W has the $\mathbb{F}_p[\Gamma_F]$-quotient $\mathrm{Gal}(K(\mu_{p^2})/K)$ with trivial Γ_F-action; by assumption, this quotient is non-zero, so that W itself must be the trivial representation.

□

We next explain in Proposition 5.8 how to interpolate the class z_T by a global class after allowing ramification at a finite number of additional primes. An important technical point in the proof of Proposition 5.11 requires that we impose an additional (at this point rather unmotivated) condition on our trivial primes.

Definition 5.7. Let K' be the composite of all abelian p-extensions L of K that are Galois over F and that satisfy

- L/F is unramified outside T;
- the $\mathbb{F}_p[\mathrm{Gal}(K/F)]$-module $\mathrm{Gal}(L/K)$ is isomorphic to one of the W_i.

Because the extensions L/F are unramified outside T with absolutely bounded degree, K' is a finite extension of F. Primes split in K' are of course also split in K, and K' and $K(\mu_{p^2})$ are linearly disjoint over K since no W_i is the trivial representation. In what follows, we will refer to trivial primes split in K' but not in $K(\mu_{p^2})$ as K'-trivial primes.

Proposition 5.8. *Continue to assume that* $Ш^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = 0$*, and hence by duality that* $Ш^2_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = 0$*. Then there is an* $\mathbb{F}_p$*-basis* $\{Y_i\}_{i=1}^r$ *of the cokernel of the restriction map* $\Psi_T \colon H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v \in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$*, and, for each i, a Čebotarev set* C_i *of* K'*-trivial primes* $v \notin T$*, a split maximal torus* T_i *and root* $\alpha_i \in \Phi(G^0, T_i)$*, and for each* $v \in C_i$ *a class* $h^{(v)} \in H^1(\Gamma_{F,T \cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ *such that*

- $h^{(v)}|_T = Y_i$*; and*
- $h^{(v)}(\tau_v)$ *spans* $\mathfrak{g}_{\alpha_i}$.

Remark 5.9. One can ask whether it is possible to hit the class z_T by allowing only one additional prime of ramification; this is how the analogous argument in [HR08] (for reducible two-dimensional $\bar{\rho}$) works. We have only been able to show such a statement when $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ is multiplicity-free as an $\mathbb{F}_p[\Gamma_F]$-module, and even then only at the expense of arguments considerably more technical than those given here. Proposition 5.8 allows us to avoid this image restriction.

Proof. Since $Ш^2_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = 0$, the Poitou–Tate sequence yields a short exact sequence

$$0 \to Ш^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \xrightarrow{\Psi_T} \bigoplus_{v \in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \left(H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)\right)^\vee \to 0,$$

and $\mathrm{coker}(\Psi_T) \xrightarrow{\sim} \left(H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)\right)^\vee$. In particular, if $\dim_{\mathbb{F}_p} \mathrm{coker}(\Psi_T) = r$ is non-zero, then $H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$ contains a non-zero class ψ_1. We claim that we can choose a triple $(T_1, \alpha_1, X_{\alpha_1})$ consisting of a split maximal torus T_1, a root $\alpha_1 \in \Phi(G^0, T_1)$, and a root vector $X_{\alpha_1} \in \mathfrak{g}_{\alpha_1}$ such that $\psi_1(\Gamma_{K'})$ is not contained in $(\mathbb{F}_p X_{\alpha_1})^\perp$ (note that we work with the $\mathbb{F}_p$-span of X_{α_1} rather than the full root space). Indeed, for any $\mathbb{F}_p$-subspace U not equal to the whole of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$, there is a root vector not in U. To check this, we must check that the $\mathbb{F}_p$-span, or equivalently the k-span, of all root vectors in $\mathfrak{g}^{\mathrm{der}}$ is equal to the whole of $\mathfrak{g}^{\mathrm{der}}$. This claim in turn reduces to the case in which $\mathfrak{g}^{\mathrm{der}}$ is simple, where again (using $p \gg_G 0$) it follows from irreducibility of $\mathfrak{g}^{\mathrm{der}}$ as a $k[G(k)]$-module. Thus to find the desired triple it suffices to note that $\psi_1(\Gamma_{K'})$ is non-trivial, by combining the second and third parts of Assumption 5.1.

Now we let C_1 be the collection of K'-trivial primes v such that $\psi_1(\sigma_v)$ is not in $(\mathbb{F}_p X_{\alpha_1})^\perp$, and for each $v_1 \in C_1$ let $L_{v_1} = \{\phi \in H^1(\Gamma_{F_{v_1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) : \phi(\tau_{v_1}) \in \mathbb{F}_p X_{\alpha_1}\}$. We deduce from a few applications of the Greenberg–Wiles formula the following points:

- The fact that $\dim(L_{v_1}) = 1 + \dim(L_{v_1}^{\mathrm{un}})$ (and $\dim(L_{v_1}^\perp) = \dim(L_{v_1}^{\mathrm{un},\perp}) - 1$) and the existence of ψ_1 together imply that $h^1_{L_{v_1}^\perp}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = r - 1$, where this Selmer group notation means that we impose no condition at the places in T.
- The inclusion
$$Ш^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \subseteq \ker\left(H^1_{L_{v_1}}(\Gamma_{F,T\cup v_1}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))\right)$$
is an equality. To see this, apply the Greenberg–Wiles formula to the Selmer systems $\mathcal{L}_1 = \{0\}_{v\in T} \cup \{L_{v_1}\}$ and $\mathcal{L}_2 = \{0\}_{v\in T}$ and use the previous bullet-point.
- The cokernel of the restriction map $H^1_{L_{v_1}}(\Gamma_{F,T\cup v_1}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ has dimension $r-1$: by the last bullet-point, it suffices to show that
$$h^1_{L_{v_1}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = 1 + h^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})),$$
which is immediate from the Greenberg–Wiles formula and the vanishing $Ш^1_T(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = 0$.

Now, if $r - 1 > 0$, then for each $v_1 \in C_1$ we can choose a non-zero $\psi_2 \in H^1_{L_{v_1}^\perp}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$. Note that ψ_2 depends on v_1. Then we can repeat the above argument, choosing $(T_2, \alpha_2, X_{\alpha_2})$ such that $\psi_2(\Gamma_{K'})$ is not contained in $(\mathbb{F}_p X_{\alpha_2})^\perp$, and then define a Čebotarev set $C_2(v_1)$ (the notation includes the dependence on the initial choice of v_1) as the set of K'-trivial v such that $\psi_2(\sigma_v) \notin (\mathbb{F}_p X_{\alpha_2})^\perp$. The same argument with the Greenberg–Wiles formula shows that
$$\ker\left(H^1_{L_{v_1}, L_{v_2}}(\Gamma_{F,T\cup v_1\cup v_2}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))\right)$$
equals $\ker\left(H^1_{L_{v_1}}(\Gamma_{F,T\cup v_1}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))\right)$, and consequently that the dimension of the cokernel of the restriction map $H^1_{L_{v_1}, L_{v_2}}(\Gamma_{F,T\cup v_1\cup v_2}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ is now $r-2$. Proceeding inductively, we obtain Čebotarev sets $C_s(v_{s-1})$, depending on $v_{s-1} \in C_{s-1}(v_{s-2})$ (and so on), for $s = 1, \ldots, r$, such that for all tuples $(v_1, \ldots, v_r)$ with each $v_s \in C_s(v_{s-1})$, the restriction map
$$H^1_{L_{v_1}, \ldots, L_{v_r}}(\Gamma_{F,T\cup v_1,\ldots,v_r}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$$
is surjective.

In particular, the above argument produces an $\mathbb{F}_p$-basis $\psi_1, \ldots, \psi_r$ of $H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$, a collection of root vectors $X_{\alpha_1}, \ldots, X_{\alpha_r}$, and a collection of elements $Y_1, \ldots, Y_r \in \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ that map to a basis of $\mathrm{coker}(\Psi_T)$: for Y_i, we take *any* vector in the image of $H^1_{L_{v_i}}(\Gamma_{F,T\cup v_i}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in T} H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ that is not in $\mathrm{im}(\Psi_T)$. (These still span $\mathrm{coker}(\Psi_T)$ because if $\widetilde{Y}_i$ denotes a lift to $H^1_{L_{v_i}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ of Y_i, the $\{\widetilde{Y}_i\}_i$ span $\mathrm{coker}(H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to H^1_{L_{v_1},\ldots,L_{v_r}}(\Gamma_{F,T\cup v_1,\ldots,v_r}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, since they are independent for ramification reasons.) For each i, we also can fix an $\mathbb{F}_p$-basis $\omega_{i,1}, \ldots, \omega_{i,r-1}$ of $H^1_{L^{\perp}_{v_i}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$. Now we define the following Čebotarev condition:

$$C_i = \left\{K'\text{-trivial primes } v \text{ such that } \psi_i(\sigma_v) \notin (\mathbb{F}_p X_{\alpha_i})^{\perp} \text{ and } \omega_{i,k}(\sigma_v) \in (\mathbb{F}_p X_{\alpha_i})^{\perp} \text{ for all } k = 1, \ldots, r-1\right\}.$$

We know that $v_i \in C_i$, so each C_i is in fact a non-empty Čebotarev condition (without this observation, the conditions defining C_i could be incompatible). Now, for all $v \in C_i$, we define L_v as before to be those classes $\phi \in H^1(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $\phi(\tau_v) \in \mathbb{F}_p X_{\alpha_i}$ and deduce an exact sequence

$$0 \to Ш^1_T(\Gamma_{F,T}\bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to H^1_{L_v}(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{w\in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to (H^1_{L^{\perp}_v}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*))^{\vee} \to 0;$$

indeed, we apply the same Euler-characteristic arguments as above (using that $\psi_i|_v \notin L^{\perp}_v$), note that the composite $H^1_{L_v}(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to (H^1_{L^{\perp}_v}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*))^{\vee}$ is zero (by the standard Poitou–Tate sequence for $\Gamma_{F,T\cup v}$ acting on $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$), and then deduce the exactness by counting dimensions. We claim that Y_i lies in the image of $H^1_{L_v}(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, for which it suffices to check that Y_i annihilates $H^1_{L^{\perp}_v}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$. We know that Y_i annihilates $H^1_{L^{\perp}_{v_i}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$ (using exactness of the above sequence for v_i), so it suffices (and is in fact necessary) to observe that

$$H^1_{L^{\perp}_{v_i}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = H^1_{L^{\perp}_v}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*);$$

this holds because both subspaces of $H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$ are equal to the span of $\omega_{i,1}, \ldots, \omega_{i,r-1}$. □

We will also need the following simpler variant of Proposition 5.8.

Lemma 5.10. *Continue with the hypotheses of Proposition 5.8. Let $Z \in \mathfrak{g}^{\mathrm{der}}$ be any non-zero element. There is a Čebotarev set C of K'-trivial primes and for each $v \in C$ a class $h^{(v)} \in H^1(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that*

- *the restriction $h^{(v)}|_T$ is independent of $v \in C$; and*
- *$h^{(v)}(\tau_v)$ spans the line $\mathbb{F}_p Z$.*

Proof. Recall from the discussion preceding Lemma 5.4 that we have enlarged T to ensure that for all $i \in I$, $H^1(\Gamma_{F,T}, W_i^*) \neq 0$. Since $(\mathbb{F}_p Z)^{\perp} \subset (\mathfrak{g}^{\mathrm{der}})^*$ is a proper subspace, it does not contain some isotypic piece $(W_i^*)^{\oplus m_i}$, hence it does not contain some $\Gamma_{F,T}$-equivariantly embedded $W_i^* \hookrightarrow (W_i^*)^{\oplus m_i}$, and so there is a $\psi \in H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$ such that $\psi(\Gamma_{K'})$ is not contained in $(\mathbb{F}_p Z)^{\perp}$ (namely, a ψ supported on a suitable copy of W_i^*). We can now repeat the argument of Proposition 5.8. In brief, fix a K'-trivial prime v_1 such that $\psi(\sigma_{v_1})$ does not belong to $(\mathbb{F}_p Z)^{\perp}$, and as before define L_{v_1} to be the set of $\phi \in H^1(\Gamma_{F_{v_1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $\phi(\tau_{v_1}) \in \mathbb{F}_p Z$. The same analysis shows that there is an element

$$Y \in \mathrm{im}\Big(H^1_{L_{v_1}}(\Gamma_{F,T\cup v_1}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{w\in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})\Big) \setminus \mathrm{im}(\Psi_T),$$

and that if we let $\omega_1, \ldots, \omega_s$ be a basis of the (codimension 1) subspace $H^1_{L^{\perp}_{v_1}}(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) \subset H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$, then

$$C_Z = \{K'\text{-trivial primes } v : \psi(\sigma_v) \notin (\mathbb{F}_p Z)^{\perp} \text{ and } \omega_j(\sigma_v) \in (\mathbb{F}_p Z)^{\perp} \text{ for all } i = 1, \ldots, s\}$$

is a *non-empty* (because $v_1 \in C_Z$) Čebotarev condition. Then as in Proposition 5.8, we also see that for all $v \in C_Z$ there is a class $h^{(v)} \in H^1_{L_v}(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $h^{(v)}|_T = Y$, and $h^{(v)}(\tau_v)$ spans $\mathbb{F}_p Z$. □

Now we fix any finite set of root vectors (for possibly different split maximal tori) $\{X_{\alpha_a}\}_{a\in A}$ such that

$$\sum_{a\in A} \mathbb{F}_p[\Gamma_F]X_{\alpha_a} = \mathfrak{g}^{\mathrm{der}}.$$

(Such a collection $\{X_{\alpha_a}\}$ clearly exists, since for any proper subspace U of $\mathfrak{g}^{\mathrm{der}}$, there is some root vector not in U: see the proof of Proposition 5.8.) Lemma 5.10 yields Čebotarev sets $C_a = C_{X_{\alpha_a}}$ and classes $Y_a \in \bigoplus_{w\in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that for all $v \in C_a$, there is a class $h^{(v)} \in H^1(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ satisfying $h^{(v)}(\tau_v) \in \mathbb{F}_p X_{\alpha_a} \setminus 0$ and $h^{(v)}|_T = Y_a$. Consider the class

$$z'_T = z_T - \sum_{a\in A} Y_a.$$

This new element may or may not be in the image of Ψ_T, but we can in any case invoke Proposition 5.8 to produce a finite set $\{Y_b\}_{b\in B} \subset \bigoplus_{w\in T} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ that spans $\mathrm{coker}(\Psi_T)$ over $\mathbb{F}_p$, and, for each $b \in B$, a Čebotarev set C_b of K'-trivial primes and a root vector X_{α_b}, and for each $v \in C_b$ a class $g^{(v)} \in H^1(\Gamma_{F,T\cup v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $g^{(v)}|_T = Y_b$ and $g^{(v)}(\tau_v) \in \mathbb{F}_p X_{\alpha_b} \setminus 0$ (the reason for the shift in notation to $g^{(v)}$ will become apparent at the end of this paragraph). In particular, we can write

$$z'_T = h^{\mathrm{old}} + \sum_{b\in B} c_b Y_b$$

for some class $h^{\mathrm{old}} \in H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ and some $c_b \in \mathbb{F}_p$. We discard those $b \in B$ such that $c_b = 0$. Thus, for all tuples

$$(v_a)_{a\in A} \times (v_b)_{b\in B} \in \prod_{a\in A} C_a \times \prod_{b\in B} C_b,$$

we can write

$$z_T = h^{\mathrm{old}}|_T + \sum_{a\in A} h^{(v_a)}|_T + \sum_{b\in B} c_b g^{(v_b)}|_T.$$

Note that the vectors $h^{(v_a)}(\tau_{v_a})$ are non-zero multiples of X_{α_a} for all $a \in A$, so the collection $\{h^{(v_a)}(\tau_{v_a})\}_{a\in A}$ is a set of $\mathbb{F}_p[\Gamma_F]$-generators of $\mathfrak{g}^{\mathrm{der}}$. Having made note of this, we will in the argument that follows not need to preserve the distinction between the sets A and B, so we set $N = A \cup B$. In order to preserve this uniformity of notation, for all $b \in B$ and $v \in C_b$ we set $h^{(v)} = c_b g^{(v)}$, so we can re-express the above equality as

$$z^{\mathrm{der}}_T = h^{\mathrm{old}}|_T + \sum_{n\in N} h^{(v_n)}|_T$$

for any $\underline{v} = (v_n)_{n\in N} \in \prod_{n\in N} C_n$.

We will need to argue in terms of Dirichlet densities of N-tuples of primes. In what follows, we define the Dirichlet density of a subset P of $\{\text{primes of } F\}^N$ to be (if it exists)

$$\delta(P) = \lim_{s\to 1^+} \frac{\sum_{\underline{v}\in P} N(\underline{v})^{-s}}{\sum_{\text{all } \underline{v}} N(\underline{v})^{-s}},$$

where $N(\underline{v}) = \prod_{n\in N} N(v_n)$. In particular, the density of a product $P = \prod_{n\in N} P_n$ of sets P_n of primes exists if each P_n has a density, and in this case $\delta(P) = \prod_{n\in N} \delta(P_n)$. We make a corresponding definition of upper Dirichlet density $\delta^+(P)$ of a set of N-tuples of primes. In particular, the preceding discussion yields a Čebotarev set $C = \prod_{n\in N} C_n$ of positive Dirichlet density.

The following argument substantially uses global duality, and we need to preface with a technical clarification of what coefficients we can take in the duality pairings. We have the $\mathbb{F}_p[\Gamma_F]$-isotypic decomposition

$$\bar{\rho}(\mathfrak{g}^{\mathrm{der}}) = \bigoplus_{i \in I} V_i = \bigoplus_{i \in I} W_i^{\oplus m_i},$$

where the various W_i are mutually non-isomorphic irreducible $\mathbb{F}_p[\Gamma_F]$-modules with endomorphism algebras $k_{W_i} = \mathrm{End}_{\mathbb{F}_p[\Gamma_F]}(W_i)$ (a finite extension of $\mathbb{F}_p$). We may (and do) fix an isomorphism of V_i with $W_i \otimes_{\mathbb{F}_p} \mathbb{F}_{p^{m_i}}$ as $k_{W_i}[\Gamma_F]$-modules (with trivial Galois action on $\mathbb{F}_{p^{m_i}}$). This gives V_i the structure of an $A_i[\Gamma_F]$-module, where $A_i := k_{W_i} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^{m_i}}$, with V_i being finite free as an A_i-module. In the sequel, duals and duality pairings will be considered with respect to this fixed structure.[3]

Proposition 5.11. *There is a finite set of K'-trivial primes Q disjoint from T and a class $h \in H^1(\Gamma_{F,T\cup Q}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that*

- *$h|_T = z_T$.*
- *For all $w \in Q$ there is a pair (T_w, α_w) of a split maximal torus T_w of G^0 and a root $\alpha_w \in \Phi(G^0, T_w)$ such that $(1 + \varpi h)\rho_2(\tau_w) = u_{\alpha_w}(X_w)$ for some α_w-root vector X_w, and $(1 + \varpi h)\rho_2(\sigma_w)$ takes on any desired value in $\widehat{G}(\mathcal{O}/\varpi^2)$, subject to having multiplier μ.*

Proof. We have seen that there is a class $h^{\mathrm{old}} \in H^1(\Gamma_{F,T}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that for any N-tuple $\underline{v} = (v_n)_{n \in N} \in C = \prod_{n \in N} C_n$, the global class $h(\underline{v}) = h^{\mathrm{old}} + \sum_{n \in N} h^{(v_n)}$ satisfies $h(\underline{v})|_T = z_T$. Since we cannot say anything about the restrictions $h(\underline{v})|_{v_n}$, we will use the "doubling method" of [KLR05] to find the desired Q and h. To that end, for any two N-tuples $\underline{v}, \underline{v}' \in C$, we consider the class

$$h = h^{\mathrm{old}} - \sum_{n \in N} h^{(v_n)} + 2 \sum_{n \in N} h^{(v'_n)} \in H^1(\Gamma_{F,T\cup\{v_n\}\cup\{v'_n\}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})), \tag{3}$$

which still satisfies $h|_T = z_T$ (and the inertial conditions dictated by the construction of the classes $h^{(v_n)}$). The argument will show that for a suitable choice of $\underline{v}$ and $\underline{v}'$, h will satisfy the conclusion of the Proposition with the set Q equal to $\{v_n\}_{n \in N} \cup \{v'_n\}_{n \in N}$.

We first restrict to a positive upper-density subset $\mathfrak{l} \subset C$ (now no longer necessarily a product of Čebotarev sets) such that the N-tuples $(\sum_{n \in N} h^{(v_n)}(\sigma_{v_m}))_{m \in N}$, $(h^{\mathrm{old}}(\sigma_{v_m}))_{m \in N}$, and $(h^{(v_n)}(\tau_{v_n}))_{n \in N}$ are independent of the choice of $\underline{v} \in \mathfrak{l}$; this is possible since as we vary over C, these N-tuples take on only finitely many values. In particular, we write X_n for the now independent-of-$\underline{v}$ value $h^{(v_n)}(\tau_{v_n})$ (for all $n \in N$, this is a non-zero multiple of X_{α_n}). Recall the decomposition $\bar{\rho}(\mathfrak{g}^{\mathrm{der}}) = \bigoplus_{i \in I} V_i = \bigoplus_{i \in I} W_i^{\oplus m_i}$ into $\mathbb{F}_p[\Gamma_F]$-isotypic components. For $i \in I$ we let $X_{n,i}$ denote the V_i-component of X_n. By construction, therefore, we have

$$\sum_{n \in N} \mathbb{F}_p[\Gamma_F] X_{n,i} = V_i.$$

We will show that for any fixed N-tuples $(C_m)_{m \in N}$ and $(C'_m)_{m \in N}$ of elements of $\mathfrak{g}^{\mathrm{der}}$, there exist $\underline{v}, \underline{v}' \in \mathfrak{l}$ such that

$$\sum_{n \in N} h^{(v'_n)}(\sigma_{v_m}) = C_m, \tag{4}$$

$$\sum_{n \in N} h^{(v_n)}(\sigma_{v'_m}) = C'_m, \tag{5}$$

[3] We use the trace maps to identify duals over the various etale $\mathbb{F}_p$-algebras that we consider.

for all $m \in N$. This will suffice to prove the Proposition, since, by Equation (3), it will allow us to prescribe the values $h(\sigma_{v_m})$ and $h(\sigma_{v'_m})$ for all $m \in N$; we then choose the C_m and C'_m such that the values $(1+\varpi h)\rho_2(\sigma_{v_m})$ and $(1+\varpi h)\rho_2(\sigma_{v'_m})$ take on whatever values we wish to prescribe.

We will now study the condition, for *fixed* $\underline{v} = (v_n)_{n \in N} \in \mathfrak{l}$, imposed on $\underline{v}'$ by Equations (4) and (5), beginning with Equation (5). For each $n \in N$, consider the maximal Galois extension $K^{(v_n)}$ of F inside $K_{h^{(v_n)}}$ that is unramified at v_n; this contains K, and

$$\sum_{n \in N} h^{(v_n)}(\mathrm{Gal}(K_{h^{(v_n)}}/K^{(v_n)})) = \mathfrak{g}^{\mathrm{der}}, \tag{6}$$

since the n-component of this sum contains X_n and is $\mathbb{F}_p[\Gamma_F]$-stable (only the $n \in A$ are needed to guarantee Equation (6) holds).

For each $m \in N$, we consider the Čebotarev condition $\mathfrak{w}_m$ on trivial primes w requiring that w split in all $K^{(v_n)}$ and that

$$\sum_{n \in N} h^{(v_n)}(\sigma_w) = C_m.$$

Since the composite of the fields $K^{(v_n)}$ is still unramified at each v_n, Equation (6) implies that this condition is non-empty. Moreover, since $K_{h^{(v_n)}} \cap K' = K^{(v_n)}$, $\mathfrak{w}_m$ induces a non-empty Čebotarev condition $\mathfrak{w}'_m$ where we further impose the condition that all primes in $\mathfrak{w}'_m$ are split in K'.

Now we turn to the condition needed to satisfy Equation (4). For all $m \in N$ and $i \in I$, letting $\{\eta_{i,j}^{(v_m)}\}_{j=1}^{d_i}$ be elements of $H^1(\Gamma_{F,T \cup v_m}, W_i^*)$ that lift a k_{W_i}-basis of $H^1(\Gamma_{F,T\cup v_m}, W_i^*)/H^1(\Gamma_{F,T}, W_i^*)$, we have for all m, n, i, j the global duality relation (writing $h^{(v)} = \sum_{i \in I} h_i^{(v)}$ for the decomposition into V_i-components)

$$\begin{aligned} \langle \eta_{i,j}^{(v_m)}(\tau_{v_m}), h_i^{(v'_n)}(\sigma_{v_m})\rangle &= -\sum_{x \in T} \langle \eta_{i,j}^{(v_m)}, h_i^{(v'_n)}\rangle_x - \langle \eta_{i,j}^{(v_m)}(\sigma_{v'_n}), h_i^{(v'_n)}(\tau_{v'_n})\rangle, \\ &= -\sum_{x \in T} \langle \eta_{i,j}^{(v_m)}, h_i^{(v_n)}\rangle_x - \langle \eta_{i,j}^{(v_m)}(\sigma_{v'_n}), X_{n,i}\rangle, \end{aligned}$$

where we systematically work with the A_i-linear pairings. Summing over n, we want to show that for all $m \in N$, $i \in I$, $j = 1, \dots, d_i$, we can prescribe by a Čebotarev condition (depending on our fixed $\underline{v}$) on $\underline{v}' \in \mathfrak{l}$ the values

$$\sum_{n \in N} \langle \eta_{i,j}^{(v_m)}(\sigma_{v'_n}), X_{n,i}\rangle \in A_i,$$

for then we can achieve the same for the values

$$\langle \eta_{i,j}^{(v_m)}(\tau_{v_m}), \sum_{n \in N} h_i^{(v'_n)}(\sigma_{v_m})\rangle.$$

Prescribing these values for varying m, i, j will allow us to achieve the equality of Equation (4).

The splitting fields $K_{\eta_{i,j}^{(v_m)}}$ are strongly linearly disjoint over K as we vary $m \in N$, $i \in I$, and $j = 1, \dots, d_i$,[4] so it suffices for this last claim to note that for any fixed non-zero vector $w_i^* \in W_i^*$ we have

$$\sum_{n \in n} \langle W_i^*, X_{n,i}\rangle = \sum_{n \in n} \langle \mathbb{F}_p[\Gamma_F] w_i^*, X_{n,i}\rangle = \sum_{n \in n} \langle w_i^*, \mathbb{F}_p[\Gamma_F] X_{n,i}\rangle = \langle w_i^*, V_i\rangle = A_i.$$

[4] Since the W_i^* are irreducible, the fields are disjoint as m varies because $K_{\eta_{i,j}^{(v_m)}}$ is ramified at v_m (and not at $v_{m'}$ for $m' \neq m$); they are disjoint as i varies because the W_i^* are mutually non-isomorphic; and they are disjoint as j varies by Lemma 5.6.

The Čebotarev condition on $\underline{v}'$ thus obtained is independent from the Čebotarev condition $\prod_{m\in N}\mathfrak{w}'_m$ determined above, since K' and the $K_{h^{(v_n)}}$ are strongly linearly disjoint from the $K_{\eta_{i,j}^{(v_m)}}$ ($\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ have no $\mathbb{F}_p[\Gamma_F]$-subquotient in common). In sum, we obtain a non-empty Čebotarev condition $\mathfrak{l}_{\underline{v}}$ on tuples $\underline{v}'$ of K'-trivial primes such that the desired equalities

$$\sum_{n\in N} h^{(v'_n)}(\sigma_{v_m}) = C_m$$
$$\sum_{n\in N} h^{(v_n)}(\sigma_{v'_m}) = C'_m$$

hold for any $\underline{v}' \in \mathfrak{l}\cap\mathfrak{l}_{\underline{v}}$. We have no assurance that this intersection is non-empty, so now we must invoke the limiting logic of [KLR05] and [HR08] that allows the doubling method to succeed. If for each member of a finite subset $\{\underline{v}_1,\dots,\underline{v}_s\}\subset\mathfrak{l}$, the intersection $\mathfrak{l}\cap\mathfrak{l}_{\underline{v}_k}$ is empty, then $\mathfrak{l}\setminus\{\underline{v}_1,\dots,\underline{v}_s\}$ is contained in $\mathfrak{l}\cap\bigcap_{k=1}^{s}\overline{\mathfrak{l}_{\underline{v}_k}}$. We will control the upper-density of this latter intersection. For each $k=1,\dots,s$, let $K_{h^{(\underline{v}_k)}}$ denote the composite of the fields $K_{h^{(\underline{v}_{k,n})}}$ for $n\in N$, and let $K_{\eta^{(\underline{v}_k)}}$ denote the composite of the fields $K_{\eta_{i,j}^{(\underline{v}_{k,n})}}$ for $n\in N$, $i\in I$, $j=1,\dots,d_i$. For fixed k, the fields $K_{h^{(\underline{v}_k)}}$ and $K_{\eta^{(\underline{v}_k)}}$ are linearly disjoint over K, and $\mathfrak{l}_{\underline{v}_k}$ is a Čebotarev condition in their composite. As k varies, the $K_{\eta^{(\underline{v}_k)}}$ will be strongly linearly disjoint, but the $K_{h^{(\underline{v}_k)}}$ may not be disjoint over K. This is where the field K' becomes significant.

We now replace the fields $K_{h^{(\underline{v}_k)}}$ and $K_{\eta^{(\underline{v}_k)}}$ with their composites $K'_{h^{(\underline{v}_k)}}$ and $K'_{\eta^{(\underline{v}_k)}}$ with K'. We will finally be able to make the limiting argument by observing that, even as k varies, the fields $K'_{h^{(\underline{v}_k)}}$ and $K'_{\eta^{(\underline{v}_k)}}$ are now all strongly linearly disjoint over K'. In the intersection $C\cap\bigcap_{k=1}^{s}\overline{\mathfrak{l}_{\underline{v}_k}}$, each term $C\cap\overline{\mathfrak{l}_{\underline{v}_k}}$ is for some finite extension L_k/K' a Čebotarev condition on primes in F picking out a proper subset of elements of $\mathrm{Gal}(L_k/K')$: for the properness, note that each $\overline{\mathfrak{l}_{\underline{v}_k}}$ is a union of complements of the proper conditions we have imposed on each $K'_{h^{(\underline{v}_{k,n})}}$ for $n\in N$ and $K'_{\eta^{(\underline{v}_k)}}$ for $n\in N$, $i\in I$, $j=1,\dots,d_i$, so we get a proper condition by the disjointness of these fields over K'. Also note that the degrees of the extensions L_k/K' are bounded independently of $\underline{v}_k\in\mathfrak{l}$, in terms of $|\mathfrak{g}^{\mathrm{der}}$—, $|N|$, and $\sum_{i\in I} d_i$. Finally, we can conclude that

$$\delta^{+}(\mathfrak{l}\setminus\{\underline{v}_1,\dots,\underline{v}_s\}) \le \delta(C)(1-\varepsilon)^s$$

for some $\varepsilon>0$. Letting s tend to infinity, we see that $\delta^{+}(\mathfrak{l})$ is less than any positive number, contradicting the fact that $\mathfrak{l}$ has positive upper-density. We conclude that for some $\underline{v}\in\mathfrak{l}$, there is a $\underline{v}'\in\mathfrak{l}\cap\mathfrak{l}_{\underline{v}}$, and so the proof is complete. $\square$

In our application of this result (see Proposition 5.15), there will be two ways we choose the values $(1+\varpi h)\rho_2(\sigma_w)$ for $w\in Q$; we make these choices explicit here:

Lemma 5.12. *In the conclusion of Proposition 5.11, we can prescribe the values* $t_w := (1+\varpi h)\rho_2(\sigma_w)$*, for* $w\in Q$*, such that:*

- *If* $e\ge 2$*, then* t_w *lies in* $Z_{G^0}(\mathcal{O}/\varpi^2)$ *(with value determined by* μ*), and in particular* $\alpha_w(t_w)\equiv 1\equiv N(w) \pmod{\varpi^2}$.
- *If* $e=1$*, then for all roots* $\beta\in\Phi(G^0,T_w)$*,* $\beta(t_w)\not\equiv 1 \pmod{\varpi^2}$*, and* $\alpha_w(t_w)\equiv N(w) \pmod{\varpi^2}$.

(In both cases the lifts belong to $\mathrm{Lift}_{\bar{\rho}|_{\Gamma_{F_w}}}^{\alpha_w,\mu}(\mathcal{O}/\varpi^2)$*, but we have imposed some precise conditions that will be useful in the application.)*

Proof. The $e \geq 2$ case is obvious. The $e = 1$ case is just a matter of checking that the condition $p \gg_G 0$ can be arranged to guarantee that elements t_w in sufficiently general position exist. Fix a trivial prime $w \in Q$, with associated (T_w, α_w), and let $q_w = N(w)$. Let $q_w^{1/2}$ denote the square-root of q_w in $\mathcal{O}/p^2$ that is congruent to 1 modulo p. Consider elements of $\widehat{T_w^{\mathrm{der}}}(\mathcal{O}/p^2)$ of the form $t_b = (1 + pb)\alpha_w^\vee(q_w^{1/2})$ for $b \in \ker(\alpha)$ and having fixed (specified by μ) projection to $\mathfrak{z}_G$ under the decomposition $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \oplus \mathfrak{z}_G$. By factoring out a term $(1 + pz)$ with $z \in \mathfrak{z}_G$, it will suffice to treat the case $b \in \mathfrak{g}^{\mathrm{der}}$. Any such t_b satisfies $\alpha(t_b) = q_w$ and reduces to $1 \in G(k)$; we will find b such that $\beta(t_b) \neq 1$ for all $\beta \in \Phi := \Phi(G^0, T_w)$. Decompose $\mathfrak{g}^{\mathrm{der}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ is the simple factor supporting α_w, and $\mathfrak{g}_2$ is the sum of the remaining simple factors. Correspondingly decompose $\mathrm{Lie}(T_w) = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, the set of roots $\Phi = \Phi_1 \sqcup \Phi_2$, and the element $b = b_1 + b_2$. We choose $b_2 \in \mathfrak{t}_2$ such that $\beta(\mathfrak{t}_2) \neq 0$ for all $\beta \in \Phi_2$; that this is possible requires that the union of hyperplanes $\cup_{\beta\in\Phi_2} \ker(\beta|_{\mathfrak{t}_2})$ not equal all of $\mathfrak{t}_2$, which is clearly not a problem for $p \gg_G 0$. To choose $b_1 \in \mathfrak{t}_1$, first note that, for any $\beta \in \Phi_1$, if there is no $b_\beta \in \ker(\alpha_w|_{\mathfrak{t}_1})$ such that $\beta(b_\beta) = -\frac{q_w-1}{2p}\langle\beta, \alpha^\vee\rangle$, then the condition $\beta(t_b) \neq 1$ will be satisfied automatically for any choice of $b_1 \in \ker(\alpha_w|_{\mathfrak{t}_1})$, so we now restrict to the subset Φ_1^* of Φ_1 for which such b_β do exist (and we fix one such b_β for each β). Then we choose b_1 in the complement of the union of hyperplanes

$$\bigcup_{\beta\in\Phi_1^*\setminus\{-\alpha\}} \left(b_\beta + \ker(\beta|_{\ker(\alpha_w|_{\mathfrak{t}_1})})\right)$$

inside $\ker(\alpha_w|_{\mathfrak{t}_1})$ (note that since β belongs to the same simple factor as α_w and is not equal to $-\alpha_w$, $\ker(\beta|_{\ker(\alpha_w|_{\mathfrak{t}_1})})$ is indeed a hyperplane in $\ker(\alpha_w|_{\mathfrak{t}_1})$). The total number of such hyperplanes is bounded in a way depending only on the Dynkin type of G^{der}, so for $p \gg_G 0$, this complement is non-empty. Clearly for such a b_1, and $b = b_1 + b_2$, we have $\beta(t_b) \neq 1$ for all $\beta \in \Phi$ and $\alpha(t_b) = q_w$. $\square$

The arguments of this section yield a generalization to any reductive group of the main theorem of [KLR05]. We sketch here a somewhat simplified version:

Corollary 5.13. *Let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ satisfy Assumption 5.1, except we do not require that K does not contain μ_{p^2}. (In particular, the results of Appendix A will show that for $p \gg_G 0$, it suffices here to assume $\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible, and $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$, for the integer a_G arising in Lemma A.6.) Fix a lift $\mu\colon \Gamma_{F,S} \to G/G^{\mathrm{der}}(\mathcal{O})$ of $\bar{\mu} = \bar{\rho} \pmod{G^{\mathrm{der}}}$, and assume that for all $v \in S$, there are lifts $\rho_v\colon \Gamma_{F_v} \to G(\mathcal{O})$ of $\bar{\rho}|_{\Gamma_{F_v}}$ with multiplier μ. Then there exists an infinitely ramified lift*

$$\begin{tikzcd} & G(\mathcal{O}) \arrow[d] \\ \Gamma_F \arrow[r, "\bar{\rho}"'] \arrow[ur, dashed, "\rho"] & G(k) \end{tikzcd}$$

such that $\rho|_{\Gamma_{F_v}} = \rho_v$ modulo $\widehat{G^{\mathrm{der}}}(\mathcal{O})$-conjugacy for all $v \in S$, and the reduction modulo ϖ^2 of $\rho(\Gamma_F)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O}/\varpi^2)$. In particular, if for $\mathcal{O}$ we can take the ring of Witt vectors $W(k)$, then $\rho(\Gamma_F)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O})$.

Remark 5.14. In this degree of generality, it is not known, but certainly expected, that local lifts ρ_v as above always exist.

Proof. See Remark 5.3 for an explanation of the slight modification of our hypotheses. For any $G(\mathcal{O})$-valued representation λ, write λ_n for its reduction modulo ϖ^n. Applying Proposition 5.11, we can find a lift $\rho_2 \colon \Gamma_{F,T\cup Q_1} \to G(\mathcal{O}/\varpi^2)$ such that $\rho_2|_{\Gamma_{F_v}} = \rho_{v,2}$ modulo $\widehat{G^{\mathrm{der}}}(\mathcal{O})$-conjugacy (no different from $\widehat{G}(\mathcal{O})$-conjugacy) for all $v \in S$, and for all $v \in T \cup Q_1 \setminus S$, $\rho_2|_{\Gamma_{F_v}}$ admits a lift ρ_v to $G(\mathcal{O})$ (by Lemma 3.2). For all $v \in T \cup Q_1$, let $g_{v,1} \in \widehat{G^{\mathrm{der}}}(\mathcal{O})$ satisfy $g_{v,1}\rho_2|_{\Gamma_{F_v}} g_{v,1}^{-1} = \rho_{v,2}$ (taking $g_{v,1} = 1$ for $v \in T \cup Q_1 \setminus S$). We then iterate the argument of Proposition 5.11: there are no obstructions to lifting ρ_2 to $G(\mathcal{O}/\varpi^3)$, and then by introducing further trivial primes Q_2 of ramification we may find a lift $\rho_3 \colon \Gamma_{F,T\cup Q_1 \cup Q_2} \to G(\mathcal{O}/\varpi^3)$, and for all $v \in T \cup Q_1$, an element $g_{v,2} \in \ker(G^{\mathrm{der}}(\mathcal{O}) \to G^{\mathrm{der}}(\mathcal{O}/\varpi^2)$ such that $g_{v,2}\rho_3|_{\Gamma_{F_v}} g_{v,2}^{-1} = g_{v,1}^{-1}\rho_{v,3} g_{v,1}$ for all $v \in T \cup Q_1$, and for $v \in Q_2$, $\rho_3|_{\Gamma_{F_v}}$ lies on some $\mathrm{Lift}^{\alpha}_{\bar{\rho}|_{\Gamma_{F_v}}}$, so again by Lemma 3.2 admits a lift ρ_v to $G(\mathcal{O})$ (and we then set $g_{v,2} = g_{v,1} = 1$ for $v \in Q_2$). We thus inductively construct $\rho = \varprojlim_n \rho_n$ having the desired properties, noting that the infinite products $\prod_{i=1}^{\infty} g_{v,i}$ converge. The first statement about $\mathrm{im}(\rho)$ is immediate from the construction of ρ_2; the second statement, when $e = 1$, is proven by inductively showing that $\mathrm{im}(\rho_2)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O}/p^2)$ implies $\mathrm{im}(\rho_n)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O}/p^n)$ for all $n \geq 2$, with the base case $n = 2$ coming from the construction of ρ_2. Suppose the claim is known for n. We will show that every element of the kernel of $\widehat{G^{\mathrm{der}}}(\mathcal{O}/p^{n+1}) \to \widehat{G^{\mathrm{der}}}(\mathcal{O}/p^n)$ is in any subgroup H surjecting onto $\widehat{G^{\mathrm{der}}}(\mathcal{O}/p^n)$. Embedding G into some GL_N, we will argue with matrices. Let $s = 1 + p^n X$ be in the above kernel. By assumption there is some element $y \in H$ of the form $y = 1 + p^{n-1}X + p^n Y$. Then H also contains

$$y^p = (1 + p^{n-1}X + p^n Y)^p = \sum_{i=0}^{p} \binom{p}{i} (p^{n-1}X + p^n Y)^i = 1 + p^n X,$$

since $n \geq 2$, and we are working modulo p^{n+1}. □

5.2. **Constructing the mod ϖ^N lift.** In the proof of the main theorem (§6) we will prove a lifting theorem that requires as input a carefully-constructed mod ϖ^N representation for some N depending on both local (the restrictions $\bar{\rho}|_{\Gamma_{F_v}}$ for $v \in S$) and global (the image of $\bar{\rho}$) properties of our given residual representation $\bar{\rho}$. To that end, in this section we extend Proposition 5.11 to prove the following:

Theorem 5.15. *Let $p \gg_G 0$ be a prime. Let F be any number field, and let $\bar{\rho} \colon \Gamma_{F,S} \to G(k)$ be a continuous representation such that $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible.*[5] *Assume that $[\widetilde{F}(\zeta_p) : \widetilde{F}]$ is strictly greater than the constant a_G of Lemma A.6. Fix a lift μ of the multiplier character $\bar{\mu} = \bar{\rho}$ (mod G^{der}). Moreover assume that for all $v \in S$ there are lifts $\rho_v \colon \Gamma_{F_v} \to G(\mathcal{O})$ with multiplier μ. Let $T \supset S$ be the set constructed in §5.1 in the discussion preceding Construction 5.5, and likewise fix unramified lifts ρ_v for each $v \in T \setminus S$, such that ρ_v (mod ϖ^2) is the lift λ_v specified in Construction 5.5.*

Then for any given integer $M \geq \max\{3, e+1\}$ and any $N \gg M$ (in particular, $N > 2M + \max\{2e, 4\}$ suffices), there exist a finite set of primes T_N containing T and a lift $\rho_N \colon \Gamma_{F,T_N} \to G(\mathcal{O}/\varpi^N)$ of $\bar{\rho}$ with multiplier μ, such that, letting $\rho_M := \rho_N$ (mod ϖ^M) be the reduction:

(1) If $w \in T_N \setminus T$ is ramified in ρ_N, then it satisfies the following properties:

[5] An inspection of the proof shows that we only use the following: $\bar{\rho}$ satisfies Assumption 5.1 and the field K is linearly disjoint over $\widetilde{F}(\mu_p)$ from $\widetilde{F}(\mu_{p^\infty})$.

(a) For some $s \in \{1, 2, e\}$ we have $\rho_s := \rho_N \pmod{\varpi^s}$ is trivial (mod center) on Γ_{F_w}, $N(w) \equiv 1 \pmod{\varpi^s}$ and $N(w) \not\equiv 1 \pmod{\varpi^{s+1}}$.

(b) There is a split maximal torus and root (T, α) (these as usual depend on w, but we omit the dependence from the notation) such that $\rho_M|_{\Gamma_{F_w}} \in \mathrm{Lift}_{\bar{\rho}}^{\mu,\alpha}(\mathcal{O}/\varpi^M)$, $\rho_M(\sigma_w) \in T(\mathcal{O}/\varpi^M)$, and $\beta(\rho_M(\sigma)) \not\equiv 1 \pmod{\varpi^{s+1}}$ for all roots $\beta \in \Phi(G^0, T)$.

(c) $\rho_N|_{\Gamma_{F_w}} \in \mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^N)$, and $\rho_N(\sigma_w) \in T(\mathcal{O}/\varpi^N)$.

(d) For any $m \geq N - M > M$, and any $\rho_{m,w} \in \mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^m)$ lifting $\rho_{N-M}|_{\Gamma_{F_w}}$, for any $1 \leq r \leq M$ the fiber of $\mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^{m+r}) \to \mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^m)$ over ρ_m is non-empty and stable under the subspace Z_r^α of cocycles introduced in Lemma 3.5.

(2) For all $v \in T$, $\rho_N|_{\Gamma_{F_v}}$ is strictly equivalent to $\rho_v \pmod{\varpi^N}$.

(3) The image $\rho_N(\Gamma_F)$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^N)$.

Proof. By Corollary A.7, Assumption 5.1 holds for $\bar{\rho}$, and we can apply the results and techniques of §5.1. We inductively lift $\bar{\rho}$ to a $\rho_n \colon \Gamma_{F,T_n} \to G(\mathcal{O}/\varpi^n)$, for each $n = 2, \dots, N$, at each stage increasing the ramification set T_n. The case $n = 2$ is settled by Proposition 5.11, which produces an auxiliary set of primes $T_2 \supset T$ and a lift $\rho_2 \colon \Gamma_{F,T_2} \to G(\mathcal{O}/\varpi^2)$ such that for each $w \in T_2 \setminus S$, one of the following holds:

- Either w is one of the primes (with corresponding fixed lifts) in $T \setminus S$ introduced in the discussion up to and including Construction 5.5, so that w is a trivial prime, and the lift $\rho_2|_{\Gamma_{F_w}}$ is unramified;
- or w is one of the primes introduced in the proof of Proposition 5.11. That is, w splits in K', $N(w) \equiv 1 \pmod{\varpi^e}$ but $N(w) \not\equiv 1 \pmod{\varpi^{e+1}}$, and there is a pair (T, α) (depending on w) for which $\rho_2|_{\Gamma_{F_w}} \in \mathrm{Lift}_{\bar{\rho}|_{\Gamma_{F_w}}}^{\mu,\alpha}(\mathcal{O}/\varpi^2)$, and moreover $\rho_2(\sigma_w) \in T(\mathcal{O}/\varpi^2)$ is
  - trivial modulo center if $e \geq 2$; and
  - satisfies $\beta(\rho_2(\sigma_w)) \not\equiv 1 \pmod{\varpi^2}$ for all roots β if $e = 1$.

  (This precise conclusion follows from combining Lemma 5.12 with Proposition 5.11.)

We now carry out the induction step, showing how, for $n \geq 3$, to pass from a lift $\rho_{n-1} \colon \Gamma_{F,T_{n-1}} \to G(\mathcal{O}/\varpi^{n-1})$ to ρ_n. For each $v \in T$, fix a lift (with multiplier μ) $\lambda_v \colon \Gamma_{F_v} \to G(\mathcal{O}/\varpi^n)$ of $\rho_{n-1}|_{\Gamma_{F_v}}$ such that λ_v is strictly equivalent to $\rho_{v,N}$ (this is clearly possible, since $\widehat{G}$ is formally smooth, and we are given the characteristic zero lift ρ_v). For each $w \in T_{n-1} \setminus T$, fix a lift λ_w of $\rho_{n-1}|_{\Gamma_{F_w}}$ satisfying

- $\lambda_w \in \mathrm{Lift}_{\bar{\rho}|_w}^{\mu,\alpha_w}(\mathcal{O}/\varpi^n)$;
- if $n \leq e$, then $\lambda_w(\sigma_w) = 1$ modulo center, and $\lambda_w(\tau_w) \in U_\alpha(\mathcal{O}/\varpi^n)$ (these conditions define a representation of Γ_{F_w} since $N(w) \equiv 1 \pmod{\varpi^n}$);
- and if $n \geq e + 1$, then $\lambda_w(\sigma_w) \in T(\mathcal{O}/\varpi^n)$, $\lambda_w(\tau_w) \in U_\alpha(\mathcal{O}/\varpi^n)$, and for all roots $\beta \in \Phi(G^0, T)$, $\beta(\lambda_w(\sigma_w)) \not\equiv 1 \pmod{\varpi^{e+1}}$ if $e > 1$ and $\beta(\lambda_w(\sigma_w)) \not\equiv 1 \pmod{\varpi^3}$ if $e = 1$.

This is possible by the formal smoothness Lemma 3.2. We also, without altering the notation, enlarge T (simply for bookkeeping convenience, we index these primes as part of T rather than the larger set T_{n-1}) by a finite set of primes split in $K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))$ and introduce at these w any unramified $\lambda_w \colon \Gamma_{F_w} \to G(\mathcal{O}/\varpi^n)$ with multiplier μ (which we may assume trivial by imposing a further splitting condition) such that $\lambda_w \pmod{\varpi^{n-1}}$ is trivial, and the elements $\lambda_w(\sigma_w)$ generate

$\ker(G^{\mathrm{der}}(\mathcal{O}/\varpi^n) \to G^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1}))$.[6] We also (for subsequent steps in the induction) at such primes fix any (multiplier μ) unramified lift $\rho_w\colon \Gamma_{F_w} \to G(\mathcal{O})$ of λ_w.

Since there are no local obstructions to lifting ρ_{n-1}, and $\text{Ш}^2_{T_{n-1}}(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = 0$ (by global duality and the vanishing of $\text{Ш}^1_{T_{n-1}}(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$), there is some lift $\rho'_n\colon \Gamma_{F,T_{n-1}} \to G(\mathcal{O}/\varpi^n)$ of multiplier μ. We wish to correct ρ'_n to match the $\{\lambda_w\}_{w\in T_{n-1}}$ locally, and so we again apply the method of Proposition 5.11. Let $z_{T_{n-1}} = (z_w)_{w\in T_{n-1}} \in \bigoplus_{w\in T_{n-1}} H^1(\Gamma_{F_w}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ be the collection of cohomology classes such that $(1+\varpi^{n-1}z_w)\rho'_n$ is equivalent to λ_w for all $w \in T_{n-1}$; that is, we can choose representative cocycles so that these two lifts of $\rho_{n-1}|_{\Gamma_{F_w}}$ are actually equal.

Now, in the proof of Proposition 5.11 (building on Proposition 5.8 and Lemma 5.10), we have produced a finite set N and for each $i \in N$ a positive-density Čebotarev set C_i with the following properties:

- There is a class $h^{\mathrm{old}} \in H^1(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, and, for each N-tuple $\underline{v} = (v_i)_{i\in N} \in \prod_{i\in N} C_i$, classes $h^{(v_i)} \in H^1(\Gamma_{F,T_{n-1}\cup v_i}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $h^{(v_i)}|_{T_{n-1}}$ is independent of the choice of $v_i \in C_i$, and
$$z_{T_{n-1}} = h^{\mathrm{old}}|_{T_{n-1}} + \sum_{i\in N} h^{(v_i)}|_{T_{n-1}}.$$
- Each $h^{(v_i)}(\tau_{v_i})$ spans a root space of $\mathfrak{g}^{\mathrm{der}}$, depending only on the class C_i, and the $\mathbb{F}_p[\Gamma_F]$-span of these root spaces is all of $\mathfrak{g}^{\mathrm{der}}$.
- Each C_i is a Čebotarev condition defined in the composite of the fields K', $F(\mu_{p^2})$, and splitting fields (over K) of dual Selmer classes $\psi \in H^1(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$.

This last point is crucial, and we explain now how to modify the construction of the C_i for our present purposes. Define K'_{n-1} to be the composite of abelian p-extensions L/K that are Galois over F, unramified outside T_{n-1}, and have $\mathrm{Gal}(L/K)$ isomorphic as Γ_F-module to one of the simple factors W_j of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ (i.e., K'_{n-1} is the unramified-outside-T_{n-1} analogue of K'). We will replace our previous Čebotarev condition with a Čebotarev condition in the composite of the fields K'_{n-1}, $K(\mu_{p^n})$, $K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))$,[7] and the splitting fields K_ψ of classes $\psi \in H^1(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$. Namely, we claim that the conditions in the fields K_ψ (dictated by the proofs of Proposition 5.8 and Lemma 5.10) are compatible with the conditions of being

- if $e = 1$: split in K'_{n-1}, split in $K(\rho_2(\mathfrak{g}^{\mathrm{der}}))$ and $K(\mu_{p^2})$, but otherwise equal to any classes we choose in $\mathrm{Gal}(K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))/K(\rho_2(\mathfrak{g}^{\mathrm{der}})))$ and $\mathrm{Gal}(K(\mu_{p^n})/K(\mu_{p^2}))$; and
- if $e > 1$: split in K'_{n-1}, split in $K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}}))$, but otherwise equal to any classes we choose in $\mathrm{Gal}(K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))/K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}})))$ and $\mathrm{Gal}(K(\mu_{p^n})/K(\mu_p))$.

First we note that the intersection of K with the cyclotomic $\mathbb{Z}_p$-extension of F is trivial by Lemma A.6 and our assumptions that $\bar{\rho}$ is absolutely irreducible and $p \gg_G 0$; thus $K \cap F(\mu_{p^\infty}) = F(\mu_p)$.

[6] This is the analogue of the enlargement of the initial set T in Construction 5.5. Both there and here, the only reason for introducing these primes is that if, without them, the localization map Ψ_T defined after Construction 5.5 had image containing z_T, then the global lifts we produce wouldn't automatically have image containing $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^n)$. If Ψ_T missed z_T, in which case we have to run the argument of Proposition 5.11, then this big image condition would be automatic from the proof of Proposition 5.11, namely from the fact that the root spaces $\mathfrak{g}_{\alpha_i}$ associated with the auxiliary Čebotarev sets C_i satisfy $\sum_{i\in N} \mathbb{F}_p[\Gamma_F]\mathfrak{g}_{\alpha_i} = \mathfrak{g}^{\mathrm{der}}$, and the image of ρ_n would by construction contain each
$$\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}^{\mathrm{der}} \cong \ker\left(\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^n) \to \widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1})\right).$$
We additionally remark that when $e = 1$ the fact that $\mathrm{im}(\rho_{n-1})$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1})$ formally implies the corresponding statement for $\mathrm{im}(\rho_n)$, as noted in the proof of Corollary 5.13.

[7] By this we mean the fixed field of the adjoint action of ρ_{n-1} on the mod ϖ^{n-1} Lie algebra $\mathfrak{g}^{\mathrm{der}}$.

That the conditions on the K_ψ are independent from the other conditions is clear from our familiar arguments ($\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ has no subquotients in common with $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ or with the trivial Galois module). For compatibility of the other conditions, note that since $\mathrm{Gal}(K'_{n-1}(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))/K)$ is as Γ_F-module an iterated extension of simple factors of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$, and therefore has no trivial sub-quotient, $K'_{n-1}(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))$ is linearly disjoint over K from the extension $K(\mu_{p^n})$. The splitting conditions in K'_{n-1} and $K(\rho_2(\mathfrak{g}^{\mathrm{der}}))$ (for $e = 1$) or $K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}}))$ (for $e > 1$) are of course compatible, and we claim that K'_{n-1} and $K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))$ are linearly disjoint over $K(\rho_2(\mathfrak{g}^{\mathrm{der}}))$ (and in the case $e > 1$, the same reasoning shows K'_{n-1} and $K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}}))$ are linearly disjoint over $K(\rho_2(\mathfrak{g}^{\mathrm{der}}))$, and thus the composite $K'_{n-1}K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}}))$ is linearly disjoint from $K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))$ over $K(\rho_{\min\{e,n-1\}}(\mathfrak{g}^{\mathrm{der}}))$). Indeed, we may inductively assume (this is something we will have to check persists at each step) that $\mathrm{im}(\rho_{n-1})$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1})$, so that $\mathrm{Gal}(K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}}))/K)$ is isomorphic to $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1})$. The abelianization of the latter group is isomorphic to $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^2)$ (this is proven by induction, the key point being that $\mathfrak{g}^{\mathrm{der}} = [\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}]$), and since K'_{n-1}/K is abelian, we see that $K'_{n-1} \cap K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}})) = K(\rho_2(\mathfrak{g}^{\mathrm{der}}))$.

Simply repeating the preliminaries to Proposition 5.11, we can therefore arrange that our new Čebotarev sets C_i (we reuse the notation) consist of primes v_i split in K'_{n-1} and satisfying:

- There is a class $h^{\mathrm{old}} \in H^1(\Gamma_{F,T_{n-1}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$, and, for each N-tuple $\underline{v} = (v_i)_{i \in N} \in \prod_{i \in N} C_i$, classes $h^{(v_i)} \in H^1(\Gamma_{F,T_{n-1}\cup v_i}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ such that $h^{(v_i)}|_{T_{n-1}}$ is independent of the choice of $v_i \in C_i$, and

$$z_{T_{n-1}} = h^{\mathrm{old}}|_{T_{n-1}} + \sum_{i \in N} h^{(v_i)}|_{T_{n-1}}.$$

- Each $h^{(v_i)}(\tau_{v_i})$ spans a root space $\mathfrak{g}_{\alpha_i}$ of $\mathfrak{g}^{\mathrm{der}}$ (with respect to a split maximal torus T_{α_i}, depending only on the class C_i, and the $\mathbb{F}_p[\Gamma_F]$-span of these root spaces is all of $\mathfrak{g}^{\mathrm{der}}$.
- If $e = 1$, then $N(v_i) \equiv 1 \pmod{\varpi^2}$ but $N(v_i) \not\equiv 1 \pmod{\varpi^3}$; and if $e > 1$, then $N(v_i) \equiv 1 \pmod{\varpi^e}$ and $N(v_i) \not\equiv 1 \pmod{\varpi^{e+1}}$.
- $\rho_{n-1}|_{\Gamma_{F_{v_i}}}$ belongs to $\mathrm{Lift}^{\mu,\alpha_i}_{\bar{\rho}|_{\Gamma_{F_{v_i}}}}(\mathcal{O}/\varpi^{n-1})$, is unramified with $\rho_{n-1}(\sigma_{v_i}) \in T_{\alpha_i}(\mathcal{O}/\varpi^{n-1})$, and
  - if $e = 1$, then $\rho_{n-1}(\sigma_{v_i})$ is trivial modulo ϖ^2 and (for $n-1 \geq 3$) general modulo ϖ^3, i.e., $\beta(\rho_3(\sigma_{v_i})) \not\equiv 1 \pmod{\varpi^3}$ for all roots β;
  - if $e > 1$, then $\rho_{n-1}(\sigma_{v_i})$ is trivial modulo $\varpi^{\min\{n-1,e\}}$ and (for $n-1 \geq e+1$) general modulo ϖ^{e+1}.

The proof of Proposition 5.11 now applies *mutatis mutandis*. Indeed, we as before fix a positive upper-density subset $\mathfrak{l} \subset \prod_{i \in N} C_i$ such that the N-tuples $(\sum_{i \in N} h^{(v_i)}(\sigma_{v_j}))_{j \in N}$, $(h^{\mathrm{old}}(\sigma_{v_j}))_{j \in N}$, and $(h^{(v_i)}(\tau_{v_i}))_{i \in N}$ are independent of $\underline{v} \in \mathfrak{l}$. Then having fixed a $\underline{v} \in \mathfrak{l}$, the Čebotarev conditions $\mathfrak{w}'_j$ ($j \in N$) described there can now be replaced with the condition that the primes $w \in \mathfrak{w}'_j$ should split in K'_{n-1} and satisfy the identity $\sum_{i \in N} h^{(v_i)}(\sigma_w) = C_j$. We likewise replace the condition $\mathfrak{l}_{\underline{v}}$ with the intersection of the conditions $\prod_{j \in N} \mathfrak{w}'_j$ and the conditions previously defined in the extension field $K_{\eta^{(\underline{v})}}$ (see Proposition 5.11). In particular, the Equations (4) and (5) will be satisfied by any pair $\underline{v}$, $\underline{v}'$ with $\underline{v} \in \mathfrak{l}$, $\underline{v}' \in \mathfrak{l} \cap \mathfrak{l}_{\underline{v}}$. In the limiting argument of the doubling method, we instead work with the composites of K'_{n-1} with the fields $K_{h^{(\underline{v}_k)}}$ and $K_{\eta^{(\underline{v}_k)}}$. Everything then works as before, and we find that for fixed elements $C_j, C'_j \in \mathfrak{g}^{\mathrm{der}}$, there exists a pair $\underline{v}$, $\underline{v'}$ of elements of $\mathfrak{l}$ such that

$$\sum_{i \in N} h^{(v'_i)}(\sigma_{v_j}) = C_j$$
$$\sum_{i \in N} h^{(v_i)}(\sigma_{v'_j}) = C'_j.$$

By appropriate choice of the N-tuples $(C_j)_{j\in N}$, $(C'_j)_{j\in N}$, we can then arrange that the class

$$h = h^{\mathrm{old}} - \sum_{i\in N} h^{(v_i)} + 2\sum_{i\in N} h^{(v'_i)}$$

satisfies $h|_{T_{n-1}} = z_{T_{n-1}}$; for all $i \in N$, $h(\tau_{v_i})$ and $h(\tau_{v'_i})$ span root spaces $\mathfrak{g}_{\alpha_i}$ of $\mathfrak{g}$; and $h(\sigma_{v_i})$ and $h(\sigma_{v'_i})$ take any values we wish to prescribe in $\mathfrak{g}^{\mathrm{der}}$.

To conclude the proof of the theorem, we specify these values so that the modified lift $\rho_n = (1+\varpi^{n-1}h)\rho'_n$ belongs to $\mathrm{Lift}^{\mu,\alpha_i}_{\bar{\rho}|_{v_i}}(\mathcal{O}/\varpi^n)$ (and likewise for v'_i) for all $i \in N$, with $\rho_n(\sigma_{v_i}) \in T_{\alpha_i}(\mathcal{O}/\varpi^n)$; and that moreover these lifts either still be trivial on σ_{v_i} (when $n \leq e$), or be in general position in the sense that $\beta(\rho_n(\sigma_{v_i})) \not\equiv 1 \pmod{\varpi^n}$ for all roots β (and likewise for v'_n). This second point is possible because of the conditions we have already imposed on v_i in the extensions $K(\rho_{n-1}(\mathfrak{g}^{\mathrm{der}})$ and $K(\mu_{p^n})$. We set $T_n = T_{n-1} \cup \{v_i\}_{i\in N} \cup \{v'_i\}_{i\in N}$, and we then claim that the conclusions of the Theorem are satisfied by $\rho_n \colon \Gamma_{F,T_n} \to G(\mathcal{O}/\varpi^n)$. Parts (1a)-(1c) are evident from the construction; note that the conditions that the various $\rho_n(\sigma_w)$ actually lie in the specified tori $T(\mathcal{O}/\varpi^n)$ persists because of the manner in which we have chosen the λ_w. Part (1d) (and more) follows from Lemma 3.5, and it is here that we must take $N \gg_M 0$. Part (2) is again immediate from the choice, at each stage, of the new local lifts λ_w for $w \in T$. Part (3) follows inductively since when we enlarged T at the beginning of the inductive step, we ensured that for some subset of T the values $\rho_n(\sigma_w) = \lambda_w(\sigma_w)$ generate $\ker(G^{\mathrm{der}}(\mathcal{O}/\varpi^n) \to G^{\mathrm{der}}(\mathcal{O}/\varpi^{n-1})$. ◻

6. Relative deformation theory

In this section we explain the relative deformation theory method and prove our main theorem, Theorem 6.9.

6.1. **Relative Selmer groups.** We need some preliminaries before proceeding to the heart of the argument. Let F be a number field and S a finite set of primes of F. Let $n \geq 1$ be an integer and let $\rho_n : \Gamma_F \to G(\mathcal{O}/\varpi^n)$ be a continuous homomorphism. Throughout this section, when we have an integer $r < n$, we will write ρ_r for the reduction $\rho_n \pmod{\varpi^r}$. For each prime $v \in S$ we assume that for $0 < r \leq n$ we have subgroups $T_{r,v} \subset Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ containing the group of boundaries $B^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$, and compatible with inclusion and reduction maps as described either in Lemma 3.5 or Proposition 4.7. We let $L_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ be the image of $T_{r,v}$ and we let $L^{\perp}_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$ be the annihilator of $L_{r,v}$ under the local duality pairing.

We define the Selmer group $H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ to be

$$\ker\left(H^1(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v\in S} \frac{H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))}{L_{r,v}}\right)$$

and we define the dual Selmer group $H^1_{\mathcal{L}_r^{\perp}}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$ analogously.

Lemma 6.1. *For any a, b such that $0 < a, b$ and $a + b \leq n$ there are exact sequences*

$$\rho_b(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F} \to H^1_{\mathcal{L}_a}(\Gamma_{F,S}, \rho_a(\mathfrak{g}^{\mathrm{der}})) \to H^1_{\mathcal{L}_{a+b}}(\Gamma_{F,S}, \rho_{a+b}(\mathfrak{g}^{\mathrm{der}})) \to H^1_{\mathcal{L}_b}(\Gamma_{F,S}, \rho_b(\mathfrak{g}^{\mathrm{der}}))$$

and

$$(\rho_b(\mathfrak{g}^{\mathrm{der}})^*)^{\Gamma_F} \to H^1_{\mathcal{L}_a^\perp}(\Gamma_{F,S}, \rho_a(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_{a+b}^\perp}(\Gamma_{F,S}, \rho_{a+b}(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_b^\perp}(\Gamma_{F,S}, \rho_b(\mathfrak{g}^{\mathrm{der}})^*)\ .$$

Proof. The exact sequence

$$0 \to \rho_a(\mathfrak{g}^{\mathrm{der}}) \to \rho_{a+b}(\mathfrak{g}^{\mathrm{der}}) \to \rho_b(\mathfrak{g}^{\mathrm{der}}) \to 0 \tag{7}$$

gives rise to an exact sequence

$$0 \to T_{a,v} \to T_{a+b,v} \to T_{b,v} \to 0\ .$$

Since all $T_{r,v}$ contain the boundaries, we get an exact sequence

$$L_{a,v} \to L_{a+b,v} \to L_{b,v} \to 0\ . \tag{8}$$

The first exact sequence in the lemma then follows from a diagram chase using the commutative diagram

$$\begin{tikzcd}
\rho_b(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F} \arrow[r] \arrow[d] & H^1(\Gamma_{F,S}, \rho_a(\mathfrak{g}^{\mathrm{der}})) \arrow[r] \arrow[d] & H^1(\Gamma_{F,S}, \rho_{a+b}(\mathfrak{g}^{\mathrm{der}})) \arrow[r] \arrow[d] & H^1(\Gamma_{F,S}, \rho_b(\mathfrak{g}^{\mathrm{der}})) \arrow[d] \\
\bigoplus_{v\in S} \rho_b(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}} \arrow[r] & \bigoplus_{v\in S} H^1(\Gamma_{F_v}, \rho_a(\mathfrak{g}^{\mathrm{der}})) \arrow[r] & \bigoplus_{v\in S} H^1(\Gamma_{F_v}, \rho_{a+b}(\mathfrak{g}^{\mathrm{der}})) \arrow[r] & \bigoplus_{v\in S} H^1(\Gamma_{F_v}, \rho_b(\mathfrak{g}^{\mathrm{der}})),
\end{tikzcd}$$

where the horizontal sequences come from the long exact sequence of cohomology of (7) and the vertical maps are coordinatewise restriction maps. The key point is that for all v, $L_{a,v} \subset H^1(\Gamma_{F_v}, \rho_a(\mathfrak{g}^{\mathrm{der}}))$ contains the image of the boundary map $(\rho_b(\mathfrak{g}^{\mathrm{der}}))^{\Gamma_{F_v}} \to H^1(\Gamma_{F_v}, \rho_a(\mathfrak{g}^{\mathrm{der}}))$, so the exact sequence (8) extends to an exact sequence

$$\rho_b(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}} \to L_{a,v} \to L_{a+b,v} \to L_{b,v} \to 0\ .$$

The sequence for dual Selmer groups follows in a similar way. Here the key point is that the surjectivity of the second map in (8) implies by duality that $L_{a,v}^\perp$ contains the image of the boundary map $(\rho_b(\mathfrak{g}^{\mathrm{der}})^*)^{\Gamma_{F_v}} \to H^1(\Gamma_{F_v}, \rho_a(\mathfrak{g}^{\mathrm{der}})^*)$ corresponding to the dual of (7) (with a and b interchanged). □

The basic object we will study in what follows is the relative (dual) Selmer group:

Definition 6.2. For $0 < r \leq n$, we define the r-th relative Selmer group to be

$$\overline{H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}}))} := \mathrm{im}\left(H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \to H^1_{\mathcal{L}_1}(\Gamma_{F,S}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))\right)$$

and the r-th relative dual Selmer group to be

$$\overline{H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)} := \mathrm{im}\left(H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_1^\perp}(\Gamma_{F,S}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)\right).$$

Given an element ϕ in a modulo ϖ^r (dual) Selmer group, we will write $\bar{\phi}$ for its image in the corresponding modulo ϖ (dual) Selmer group.

In addition, we say that the local conditions $\mathcal{L}_r$ are *balanced* if

$$\dim(H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})) = \dim(H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*))$$

for $0 < r \leq n$.

Lemma 6.3. *Suppose the local conditions $\mathcal{L}_r = \{L_{r,v}\}_{v\in S}$ are balanced and the spaces of invariants $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F}$ and $(\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\Gamma_F}$ are both zero. Then the relative Selmer and dual Selmer groups are also balanced, i.e.,*

$$\dim(\overline{H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}}))}) = \dim(\overline{H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)}).$$

Proof. This follows from the assumptions and the exact sequences in Lemma 6.1 for the case $a = n-1$ and $b = 1$. □

6.2. **Annihilating the relative (dual) Selmer group.** We now begin to work with a residual representation as in our eventual theorem. Let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ satisfy the hypotheses of Theorem 5.15. Let Γ be the inverse image in $G^{\mathrm{ad}}(\mathcal{O})$ of $\mathrm{im}(\mathrm{Ad}\circ\bar{\rho}) \subset G^{\mathrm{ad}}(k)$. We apply Lemma B.2 to the group Γ, the module $M = \mathrm{Lie}(\Gamma) = \mathfrak{g}^{\mathrm{der}}$, and the integer $m = 1$ to deduce that the image of the reduction map

$$H^1(\Gamma, \mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^M) \to H^1(\Gamma, \mathfrak{g}^{\mathrm{der}} \otimes k)$$

is zero for all M greater than some integer M_1 (depending only on $\mathrm{im}(\bar{\rho})$).

We now assume that we have integers $M \geq M_1$ and $N \geq 4M$ and a homomorphism $\rho_N\colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$ lifting our given $\bar{\rho}$ such that $\mathrm{im}(\rho_N)$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^N)$ (for instance, any ρ_N produced by an application of Theorem 5.15). We may and do assume that M is divisible by e. For any $1 \leq r \leq N$, we write F_r for the splitting field $\widetilde{F}(\rho_r(\mathfrak{g}^{\mathrm{der}}))$. As in §5.1, we write $K = \widetilde{F}(\bar{\rho}(\mathfrak{g}^{\mathrm{der}}), \mu_p)$. We also let $F^*_M = F_M(\mu_{p^{M/e}})$ and $F^*_N = F_N(\mu_{p^{M/e+1}})$.

Lemma 6.4. *Provided M is sufficiently large, in a manner depending only on* $\mathrm{im}(\bar{\rho})$*, we have:*

- $H^1(\mathrm{Gal}(F^*_N/F), \rho_M(\mathfrak{g}^{\mathrm{der}})^*) = 0$.
- *The map* $H^1(\mathrm{Gal}(F^*_N)/F), \rho_M(\mathfrak{g}^{\mathrm{der}})) \to H^1(\mathrm{Gal}(F^*_N/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ *is zero.*

Proof. For the first item, it suffices to prove that $H^1(\mathrm{Gal}(F^*_N/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) = 0$. Since $p \gg_G 0$, it follows from Lemma A.6 and our assumption (from Theorem 5.15) $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$ that K/F_1 is a nontrivial extension of degree prime to p. Consider the inflation-restriction sequence

(9) $$H^1(\mathrm{Gal}(K/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) \to H^1(\mathrm{Gal}(F^*_N/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) \to H^1(\mathrm{Gal}(F^*_N/K), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\mathrm{Gal}(K/F)}$$

The extension F_N/F_1 has degree a power of p so it is linearly disjoint from K/F_1. Thus, $\mathrm{Gal}(K/F_1)$ acts trivially on $\mathrm{Gal}(F^*_N/F_1)$. This implies that

$$H^1(\mathrm{Gal}(F^*_N/K), (\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\mathrm{Gal}(K/F_1)} = \mathrm{Hom}(\mathrm{Gal}(F^*_N/K), (\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\mathrm{Gal}(K/F_1)}).$$

Since $(\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\mathrm{Gal}(K/F_1)} = 0$, the last term in (9) is also zero. The first term in (9) is zero using Lemma A.6: namely, applying inflation-restriction and using the fact that $\mathrm{Gal}(K/F_1)$ has order prime to p, we reduce to checking that $((\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\mathrm{Gal}(K/F_1)} = 0$, which holds since $\kappa|_{\Gamma_{F_1}}$ is non-trivial. Thus we are done with the first claim.

For the second item, let F_N be as above, and consider the inflation-restriction exact sequence

(10) $$0 \to H^1(\mathrm{Gal}(F_N/F), \rho_M(\mathfrak{g}^{\mathrm{der}})) \to H^1(\mathrm{Gal}(F^*_N/F), \rho_M(\mathfrak{g}^{\mathrm{der}})) \to H^1(\mathrm{Gal}(F^*_N/F_N), \rho_M(\mathfrak{g}^{\mathrm{der}}))^{\mathrm{Gal}(F_N/F)}.$$

The right-hand side is zero: letting $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, any homomorphism $\mathrm{Gal}(F^*_N/F_N) \to \rho_M(\mathfrak{g}^{\mathrm{der}})$ factors through the maximal p-group quotient $\mathrm{Gal}(F^*_N \cap \mathbb{Q}_\infty)/F_N$, and as F_N is linearly disjoint from $\mathbb{Q}_\infty$ (by Lemma A.6 and the fact that $p \gg_G 0$) the $\mathrm{Gal}(F_N/F)$-action on this quotient group is trivial. Thus

$$H^1(\mathrm{Gal}(F^*_N/F_N), \rho_M(\mathfrak{g}^{\mathrm{der}}))^{\mathrm{Gal}(F_N/F)} = \mathrm{Hom}(\mathrm{Gal}(F^*_N \cap \mathbb{Q}_\infty/F_N), \rho_M(\mathfrak{g}^{\mathrm{der}})^{\mathrm{Gal}(F_N/F)}) = 0,$$

since $\rho_M(\mathfrak{g}^{\mathrm{der}})^{\mathrm{Gal}(F_N/F)} = 0$ by absolute irreducibility of $\bar{\rho}$ and Lemma A.2.

We are reduced to proving that the map $H^1(\mathrm{Gal}(F_N/F), \rho_M(\mathfrak{g}^{\mathrm{der}})) \to H^1(\mathrm{Gal}(F_N/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ is zero if $N \geq M \geq M_1$. Let Γ as above be the inverse image in $G^{\mathrm{ad}}(\mathcal{O})$ of $\mathrm{Gal}(F_N/F) \subset G^{\mathrm{ad}}(\mathcal{O}/\varpi^N)$. Using the fact that the inflation map $H^1(\mathrm{Gal}(F_N/F), A) \to H^1(\Gamma, A)$ is injective for any $\mathrm{Gal}(F_N/F)$-module A, the claim then follows from the choice of $M \geq M_1$ produced by Lemma B.2. $\square$

We continue with our fixed lift $\rho_N \colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$, with reduction ρ_M. We define the set of auxiliary primes that we will consider in annihilating the (relative) dual Selmer group:

Definition 6.5. Let Q_M be the set of primes v of F satisfying the following properties:

- $N(v) \equiv 1 \pmod{\varpi^M}$ but $N(v) \not\equiv 1 \pmod{\varpi^{M+1}}$.
- $\rho_N|_{\Gamma_{F_v}}$ is unramified (with multiplier μ), and $\rho_M|_{\Gamma_{F_v}}$ is trivial (mod center).
- There exist a split maximal torus T of G and a root $\alpha \in \Phi(G^0, T)$ such that $\rho_N(\sigma_v) \in T(\mathcal{O}/\varpi^N)$ and $\alpha(\rho_N(\sigma_v)) = \kappa(\sigma_v) = N(v)$.
- For all roots $\beta \in \Phi(G^0, T)$, $\beta(\rho_{M+1}(\sigma_v)) \not\equiv 1 \pmod{\varpi^{M+1}}$.

For these primes we will consider the functors of lifts $\mathrm{Lift}_{\bar{\rho}}^{\mu,\alpha}$ and $\mathrm{Lift}_{\rho_M}^{\mu,\alpha}$ as in Definitions 3.1 and 3.4.

Lemma 6.6. *Assume $N \geq 4M$. Then for any $v \in Q_M$, with corresponding (T, α), the following properties hold:*

- *For any $m \geq N - M$ and $1 \leq r \leq M$, the fibers of $\mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^{m+r}) \to \mathrm{Lift}_{\rho_M}^{\mu,\alpha}(\mathcal{O}/\varpi^m)$ are non-empty and stable under $Z_{r,v}^{\alpha}$, where $Z_{r,v}^{\alpha} \subset Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is the submodule produced in Lemma 3.5.*
- *Let $L_{r,v}^{\alpha}$ be the image of $Z_{r,v}^{\alpha}$ in $H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$, and let $L_{r,v}^{\alpha,\perp}$ be its annihilator in $H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$ under the local duality pairing. Then*
  - $|L_{r,v}^{\alpha}| = h^0(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})) = h^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$.
  - *The inclusion*
  $$L_{r,v}^{\alpha} \cap H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \hookrightarrow H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$$
  has cokernel isomorphic to $\mathcal{O}/\varpi^r$, and this cokernel is generated by the image of the unramified cocycle that maps σ_v to $H_\alpha = d(\alpha^\vee)(1)$, the usual coroot element in $\mathfrak{t}^{\mathrm{der}}$.
  - *The inclusion*
  $$L_{r,v}^{\alpha,\perp} \cap H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*) \hookrightarrow H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$$
  has cokernel isomorphic to $\mathcal{O}/\varpi^r$, and this cokernel is generated by the image of the unramified cocycle that maps σ_v to any element of $(\mathfrak{g}^{\mathrm{der}})^*$ whose restriction to $\mathfrak{g}_\alpha$ spans the free rank one $\mathcal{O}/\varpi^r$-module $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}_\alpha, \mathcal{O}/\varpi^r)$.

Proof. For the first point, we simply apply Lemma 3.5 with $s = M$. For the second point, we note that since ρ_M is trivial at v, and $N(v) \equiv 1 \pmod{\varpi^M}$, $H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ consists of all cocycles $\phi_X^{\mathrm{un},r}$ for $X \in \rho_r(\mathfrak{g}^{\mathrm{der}})$ (here we use the notation of Lemma 3.5, i.e. $\phi_X^{\mathrm{un},r}(\sigma_v) = X$). By the proof of Lemma 3.5, we see that $L_{r,v}^{\alpha}$ consists of the unramified cocycles $\phi_X^{\mathrm{un},r}$ for all $X \in \ker(\alpha|_{\mathfrak{t}^{\mathrm{der}}}) \oplus \bigoplus_{\beta \in \Phi(G^0,T)} \mathfrak{g}_\beta$, as well as the ramified cocycle ϕ_α^r; indeed, since $\rho_N|_{\Gamma_{F_v}}$ is unramified, the cocycles denoted $\phi_{X_\beta}^r$ in Lemma 3.5 are equal to the cocycles $\phi_{X_\beta}^{\mathrm{un},r}$. It follows immediately that the cokernel of $L_{r,v}^{\alpha} \cap H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \hookrightarrow H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is the free $\mathcal{O}/\varpi^r$-module of rank 1 spanned by (the image of) $\phi_{H_\alpha}^{\mathrm{un},r}$. The claim about the other inclusion follows from this description of $L_{r,v}^{\alpha}$, the fact that $H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})) = \mathrm{Hom}(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is isomorphic to two copies of $\rho_r(\mathfrak{g}^{\mathrm{der}})$ (via evaluation at σ_v and τ_v), and Lemma 3.7. □

The following two central results, Proposition 6.7 and Theorem 6.8, are our replacements for Ramakrishna's original Selmer-annihilation arguments. We will prove them under the restricted hypothesis that $\mathfrak{g}^{\mathrm{der}}$ consists of a single $\pi_0(G)$-orbit of simple factors; in our main theorem, Theorem 6.9, we will reduce to this case.

Proposition 6.7. *Assume that $\mathfrak{g}^{\mathrm{der}}$ consists of a single $\pi_0(G)$-orbit of simple factors. Let Q be any finite subset of Q_M, and let $\phi \in H^1_{\mathcal{L}_M}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ and $\psi \in H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$ be such that $0 \neq \overline{\phi} \in \overline{H^1_{\mathcal{L}_M}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}}))}$ and $0 \neq \overline{\psi} \in \overline{H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)}$. Then there exists a prime $v \in Q_M$, with associated torus and root (T, α), such that*
- $\overline{\psi}|_{\Gamma_{F_v}} \notin L^{\alpha,\perp}_{1,v}$*; and*
- $\phi|_{\Gamma_{F_v}} \notin L^{\alpha}_{M,v}$*.*

Proof. We first note that both $\phi|_{\Gamma_{F_N^*}}$ and $\overline{\psi}|_{\Gamma_{F_N^*}}$ are non-zero. For the former, consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(\mathrm{Gal}(F_N^*/F), \rho_M(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & H^1(\Gamma_{F,S}, \rho_M(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & H^1(\Gamma_{F_N^*}, \rho_M(\mathfrak{g}^{\mathrm{der}})) \\
& \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\
0 \longrightarrow & H^1(\mathrm{Gal}(F_N^*/F), \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & H^1(\Gamma_{F,S}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & H^1(\Gamma_{F_N^*}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})),
\end{array}$$

where the (exact) rows are inflation-restriction sequences, and where Lemma 6.4 implies the map labeled by 0 is zero. If $\phi|_{\Gamma_{F_N^*}}$ were zero, then $\overline{\phi}$ would equal zero, a contradiction. In particular, $\phi(\Gamma_{F_N^*})$ is a non-zero Γ_F-stable submodule of $\rho_M(\mathfrak{g}^{\mathrm{der}})$. Similarly, Lemma 6.4 implies that $\overline{\psi}|_{\Gamma_{F_N^*}}$ is non-zero. As usual, the fixed fields $F_N^*(\phi)$ and $F_N^*(\overline{\psi})$ of these restricted cocycles are then non-trivial and linearly disjoint extensions of F_N^*. We claim that there is a pair (T, α) consisting of a split maximal torus T and a root $\alpha \in \Phi(G^0, T)$ such that $\phi(\Gamma_{F_N^*})$ is not contained in $\ker(\alpha|_{\mathfrak{t}}) \oplus \bigoplus_\beta \mathfrak{g}_\beta$ and such that $\overline{\psi}(\Gamma_{F_N^*})$ is not contained in the annihilator of $\mathfrak{g}_\alpha$ under local duality. Granted this claim, we explain how to finish the proof. Let $\gamma_1 \in \Gamma_F$ be any element such that $\rho_N(\gamma_1)$ satisfies the conditions (on $\rho_N(\sigma_v)$) of Definition 6.5 for the pair (T, α). Such elements clearly exist, since $\rho_N(\Gamma_F)$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}/\varpi^N)$. By the claim and the linear disjointness, we can choose an element $\gamma_2 \in \mathrm{Gal}(F_N^*(\phi, \overline{\psi})/F_N^*)$ such that

$$\phi(\gamma_2\gamma_1) = \phi(\gamma_2) + \phi(\gamma_1) \notin \ker(\alpha|_{\mathfrak{t}}) \oplus \bigoplus_\beta \mathfrak{g}_\beta,$$
$$\overline{\psi}(\gamma_2\gamma_1) = \overline{\psi}(\gamma_2) + \overline{\psi}(\gamma_1) \notin \mathfrak{g}_\alpha^\perp.$$

Applying the Čebotarev density theorem, we take v to be any prime in the positive-density set of primes whose Frobenius elements are equal to the element $\gamma_2\gamma_1$ in $\mathrm{Gal}(F_N^*(\phi, \overline{\psi})/F)$.

To finish the proof, we return to the claim that such a pair (T, α) exists. First note that the images $\phi(\Gamma_{F_N^*})$ and $\overline{\psi}(\Gamma_{F_N^*})$ have non-trivial projection to each simple factor of $\mathfrak{g}^{\mathrm{der}}$: this follows from the Γ_F-equivariance of $\phi|_{\Gamma_{F_N^*}}$ and $\overline{\psi}|_{\Gamma_{F_N^*}}$ and the fact that $\mathfrak{g}^{\mathrm{der}}$ is a single $\pi_0(G)$-orbit of simple factors. We will meet the desired conditions on (T, α) if and only if we do so after replacing $\phi(\Gamma_{F_N^*})$ and $\overline{\psi}(\Gamma_{F_N^*})$ by their projections to a single simple factor of $\mathfrak{g}^{\mathrm{der}}$ (namely, the one containing $\mathfrak{g}_\alpha$; thus in the (purely Lie-theoretic) remainder of the argument we may and do assume that G is connected and simple. Fix a pair (T_1, α_1). For any $g \in G(\mathcal{O})$, we write (T_g, α_g) for $\mathrm{Ad}(g)(T, \alpha)$. To the non-zero cocycle $\overline{\psi}$ we can associate the following proper closed subscheme of G_k:

$$Y_{\overline{\psi}} = \{g \in G : \langle \overline{\psi}(\Gamma_{F_N^*}), \mathfrak{g}_{\alpha_g} \rangle = 0\}.$$

For notational convenience, we modify the initial choice (T_1, α_1) so that $1 \notin Y_{\overline{\psi}}$.[8] We let $U_{\overline{\psi}}$ be the open complement $G \setminus Y_{\overline{\psi}}$, and we let $U_{\overline{\psi},M}$ be the following set (not scheme):

$$U_{\overline{\psi},M} = \{g \in G(\mathcal{O}/\varpi^M) : g \pmod{\varpi} \in U_{\overline{\psi}}(k)\}.$$

We claim there is a constant C_G depending only on the root datum of G such that for all $p > C_G$ the intersection

$$\bigcap_{g \in U_{\overline{\psi},M}} \mathrm{Ad}(g) \left(\ker(\alpha_1|_{\mathfrak{t}_1}) \oplus \bigoplus_{\beta \in \Phi(G^0, T_1)} \mathfrak{g}_\beta \right)$$

is zero. In what follows, we write X_g for the term in this intersection corresponding to g, and for any subset S of $U_{\overline{\psi},M}$ we write X_S for $\bigcap_{g \in X_S} X_g$. Granted that $X_{U_{\overline{\psi},M}}$ is zero, we are done: there is then some $g \in U_{\overline{\psi},M}$ such that $\phi(\Gamma_{F_N^*})$ is not contained in X_g, and (T_g, α_g) is then the desired pair (T, α).

We next observe that $X_{\widehat{G}(\mathcal{O}/\varpi^M)}$ is contained in $\varpi^{M-1}\rho_M(\mathfrak{g}^{\mathrm{der}})$: indeed, for any $Z \in \rho_M(\mathfrak{g}^{\mathrm{der}})$ such that $Z \not\equiv 0 \pmod{\varpi^r}$, the span $\mathcal{O}/\varpi^M[\widehat{G}(\mathcal{O}/\varpi^M)] \cdot Z$ contains $\varpi^r \rho_M(\mathfrak{g}^{\mathrm{der}})$. If $X_{\widehat{G}(\mathcal{O}/\varpi^M)}$ were not contained in $\varpi^{M-1}\rho_M(\mathfrak{g}^{\mathrm{der}})$, then it would contain $\varpi^{M-1}\rho_M(\mathfrak{g}^{\mathrm{der}})$, which is not possible (for $p \gg_G 0$) for even a single X_g. Consequently the entire intersection $X_{U_{\overline{\psi},M}}$ is zero provided $\bigcap_{g \in U_{\overline{\psi}}(k)} \overline{X}_g$ is zero, where we write $\overline{X}_g$ for $\mathrm{Ad}(g)(X_1 \otimes k)$. It therefore suffices to show that if we fix any two non-zero elements $\overline{A} \in \mathfrak{g}^{\mathrm{der}}$ and $\overline{B} \in (\mathfrak{g}^{\mathrm{der}})^*$, then for $p \gg_G 0$ there exists $g \in G(k)$ such that $\mathrm{Ad}(g)^{-1}\overline{A} \notin \ker(\alpha_{\mathfrak{t}_1}) \oplus \bigoplus_\beta \mathfrak{g}_\beta$ and $\mathrm{Ad}(g)^{-1}\overline{B} \notin \mathfrak{g}_{\alpha_1}^\perp$: indeed, taking $\overline{B}$ to be a non-zero element of $\overline{\psi}(\Gamma_{F_N^*})$, the set $U_{\overline{\psi}}(k)$ contains the locus thus associated to $\overline{B}$, so if $\overline{A}$ were a non-zero element of $\cap_{g \in U_{\overline{\psi}(k)}} \overline{X}_g$ we would obtain a contradiction.

Thus we are reduced to showing that if $p \gg_G 0$, then for *any* pair $(\overline{A}, \overline{B})$ as above,

$$\{g \in G(k) : \mathrm{Ad}(g)\overline{A} \notin \ker(\alpha_{\mathfrak{t}_1}) \oplus \bigoplus_{\beta \in \Phi(G^0, T_1)} \mathfrak{g}_\beta \text{ and } \mathrm{Ad}(g)\overline{B} \notin \mathfrak{g}_{\alpha_1}^\perp\}$$

is non-empty. The locus in question is the set of k-points of the complement $G \setminus (Y_{\overline{A}} \cup Y_{\overline{B}})$ of two proper closed subschemes $Y_{\overline{A}}$ and $Y_{\overline{B}}$. The key point is now the following: via a faithful representation of G (the smallest dimension of which is a function of the root datum), there are integers N, r, and d effectively bounded in terms of the root datum of G such that G, $\Phi_{\overline{A}}$, and $\Phi_{\overline{B}}$ are closed subschemes of an affine space $\mathbb{A}^N$ cut out by at most r equations of degree at most d (where these constants do not depend on the choice of $\overline{A}$ and $\overline{B}$). By the Grothendieck-Lefschetz trace formula, Deligne's work on the Weil conjectures [Del80], and [Kat01, Corollary of Theorem 1], we see that there is a constant $c(G)$, depending only on G, such that

$$|G \setminus (\Phi_{\overline{A}} \cup \Phi_{\overline{B}})(k)| \geq q^{d_G} - c(G)q^{d_G - 1},$$

[8] This modification depends on $\overline{\psi}$, but the bound on p we derive will not depend on this.

where $d_G = \dim(G)$. In particular, for $p \gg_G 0$, this complement is non-empty, and the proof is complete. (Note that if we bounded in terms of G the number of irreducible components of $\Phi_{\overline{A}}$ and $\Phi_{\overline{B}}$, we could use a much more elementary argument here, *à la* Lang–Weil.) $\square$

Recall that we say a subspace $L_{M,v} \subset H^1(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$ is *balanced* if $|L_{M,v}|$ satisfies part (3) of the conclusion of Proposition 4.7.

Theorem 6.8. *Continue to assume that $\mathfrak{g}^{\mathrm{der}}$ consists of a single $\pi_0(G)$-orbit of simple factors. For any initial set $\mathcal{L}_M = \{L_{M,v}\}_{v \in S'}$ of balanced local conditions at primes $v \in S'$, there exists a finite set $Q \subset Q_M$ such that such that*

$$\overline{H^1_{\mathcal{L}_M \cup \{L_{M,v}\}_{v\in Q}}(\Gamma_{S' \cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}}))} = \overline{H^1_{\mathcal{L}_M^{\perp} \cup \{L_{M,v}^{\perp}\}_{v\in Q}}(\Gamma_{S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)} = 0,$$

where for $v \in Q$ we take $L_{M,v}$ to be the submodule (where for notational simplicity we omit the root α) in Lemma 6.6.

Proof. We will show that either the relative Selmer group in question is zero, or we can find a prime $v \in Q_M$ such that

$$|H^1_{\mathcal{L}_M \cup L_{M,v}}(\Gamma_{S'\cup\{v\}}, \rho_M(\mathfrak{g}^{\mathrm{der}}))| < |H^1_{\mathcal{L}_M}(\Gamma_{S'}, \rho_M(\mathfrak{g}^{\mathrm{der}}))|.$$

Applying this assertion inductively, we either arrive at a trivial relative Selmer group as in the conclusion of the Theorem or, even stronger, we annihilate the entire mod ϖ^M Selmer group. (The reader should note that the logic of the proof is not that at each step we decrease the size of the relative Selmer group.)

By the Greenberg–Wiles formula and the assumption that the local conditions are balanced, if $\overline{H^1_{\mathcal{L}_M}(\Gamma_{S'}, \rho_M(\mathfrak{g}^{\mathrm{der}}))}$ is non-zero, then we can find ϕ and ψ as in the hypotheses of Proposition 6.7. Let $v \in Q_M$ be any prime as in its conclusion. Since $\overline{\psi}|_{\Gamma_{F_v}}$ is the (non-zero) image of $\psi|_{\Gamma_{F_v}} \in H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$ in $H^1_{\mathrm{unr}}(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$, it follows that ψ must generate a submodule of $H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$ isomorphic to $\mathcal{O}/\varpi^M$. The choice of local condition at v (namely, the result of Lemma 6.6) and the fact that $\overline{\psi}|_{\Gamma_{F_v}} \notin L^{\perp}_{1,v}$ then implies that $\varpi^i \psi|_{\Gamma_{F_v}} \in L_{M,v}$ if and only if $\varpi^i \psi = 0$.

Let $L'_{M,v} = L_{M,v} + H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$, so (by Lemma 6.6) $L'_{M,v}/L_{M,v} \cong \mathcal{O}/\varpi^M$. Since $(L'_{M,v})^{\perp} = L^{\perp}_{M,v} \cap H^1_{\mathrm{unr}}(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$, it follows from the previous paragraph that

$$H^1_{\mathcal{L}_M^{\perp}}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)/H^1_{\mathcal{L}_M^{\perp} \cup (L'_{M,v})^{\perp}}(\Gamma_{F,S'\cup\{v\}}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$$

is also isomorphic to $\mathcal{O}/\varpi^M$. Two applications of the Greenberg–Wiles formula then imply that the inclusion

$$H^1_{\mathcal{L}_M}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}})) \hookrightarrow H^1_{\mathcal{L}_M \cup L'_{M,v}}(\Gamma_{F,S'\cup\{v\}}, \rho_M(\mathfrak{g}^{\mathrm{der}})) \tag{11}$$

is an equality. But now the fact that $\phi|_{\Gamma_{F_v}}$ does not belong to $L_{M,v}$ implies that the inclusion

$$H^1_{\mathcal{L}_M \cup L_{M,v}}(\Gamma_{F,S'\cup\{v\}}, \rho_M(\mathfrak{g}^{\mathrm{der}})) \subset H^1_{\mathcal{L}_M \cup L'_{M,v}}(\Gamma_{F,S'\cup\{v\}}, \rho_M(\mathfrak{g}^{\mathrm{der}})) = H^1_{\mathcal{L}_M}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}}))$$

is strict, and our induction argument can proceed. $\square$

6.3. **Main theorem.** We can now finally collect all of our results to prove the main theorem.

Theorem 6.9. *Let $p \gg_G 0$ be a prime. Let F be a totally real field, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a continuous representation unramified outside a finite set of finite places S containing the places above p. Let $\widetilde{F}$ denote the smallest extension of F such that $\bar{\rho}(\Gamma_{\widetilde{F}})$ is contained in $G^0(k)$, and assume that $[\widetilde{F}(\zeta_p) : \widetilde{F}]$ is strictly greater than the integer a_G arising in Lemma A.6. Fix a lift $\mu\colon \Gamma_{F,S} \to G/G^{\mathrm{der}}(\mathcal{O})$ of $\bar{\mu} = \bar{\rho} \pmod{G^{\mathrm{der}}}$, and assume that $\bar{\rho}$ satisfies the following:*

- *$\bar{\rho}$ is odd, i.e. for all infinite places v of F, $h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = \dim(\mathrm{Flag}_{G^{\mathrm{der}}})$.*
- *$\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible.*
- *For all $v \in S$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift $\rho_v\colon \Gamma_{F_v} \to G(\mathcal{O})$ of type $\mu|_{\Gamma_{F_v}}$; and that for $v \mid p$ this lift may be chosen to be de Rham and regular in the sense that the associated Hodge–Tate cocharacters are regular.*

Then there exist a finite extension K' of $K = \mathrm{Frac}(\mathcal{O})$ (whose ring of integers and residue field we denote by $\mathcal{O}'$ and k'), and depending only on the set $\{\rho_v\}_{v\in S}$; a finite set of places $\widetilde{S}$ containing S; and a geometric lift

$$\begin{tikzcd} & G(\mathcal{O}') \arrow[d] \\ \Gamma_{F,\widetilde{S}} \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & G(k') \end{tikzcd}$$

of $\bar{\rho}$ such that $\rho(\Gamma_F)$ contains $\widehat{G}^{\mathrm{der}}(\mathcal{O}')$. Moreover, if we fix an integer t and for each $v \in S$ an irreducible component defined over $\mathcal{O}'$ and containing ρ_v of:

- *for $v \in S \setminus \{v \mid p\}$, the generic fiber of the local lifting ring, $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ (where $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$ pro-represents $\mathrm{Lift}_{\bar{\rho}}|_{\Gamma_{F_v}}$); and*
- *for $v \mid p$, the lifting ring $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ whose $\overline{K}$-points parametrize lifts of $\bar{\rho}|_{\Gamma_{F_v}}$ with specified Hodge type $\mathbf{v}$ (see* [Bal12, Prop. 3.0.12] *for the construction of this ring);*

then the global lift ρ may be constructed such that, for all $v \in S$, $\rho|_{\Gamma_{F_v}}$ is congruent modulo ϖ^t to some $\widehat{G}(\mathcal{O}')$-conjugate of ρ_v, and $\rho|_{\Gamma_{F_v}}$ belongs to the specified irreducible component for every $v \in S$.[9]

Remark 6.10. A number of our preliminary results either take in or produce Galois representations modulo ϖ^M and ϖ^N for integers M and N that satisfy various bounds, both absolute and relative to each other. In the proof of the present theorem, we will finally gather these statements together and use certain fixed values of M and N. The requirements on M result from the hypotheses of Proposition 5.15 and the conclusion of Lemma B.2; the requirements on N are more complex and are described in the paragraph leading up to Equation (12) below.

Proof. We begin with a preliminary reduction:

Claim 6.11. It suffices to prove the theorem when G^0 is adjoint, and $\mathfrak{g}$ (now equal to $\mathfrak{g}^{\mathrm{der}}$) is equal to a single $\pi_0(G)$-orbit of simple factors.

[9] To be clear, the set $\widetilde{S}$ may depend on the integer t, but the extension $\mathcal{O}'$ does not depend on t.

Proof of Claim: Writing $\bar{\rho}_Z$ for the image of $\bar{\rho}$ in G/Z_{G^0}, reduction modulo Z_{G^0} induces morphisms of functors $\mathrm{Lift}^{\mu}_{\bar{\rho}} \to \mathrm{Lift}_{\bar{\rho}_Z}$ and $\mathrm{Def}^{\mu}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}_Z}$; that the obvious functor on lifts passes to deformation classes follows from the fact that $\widehat{G} = \widehat{G^0}$. Moreover, both of these morphisms are isomorphisms, since they induce isomorphisms of both tangent and obstruction spaces.

In the remainder of the proof, we assume that G^0 is an adjoint group. It is then a product of simple adjoint groups, and we partition this product into the set Σ of $\pi_0(G)$-orbits, writing $G^0 = \prod_{s\in\Sigma} G_s$, where $\pi_0(G)$ transitively permutes the simple factors of G_s. We also write $G_{\neq s}$ for $\prod_{t\neq s} G_t$. Then the embedding $G \to G/G_s \times G/G_{\neq s}$ induces an isomorphism of G onto the subgroup of pairs $(x_{\neq s}, x_s)$ such that $x_{\neq s}$ and x_s have the same image in $\pi_0(G)$ (consider the components under the orbit decomposition of $x_s^{-1}x_{\neq s} \in G^0$ to find suitable modifications of x_s and $x_{\neq s}$ modulo $G_{\neq s}$ and G_s, respectively). Consequently, G is isomorphic to the subgroup of $\prod_{s\in S} G/G_{\neq s}$ of elements with common projection to $\pi_0(G)$, and (since all lifts over $R \in \mathcal{C}^f_{\mathcal{O}}$ have $\pi_0(G)$-projection determined by the residual representation) we conclude $\mathrm{Lift}_{\bar{\rho}}$ and $\mathrm{Def}_{\bar{\rho}}$ themselves canonically decompose into the product of the corresponding functors with $\bar{\rho}$ replaced by its image $\bar{\rho}_s$ in each $G/G_{\neq s}$, each of which is a single G-orbit of simple factors.

We therefore need only check that the hypotheses of our theorem still hold for each $\bar{\rho}_s$, and in turn that the conclusion of the theorem for each $\bar{\rho}_s$ then implies it for the original $\bar{\rho}$. The hypotheses of absolute irreducibility and existence of local lifts clearly still hold for each $\bar{\rho}_s$. To see that each $\bar{\rho}_s$ is odd, note that for all $v \mid \infty$,

$$h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g})) = \sum_{s\in\Sigma} h^0(\Gamma_{F_v}, \bar{\rho}_s(\mathrm{Lie}(G_s))) \geq \sum_{s\in\Sigma} \dim(\mathrm{Flag}_{G_s}) = \dim(\mathrm{Flag}_G) = h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g})),$$

so in each term ($s \in \Sigma$) in the sum, equality must hold, and therefore each $\bar{\rho}_s$ is odd. Finally, given at each $v \in S$ a local lift ρ_v of $\bar{\rho}|_{\Gamma_{F_v}}$, if we can approximate $\rho_v \pmod{G_{\neq s}}$ (modulo $\widehat{G_s}$-conjugacy) to any desired accuracy for all $s \in \Sigma$, then we have succeeded in approximating ρ_v itself (modulo $\widehat{G}$-conjugacy). ∎

For the remainder of the proof of the theorem, we therefore assume that G^0 is an adjoint group, and that $\mathfrak{g}^{\mathrm{der}} = \mathfrak{g}$ consists of a single $\pi_0(G)$-orbit of simple factors.

First we apply Lemma 6.4 to $\bar{\rho}$, producing an integer M as in the conclusion of that lemma (which we assume enlarged to be a multiple of e, and to be at least $\max\{3, e+1\}$ so that the hypotheses of Proposition 5.15 are satisfied). Next, for all $v \in S$, fix as in the theorem statement irreducible components $\overline{R}_v[1/\varpi]$, containing the specified lifts ρ_v, of $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ (for v not above p) or $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ (for v above p); we may if desired enlarge $\mathcal{O}$ and make the component choice after extending scalars. Denote by $\overline{R}_v$ the scheme-theoretic closure of this component in $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$. Building on work of Kisin ([Kis08]), Bellovin-Gee ([BG17, Theorem 3.3.3]) have shown that

$$\dim(\overline{R}_v[1/\varpi]) = \begin{cases} \dim(\mathfrak{g}^{\mathrm{der}}) \text{ for } v \text{ not above } p; \\ \dim(\mathfrak{g}^{\mathrm{der}}) + \dim(\mathrm{Flag}_{G^{\mathrm{der}}}) \text{ for } v \text{ above } p, \end{cases}$$

and that $\overline{R}_v[1/\varpi]$ has a dense set of formally smooth closed points. (For v above p, the dimension cited here results from the fact that the given lift ρ_v is regular, so the parabolic associated to any Hodge–Tate cocharacter is in fact a Borel.) Recall from Lemma 4.9 that there is moreover a finite extension $\mathcal{O}'$ of $\mathcal{O}$, independent of the choice of t, and lifts ρ'_v of $\bar{\rho}|_{\Gamma_{F_v}}$ corresponding to $\mathcal{O}'$-points of the $\overline{R}_v$ such that $\rho'_v \equiv \rho_v \pmod{\varpi^t}$ and ρ'_v defines a formally smooth point on $\overline{R}_v[1/\varpi]$.

We replace ρ_v by this approximation ρ'_v and the ring $\mathcal{O}$ by $\mathcal{O}'$; for simplicity in the remainder of the proof, we will retain the original notation $\rho_v, \mathcal{O}$. We then apply Proposition 4.7 with $r_0 = M$ to each pair $(\overline{R}_v[1/\varpi], \rho_v)$. Let N_0 be the maximum (over all $v \in S$) of the integers n_0 thus produced by Proposition 4.7, let N_1 be the integer produced by Lemma 6.13, and let

$$N = \max\{N_0 + M, N_1 + M, t + M, 4M\}. \tag{12}$$

We next apply Theorem 5.15 to $\bar{\rho}$ and the fixed local lifts ρ_v to produce, for some finite set $S' \supset S$, a $\rho_N \colon \Gamma_{F,S'} \to G(\mathcal{O}/\varpi^N)$ satisfying the conclusions of Theorem 5.15. At each $v \in S'$, we let $L_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ be the subspace arising from the proof of Theorem 5.15 and the results of §3 and §4 as follows:

- If $v \in S$, recall that Theorem 5.15 shows that $\rho_N \equiv \rho_v \pmod{\varpi^N}$ modulo $\widehat{G}(\mathcal{O})$-conjugacy. We *replace* ρ_v with its suitable $\widehat{G}(\mathcal{O})$-conjugate, noting as in Remark 4.8 that Proposition 4.7 applies–with the same quantitative bounds–to this conjugate, and then we let $L_{r,v}$, for $1 \leq r \leq M$, be the subspaces of $H^1(\Gamma_{F_v}, \rho_v(\mathfrak{g}^{\mathrm{der}}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r)$ produced by applying Proposition 4.7 to our new ρ_v.
- If $v \in S' \setminus S$, and $\rho_N|_{\Gamma_{F_v}}$ is unramified, then we simply take $L_{r,v}$ to be the (image in cohomology of the) subspace of unramified cocycles.
- If $v \in S' \setminus S$, and $\rho_N|_{\Gamma_{F_v}}$ is ramified, then the space $L_{r,v}$ is the one constructed in Lemma 3.5.

We can then consider the (balanced) relative Selmer and dual Selmer groups $\overline{H^1_{\mathcal{L}_M}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}}))}$ and $\overline{H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)}$. We apply Theorem 6.8 (here is where we use the reduction of Claim 6.3) to produce a finite set $Q \subset \mathcal{Q}_M$ of primes such that the $\mathcal{Q}_M$-new relative (dual) Selmer group $\overline{H^1_{\mathcal{L}_M^\perp \cup \{L_{M,v}^\perp\}_{v \in Q}}(\Gamma_{F,S' \cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)}$ vanishes. For all $v \in S' \cup Q$ and all $n \geq N - M$, we let $D_{n,v} \subset \mathrm{Lift}_{\bar{\rho}|_{\Gamma_{F_v}}}(\mathcal{O}/\varpi^n)$ be the class of lifts produced by applying Proposition 4.7 (for $v \in S$), Theorem 5.15 (for $v \in S' \setminus S$ that are ramified in ρ_N), or Lemma 6.6 (for $v \in Q$), or by simply taking $D_{n,v}$ to be the unramified lifts (for $v \in S' \setminus S$ that are unramified in ρ_N). In particular, for all $1 \leq r \leq M$, the fibers of $D_{n+M,v} \to D_{n,v}$ are stable under the preimages in $Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ of the subspaces $L_{r,v}$. (The only case in which we have not discussed this explicitly is the unramified case; but there it is straightforward.) The Theorem now results from the following:

Claim 6.12. For each $n \geq N - M$, we have pairs of (multiplier μ) liftings (τ_n, ρ_{n+M}), where $\tau_n \colon \Gamma_{F,S' \cup Q} \to G(\mathcal{O}/\varpi^n)$ and $\rho_{n+M} \colon \Gamma_{F,S' \cup Q} \to G(\mathcal{O}/\varpi^{n+M})$, and $\tau_n = \rho_{n+M} \pmod{\varpi^n}$, with the following properties:

1. For each $v \in S' \cup Q$, $\tau_n|_{\Gamma_{F_v}}$ belongs to $D_{n,v}$, and $\rho_{n+M}|_{\Gamma_{F_v}}$ belongs to $D_{n+M,v}$.
2. $\tau_{n+1} = \rho_{n+M} \pmod{\varpi^{n+1}}$.
3. $\tau_n = \tau_{n+1} \pmod{\varpi^n}$.

Proof of Claim: The proof will be by induction on n. We start the induction by setting τ_{N-M} to be $\rho_N \pmod{\varpi^{N-M}}$. The assumptions on ρ_N ensure that first property holds for the initial pair (τ_{N-M}, ρ_N).

We now assume that we have constructed a pair of representations (τ_n, ρ_{n+M}) satisfying the first property. We then set $\tau_{n+1} = \rho_{n+M} \pmod{\varpi^{n+1}}$, so the second property also holds for this n. Recall that we have arranged the vanishing of $Ш^1_{S'}(\Gamma_{F,S'}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)$, which implies the same for $Ш^1_{S' \cup Q}(\Gamma_{F,S'}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*) \cong Ш^2_{S' \cup Q}(\Gamma_{F,S'}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))^\vee$, so ρ_{n+M} lifts to a homomorphism ρ'_{n+M+1} : $\Gamma_{F,S' \cup Q} \to G(\mathcal{O}/\varpi^{n+M+1})$. Viewing the restriction of ρ'_{n+M+1} to Γ_{F_v} for $v \in S' \cup Q$ as a lift of the restriction of τ_{n+1} to Γ_{F_v} and comparing with an element of the fiber of $D_{n+M+1,v} \to D_{n+1,v}$ over $\tau_{n+1}|_{\Gamma_{F_v}}$, we get an element

$$(f_v)_{v\in S'\cup Q} \in \bigoplus_{v\in S'\cup Q} \frac{H^1(\Gamma_{F_v}, \rho_M(\mathfrak{g}^{\mathrm{der}}))}{L_{M,v}}.$$

Since ρ'_{n+M+1} can be viewed as a lift of $\rho_{n+M} \pmod{\varpi^{n+1}}$, and $\rho'_{n+M+1} \mod \varpi^{n+M} = \rho_{n+M}$ is at each v in the fiber of $D_{n+M,v} \to D_{n+1,v}$ over $\rho_{n+M} \pmod{\varpi^{n+1}} = \tau_{n+1}$, the image of (f_v) in $\bigoplus_{v\in S'\cup Q} \frac{H^1(\Gamma_{F_v}, \rho_{M-1}(\mathfrak{g}^{\mathrm{der}}))}{L_{M-1,v}}$ vanishes.

We have a commutative diagram

$$\begin{array}{ccccc}
H^1(\Gamma_{S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & \bigoplus_{v\in S'\cup Q} \frac{H^1(\Gamma_{F_v},\, \rho_M(\mathfrak{g}^{\mathrm{der}}))}{L_{M,v}} & \longrightarrow & H^1_{\mathcal{L}_M^\perp \cup \{L_{M,v}^\perp\}_{v\in Q}}(\Gamma_{S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)^\vee \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\Gamma_{S'\cup Q}, \rho_{M-1}(\mathfrak{g}^{\mathrm{der}})) & \longrightarrow & \bigoplus_{v\in S'\cup Q} \frac{H^1(\Gamma_{F_v},\, \rho_{M-1}(\mathfrak{g}^{\mathrm{der}}))}{L_{M-1,v}} & \longrightarrow & H^1_{\mathcal{L}_{M-1}^\perp \cup \{L_{M-1,v}^\perp\}_{v\in Q}}(\Gamma_{S'\cup Q}, \rho_{M-1}(\mathfrak{g}^{\mathrm{der}})^*)^\vee
\end{array}$$

in which the rows come from (part of) the Poitou–Tate exact sequence, and the vertical maps are induced by the reduction map $\rho_M(\mathfrak{g}^{\mathrm{der}}) \to \rho_{M-1}(\mathfrak{g}^{\mathrm{der}})$; commutativity of the left-hand square is obvious and of the right-hand square follows from the properties of cup-product and the local duality pairings. Lemma 6.1 and the vanishing of the relative dual Selmer group $\overline{H^1_{\mathcal{L}_M^\perp \cup \{L_{M,v}^\perp\}_{v\in Q}}(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)}$ together imply that the map

$$H^1_{\mathcal{L}_{M-1}^\perp}(\Gamma_{F,S'}, \rho_{M-1}(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$$

is surjective, so the last vertical map in the diagram is injective. The commutativity of the diagram then implies that $(f_v)_{v\in S'\cup Q}$ maps to zero in $H^1_{\mathcal{L}_M^\perp \cup \{L_{M,v}^\perp\}_{v\in Q}}(\Gamma_{S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)^\vee$, so ρ'_{n+M+1} can be modified by a cocycle in $H^1(\Gamma_{F,S'\cup Q}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)$ to get a lift ρ_{n+M+1} of τ_{n+1} such that for all $v \in S' \cup Q$, $\rho_{n+M+1}|_{\Gamma_{F_v}}$ belongs to $D_{n+M+1,v}$[10]; here crucially we use that for n in the considered range the fibers of $D_{n+M+1,v} \to D_{n+1,v}$ are stable under all cocycles (with image) in $L_{M,v}$. This completes the induction step. □

Having established the Claim, we set $\rho = \varprojlim \tau_n \colon \Gamma_{F,S'\cup Q} \to G(\mathcal{O})$. Since ρ lifts ρ_{N-M}, we have achieved the desired (modulo ϖ^t) approximation of our fixed local lifts, and Lemma 6.13 implies that $\mathrm{im}(\rho)$ contains $\widehat{G^{\mathrm{der}}}(\mathcal{O})$, concluding the proof of the theorem. □

Lemma 6.13. *There is an integer N_1 depending only on G^{der} such that any closed subgroup K of $\widehat{G^{\mathrm{der}}}(\mathcal{O})$ whose reduction modulo ϖ^{N_1} equals $\widehat{G^{\mathrm{der}}}(\mathcal{O}/\varpi^{N_1})$ must in fact equal $\widehat{G^{\mathrm{der}}}(\mathcal{O})$.*

Proof. Since $H := \widehat{G^{\mathrm{der}}}(\mathcal{O})$ is a p-adic Lie group, the subgroup H^p generated by p^{th} powers is open, so there exists an integer N_1 such that H^p contains the kernel of reduction modulo ϖ^{N_1}. In particular, if $K \pmod{\varpi^{N_1}}$ is equal to $\widehat{G^{\mathrm{der}}}(\mathcal{O}/\varpi^{N_1})$, then K surjects onto H/H^p. Since the Frattini subgroup of a finite p-group contains the subgroup generated by p^{th} powers, Frattini's theorem for finite groups implies that K surjects onto any finite quotient of H. Since K is closed, we must in fact have $K = H$. □

[10] Note that we do not claim that ρ_{n+M+1} is a lift of ρ_{n+M}.

Remark 6.14. When $G = \mathrm{GL}_n$, GSp_{2n}, or GO_n, local lifts ρ_v as in the theorem statement are known to exist for $v \nmid p$, by [CHT08, §2.4.4] and [Boo18, §7] (after possibly replacing k with a finite extension). Many other cases for general G can be worked out by hand (e.g., [Pat16, §4] and [Pat17, §4]), but there is as yet no general result. We expect that it is always possible to find such ρ_v.

For $v \mid p$, much less is known, but forthcoming work of Emerton and Gee ([EG19]) will show that such lifts ρ_v always exist when $G = \mathrm{GL}_n$. It is to be hoped that their methods will eventually eliminate this hypothesis entirely for arbitrary G.

Remark 6.15. We make some remarks on the effectivity of the bound $p \gg_G 0$ in the theorem. The possible need to increase p arises at several points in the paper. In §3, p is any prime. In §5.1 §5.2 we have not computed an explicit bound on p, but we could easily derive one by following the arguments of those sections; the bounds coming from these sections essentially amount to the condition that certain $\mathbb{F}_p$-vector spaces not be covered by a finite (bounded absolutely in terms of G) number of hyperplanes. In deducing the image hypothesis of §5.1 and §5.2 from the irreducibility hypothesis of Theorem 6.9, there is an explicit bound ensuring the cohomology (H^0 and H^1) vanishing, and an explicit bound (see Remark A.4) to ensure disjointness of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$. The same remark and an inspection of the proof yields an effective bound for the integer a_G in Lemma A.6. In sum, the bound in Theorem 6.9 can be made effective.

Remark 6.16. If we are instead given a homomorphism $\rho_n \colon \Gamma_{F,S} \to G(\mathcal{O}/\varpi^n)$ such that $\bar{\rho}$ satisfies the hypotheses of the theorem, and moreover (for all $v \in S$) $\rho_n|_{\Gamma_{F_v}}$ has a lift ρ_v as in the theorem statement, then we can produce a geometric lift of ρ_n (moreover approximating the given ρ_v's).

Remark 6.17. When $G^0 = \mathrm{GL}_n$ (or some minor variant thereof), we compare our results to the lifting results coming from potential automorphy theorems. The main lifting theorem of [BLGGT14, Theorem 4.3.1] implies the existence of lifts with prescribed local behavior of an odd homomorphism $\bar{\rho} \colon \Gamma_{F,S} \to \mathcal{G}_n(k)$ such that $\bar{\rho}|_{\Gamma_{\tilde{F}(\zeta_p)}}$ is absolutely irreducible, $p \geq 2(n+1)$; to be precise, one asks that:

- for $v \nmid p$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift on a given irreducible component of $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$; and
- for $v \mid p$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a potentially diagonalizable lift on a given irreducible component of some $R^{\square,\mu,\mathbf{v}}_{\bar{\rho}|_{\Gamma_{F_v}}}$ (for a regular p-adic Hodge type $\mathbf{v}$);

and one concludes that $\bar{\rho}$ has a global lift ρ such that $\rho|_{\Gamma_{F_v}}$ lies on the given irreducible components for all $v \in S$. Our result (in addition to applying to general G) strengthens this in two ways: at $v \mid p$, we do not require the local lift ρ_v to be potentially diagonalizable (we remark that the recent preprint [CEG18] improves "potentially diagonalizable" to "globally realizable"); and for all $v \in S$, we can approximate (modulo $\widehat{G}(\mathcal{O})$-conjugacy) the fixed local lifts to any desired degree of precision. What we lose is some sharpness in the bound on allowable p, and, more important, minimality of the lifts (i.e., our lifts are not unramified outside S). And of course we do not establish potential automorphy!

Remark 6.18. It would be interesting to pursue an analogue of our main theorem for *reducible* $\bar{\rho}$; indeed, in some sense the seed of our project was the study of the paper [HR08], which produces irreducible lifts of (certain) reducible $\Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(k)$. Our methods will certainly adapt to cover many reducible cases as well, and we intend to pursue this problem in the future.

We also note that the method of proof allows us, without assuming oddness of $\bar{\rho}$, to construct possibly non-geometric p-adic deformations, since the arguments of Theorem 6.9 only require that whenever we have a non-trivial dual Selmer class, we can also find a non-trivial Selmer class,:

Theorem 6.19. *Let $p \gg_G 0$ be a prime, let F be any number field, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a continuous representation unramified outside a finite set of places S containing those above p. Fix a lift $\mu\colon \Gamma_{F,S} \to G/G^{\mathrm{der}}(\mathcal{O})$ of $\bar{\rho}$ $\pmod{G^{\mathrm{der}}}$. Let $\widetilde{F}$ be as in Theorem 6.9, assume that $[\widetilde{F}(\zeta_p) : \widetilde{F}] = p-1$, and that $\bar{\rho}$ satisfies the following:*

- *$\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible.*
- *For all $v \in S$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift ρ_v of type $\mu|_{\Gamma_{F_v}}$, and for $v \mid p$ this lift can be chosen to correspond to a formally smooth point on an irreducible component of $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi]$ of dimension $(1 + [F_v : \mathbb{Q}_p])\dim_k(\mathfrak{g}^{\mathrm{der}})$. (For instance, this hypothesis holds if $H^2(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = 0$.)*

Then for some finite set of primes $\widetilde{S} \supset S$ and finite extension $\mathcal{O}'$ of $\mathcal{O}$, $\bar{\rho}$ admits a lift $\rho\colon \Gamma_{F,\widetilde{S}} \to G(\mathcal{O}')$, and ρ may be arranged constructed such that, for all $v \in S$, $\rho|_{\Gamma_{F_v}}$ is congruent modulo ϖ^t to some $\widehat{G}(\mathcal{O}')$-conjugate of ρ_v.

Remark 6.20. There is not to our knowledge a known result, analogous to the results of [BG17] on the generic fibers of the unrestricted lifting rings $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$, and such results do not follow formally from the methods of [BG17].

Proof. The argument is the same as that of Theorem 6.9, except at places $v \mid p$ we consider all lifts of $\bar{\rho}|_{\Gamma_{F_v}}$ and choose an irreducible component $\overline{R}_v[1/\varpi]$ of the full local lifting ring $R^{\square,\mu}_{\bar{\rho}|_{\Gamma_{F_v}}}$ as specified in the hypotheses of the theorem. The corresponding subspaces $L_{r,v}$ produced by an analogue of Proposition 4.7 have order $|\rho_r(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}}| \cdot |\mathcal{O}/\varpi^r|^{\dim_k(\mathfrak{g}^{\mathrm{der}})[F_v:\mathbb{Q}_p]}$, and the corresponding application of the Greenberg–Wiles formula shows that

$$|H^1_{\mathcal{L}_M}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}}))|/|H^1_{\mathcal{L}_M^\perp}(\Gamma_{F,S'}, \rho_M(\mathfrak{g}^{\mathrm{der}})^*)| \geq 1$$

(equality holds when F is totally real, and $\bar{\rho}(c_v) = 1$ for all complex conjugations c_v), with analogous conclusions for the relative Selmer and dual Selmer groups. This inequality suffices to proceed as in the proof of Theorem 6.9. □

7. Examples

In this section we gather a few examples to which our methods apply.

7.1. **The principal** SL_2**.** In [Pat16] (and [Pat17]) it was shown how the original lifting argument of [Ram02] and [Tay03] could be adapted to prove lifting results for $\bar{\rho}\colon \Gamma_F \to G(k)$ whose image was (approximately) a principal SL_2. In fact, the argument in that paper was carried out for the exceptional groups, at one point relying on a brute-force Magma computation (see [Pat16, Lemma 7.6]); for the classical Dynkin types except for D_{2n}, case-by-case matrix calculations (not carried out in [Pat16], but some of which appear in [Tan18]) complete the argument. The arguments of the present paper apply to these examples without relying on case-by-case calculation, and moreover treating type D_{2n} as well.

Let G^0 be a split connected reductive group over $\mathbb{Z}_p$. Recall that for $p \gg_{G^0} 0$, there is a unique conjugacy class of principal homomorphisms $\varphi\colon \mathrm{SL}_2 \to G^0$ defined over $\mathbb{Z}_p$ (see [Ser96]). Assume that $G = {}^L H$, the L-group of a connected reductive group H over F; that is, we choose over $\overline{F}$ a maximal torus and Borel subgroup $T_{\overline{F}} \subset B_{\overline{F}} \subset H_{\overline{F}}$ to obtain a based root datum, and then a choice of pinning allows us to define an L-group $G = {}^L H = H^\vee \rtimes \mathrm{Gal}(\widetilde{F}/F)$ for some finite extension $\widetilde{F}/F$. The principal SL_2 extends to a homomorphism $\varphi\colon \mathrm{SL}_2 \times \Gamma_F \to {}^L H$ ([Gro97, §2]), and we assume that φ extends to a homomorphism $\mathrm{GL}_2 \times \Gamma_F \to {}^L H$ (this is always the case if, eg, H is simply-connected, and in general it can be arranged by enlarging the center of H). The following crucial assumption is needed to use the principal SL_2 to produce *odd* homomorphisms valued in ${}^L H$:

Assumption 7.1. Assume that $\widetilde{F}/F$ is contained in a quadratic totally imaginary extension of the totally real field F, and that the automorphism of $H^\vee$ given by projecting any complex conjugation $c \in \Gamma_F$ to $\mathrm{Gal}(\widetilde{F}/F)$ preserves each simple factor $\mathfrak{h}_i^\vee$ of $\mathfrak{h}^\vee = \mathrm{Lie}(H^\vee)$, and acts on $\mathfrak{h}_i^\vee$ as the identity if $-1 \in W_{\mathfrak{h}_i}$ and as the opposition involution if $-1 \notin W_{\mathfrak{h}_i}$.

This assumption leads to the following archimedean calculation:

Lemma 7.2. *Let $\theta_v \in {}^L H(k)$ be the element*

$$\theta_v = \varphi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times c_v\right).$$

Then

$$\dim_k(\mathfrak{g}^{\mathrm{der}})^{\mathrm{Ad}(\theta_v)=1} = \dim_k(\mathfrak{n}),$$

where $\mathfrak{n}$ is the unipotent radical of a Borel subgroup of G (or of G^{der}).

Proof. Combining [Pat16, Lemma 4.19, 10.1], we find that $\dim_k(\mathfrak{g}^{\mathrm{der}})^{\mathrm{Ad}(\theta_v)=1} = \dim_k(\mathfrak{n})$. □

Theorem 7.3. *Let $G = {}^L H$ be constructed as above, satisfying Assumption 7.1, let $p \gg_G 0$, and assume that $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$. Let S be a finite set of places of the totally real field F containing all $v \mid p$, assuming for simplicity that all places in S are split in $\widetilde{F}/F$, and let $\bar{r}\colon \Gamma_{F,S} \to \mathrm{GL}_2(k)$ be a continuous representation satisfying the following properties:*

1. *For some subfield $k_0 \subset k$, the projective image of $\bar{r}$ contains $\mathrm{PSL}_2(k_0)$.*
2. *$\det \bar{r}(c) = -1$ for all complex conjugations $c \in \Gamma_F$.*

Then there exist a finite extension $\mathcal{O}'$ of $\mathcal{O}$ and a finite set of trivial primes Q such that $\bar{\rho} = \varphi \circ \bar{r}\colon \Gamma_{F,S\cup Q} \to G(k)$ has a geometric lift $\rho\colon \Gamma_{F,S\cup Q} \to G(\mathcal{O})$, with $\mathrm{im}(\rho)$ containing $\widehat{G^{\mathrm{der}}}(\mathcal{O})$. (The more refined local conclusions of Theorem 6.9 also hold, given local liftings ρ_v, $v \in S$, that one wants to approximate.)

Proof. For $p \gg_G 0$, it is straightforward to verify the irreducibility hypotheses of Theorem 6.9 (compare [Pat16, Theorem 7.4, Theorem 10.4]). Note that the constant a_G of Lemma A.6 can be replaced by 2, by the assumption on $\mathrm{im}(\bar{r})$. To satisfy the local hypotheses of Theorem 6.9, it will even suffice to construct local lifts $r_v\colon \Gamma_{F_v} \to \mathrm{GL}_2(\mathcal{O})$ such that, for $v \mid p$, r_v is de Rham and regular. If $v \in S \setminus \{v \mid p\}$, such lifts are known to exist even with GL_N in place of GL_2 ([CHT08, Corollary 2.4.21]). If $v \mid p$, $\bar{r}|_{\Gamma_{F_v}}$ admits a Hodge-Tate regular, potentially crystalline lift r_v by [Mul13, Theorem 2.5.3, Theorem 2.5.4]. The theorem now follows. □

Remark 7.4. In particular, starting with $\bar{r}\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ coming either from classical modular forms or elliptic curves, we can construct geometric representations $\rho\colon \Gamma_{\mathbb{Q}} \to G(\mathcal{O})$ whose image has Zariski closure containing G^{der}. This was the application of the lifting theorems in [Pat16].

7.2. **Normalizers of tori.** In this subsection we make no effort to be maximally general. For simplicity we assume that G^0/Z_{G^0} is simple. Let T be a (split) maximal torus of G^0. Residual representations valued in $N_G(T)(k)$ lift to $G(\mathcal{O})$, since (provided p does not divide $|W_{G^0}|$) the image of $\bar{\rho}$ has order prime to p. Our main theorem shows that non-trivial lifts, with image containing an open subgroup of $G^{\mathrm{der}}(\mathcal{O})$, also exist under suitable hypotheses on $\bar{\rho}$.

Theorem 7.5. *Let $p \gg_G 0$, and assume $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$. Let $\bar{\rho}\colon \Gamma_{F,S} \to N_G(T)(k)$ satisfy the following:*

- *$\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible. For instance, we could assume $\mathrm{im}(\bar{\rho})$ contains $N_G(T)(\mathbb{F}_p)$ (and $p \gg_G 0$); or that $\bar{\rho}(\Gamma_{\widetilde{F}(\zeta_p)})$ contains a regular semisimple element of T whose centralizer is T (automatic if G^0 is simply-connected), and that the projection of $\bar{\rho}(\Gamma_{\widetilde{F}(\zeta_p)})$ to the Weyl group contains a Coxeter element.*
- *$\bar{\rho}$ is odd. For instance, we can make one of the following assumptions:*
  - *If $-1 \in W_{G^0}$, then for all $v \mid \infty$, $\bar{\rho}(c_v)$ either projects to $-1 \in W_{G^0}$, or projects to $\rho^\vee(-1) \in G^{\mathrm{ad}}$ (where $\rho^\vee$ is the usual half-sum of the positive co-roots of G).*
  - *If $-1 \notin W_{G^0}$, then for all $v \mid \infty$, $\bar{\rho}(c_v)$ either equals $(w_0, \tau) \in G^0 \rtimes \pi_0(G)$, where w_0 lifts the longest element of W_{G^0}, and τ is a pinned outer automorphism of G^0 acting as the opposition involution on $T \cap G^{\mathrm{der}}$; or it projects to $(\rho^\vee(-1), \tau) \in G^{\mathrm{ad}} \rtimes \pi_0(G)$.*
- *For all $v \mid p$, $\bar{\rho}|_{\Gamma_{F_v}}$ factors through $T(k)$.*

Then for some finite set of places $T \supset S$, $\bar{\rho}$ admits a geometric lift $\rho\colon \Gamma_{F,T} \to G(\mathcal{O})$ whose image contains $\widehat{G}^{\mathrm{der}}(\mathcal{O})$.

Proof. First we check the local hypotheses of Theorem 6.9. At primes in $S \setminus \{v \mid p\}$ we may take the obvious Teichmüller lifts, since the order of $\mathrm{im}(\bar{\rho})$ is prime to p. At $v \mid p$, it is easy to lift a $\Gamma_{F_v} \to T(k)$ to a potentially crystalline and Hodge–Tate regular $\rho_v \colon \Gamma_{F_v} \to T(\mathcal{O})$. That the examples given of possible $\bar{\rho}(c_v)$ are in fact involutions follows from [Yun14, Lemma 2.3], [Pat16, Lemma 10.1], and a similar check in the case $\bar{\rho}(c_v) = (w_0, \tau)$. To check the irreducibility hypothesis is similarly straightforward: compare the proof of [BHKT16, Proposition 10.7]. $\square$

We have certainly not optimized the explicit descriptions of the possible local or global images here. It can be difficult, for instance, to realize $N_G(T)(\mathbb{F}_p)$ as a Galois group over $\mathbb{Q}$ (the sequence $1 \to T \to N_G(T) \to W_G \to 1$ need not split), and Theorem 6.9 is easily seen to apply when $\bar{\rho}(\Gamma_{\widetilde{F}})$ equals certain somewhat smaller subgroups of $N_G(T)(\mathbb{F}_p)$. Here we give two examples from the recent literature:

Example 7.6. In [Tan18], Tang classifies those connected reductive groups G that arise as the Zariski closure of the image of a homomorphism $\Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_p)$. The main theorem of [Tan18] gives a complete answer to this question (for $p \gg 0$) modulo some elusive cases, consisting of certain simply-connected groups (e.g. E_7^{sc}) for p failing to satisfy some congruence condition (see [Tan18, Theorem 1.3]). As explained in [Tan18, Theorem 1.5, §3.4], our main theorem allows Tang to treat these remaining cases by deforming $\bar{\rho}$ valued in a "large enough" subgroup of $N_G(T)(\mathbb{F}_p)$.

Example 7.7. The main theorem of [BCE+18] produces E_6-Galois representations that arise in the cohomology of algebraic varieties (and are potentially automorphic) by studying the deformation theory of a carefully-constructed $\bar{\rho}\colon \Gamma_{\mathbb{Q}} \to E_6^{\mathrm{sc}}(\mathbb{F}_p) \rtimes \mathrm{Out}(E_6)$ whose image projects onto a 3-Sylow subgroup of the Weyl group of E_6. Our theorem applies to find geometric lifts with full image of these $\bar{\rho}$ as well, and it also can lift them with sets of Hodge–Tate weights not accessible by the methods of [BCE+18] (which rely on potential automorphy theorems after composing with the minuscule representation of E_6). Of course, our arguments do not show these lifts are motivic or potentially automorphic!

7.3. **Deforming exotic finite subgroups.** We conclude by constructing some odd irreducible representations $\bar{\rho}\colon \Gamma_{\mathbb{Q}} \to G(k)$ of a less Lie-theoretic flavor that Theorem 6.9 will lift to Zariski-dense geometric representations $\bar{\rho}\colon \Gamma_{\mathbb{Q}} \to G(\mathcal{O})$. Recall that over $\mathbb{C}$ we have an embedding $F_4(\mathbb{C}) \hookrightarrow E_6^{\mathrm{sc}}(\mathbb{C})$ given by identifying F_4 to the stabilizer of a vector in one of the 27-dimensional minuscule representations $V_{\min}$ of E_6^{sc}. Letting H and G be the split groups (over $\mathbb{Z}$) of type F_4 and E_6^{sc}, we can realize this embedding $H \hookrightarrow G$ over $R = \mathcal{O}_E[\frac{1}{N}]$ for some number field E and integer N (a quantitative refinement of this soft "spreading-out" assertion is of course possible). By [CW97, 1.1 Main Theorem], the finite groups A_6 and $\mathrm{PSL}_2(\mathbb{F}_{13})$ embed into $F_4(\mathbb{C})$, and, perhaps after replacing E (and hence R) by a finite extension, we may assume these groups are embedded into $H(R)$. This theorem also tells us the characters of A_6 and $\mathrm{PSL}_2(\mathbb{F}_{13})$ in $V_{\min}$ and the adjoint representation of E_6. Recalling the decompositions as F_4-representations

$$\mathrm{Lie}(E_6) = \mathrm{Lie}(F_4) \oplus U,$$
$$V_{\min} = \mathbb{1} \oplus U,$$

where U is the irreducible 26-dimensional representation of F_4, we compute the following decompositions of $\mathrm{Lie}(F_4)$ as A_6 and $\mathrm{PSL}_2(\mathbb{F}_{13})$-representations:

$$\mathrm{Lie}(F_4) \cong \begin{cases} \chi_4 \oplus 3\cdot\chi_5 \oplus 2\cdot\chi_7 & \text{(case } A_6);\\ \chi_4 \oplus \{\chi_5 \text{ or } \chi_6\} \oplus 2\cdot\chi_9 & \text{(case } \mathrm{PSL}_2(\mathbb{F}_{13})), \end{cases}$$

where we use the ATLAS notation ([CCN+85]) for characters. It turns out that for our purposes knowing whether χ_5 or χ_6 appears in the decomposition in the $\mathrm{PSL}_2(\mathbb{F}_{13})$ case is irrelevant. In particular, letting c denote the unique conjugacy class of order 2 in either case, the ATLAS character tables tell us that the trace of c acting on $\mathrm{Lie}(F_4)$ is $-4 = -\mathrm{rk}(F_4)$, and so

$$\dim \mathrm{Lie}(F_4)^{\mathrm{Ad}(c)=1} = \frac{\dim(F_4) - \mathrm{rk}(F_4)}{2} = \dim \mathrm{Flag}_{F_4},$$

i.e. $\mathrm{Ad}(c)$ is an odd involution of $\mathrm{Lie}(F_4)$.

Proposition 7.8. *For all sufficiently large primes, there are representations $\bar{\rho}_1\colon \Gamma_{\mathbb{Q}} \to F_4(\overline{\mathbb{F}}_p)$ and $\bar{\rho}_2\colon \Gamma_{\mathbb{Q}} \to F_4(\overline{\mathbb{F}}_p)$ that have images $\mathrm{im}(\bar{\rho}_1) \cong A_6$, $\mathrm{im}(\bar{\rho}_2) \cong \mathrm{PSL}_2(\mathbb{F}_{13})$, and that admit geometric deformations $\rho_1, \rho_2\colon \Gamma_{\mathbb{Q}} \to F_4(\overline{\mathbb{Z}}_p)$ with Zariski-dense image.*

Proof. There are Galois extensions $L_1/\mathbb{Q}$ and $L_2/\mathbb{Q}$ satisfying $\mathrm{Gal}(L_1/\mathbb{Q}) \cong A_6$, $\mathrm{Gal}(L_2/\mathbb{Q}) \cong \mathrm{PSL}_2(\mathbb{F}_{13})$, and complex conjugation c is non-trivial in each $\mathrm{Gal}(L_i/\mathbb{Q})$: the constructions of L_1 and L_2 are due to Hilbert and Shih, respectively, and both are explained in [Ser08, §4.5, Theorem 5.1.1]. It is easy to see that we can take c to be non-trivial, and note that to apply Shih's theorem we use that $\left(\frac{2}{13}\right) = -1$. Let p be any sufficiently large (in the sense of Theorem 6.9 for F_4, not dividing N, and not dividing $|\mathrm{im}(\bar{\rho}_i)|$) prime that is unramified in $L_i/\mathbb{Q}$. Reducing the inclusions $\mathrm{Gal}(L_i/\mathbb{Q}) \hookrightarrow H(R)$ modulo a prime of R above p, we obtain residual representations $\bar{\rho}_i\colon \Gamma_{\mathbb{Q}} \to H(\overline{\mathbb{F}}_p)$ satisfying the hypotheses of Theorem 6.9. Indeed, by the character calculation preceding Proposition 7.8, both $\bar{\rho}_i$ are odd. At primes $v \nmid p$ there are obvious Teichmüller lifts of $\bar{\rho}_i|_{\Gamma_{F_v}}$. At the prime p, $\bar{\rho}_i(\Gamma_{\mathbb{Q}_p})$ is valued in some torus of F_4, since p is unramified in L_i and $\mathrm{im}(\bar{\rho}_i)$ is coprime to p. Finally, L_i is linearly disjoint from $\mathbb{Q}(\mu_p)$ over $\mathbb{Q}$, and $\bar{\rho}_i$ satisfies our global image requirements: the hypotheses of Assumption 5.1 are clearly satisfied since $\mathrm{im}(\bar{\rho}_i)$ has order coprime to p, and $\mathrm{im}(\bar{\rho}_i)$ has no non-trivial cyclic quotient, substituting for the application of Lemma A.6 in §5.2 and §6. □

Remark 7.9. We note that the multiplicities of A_6 and $\mathrm{PSL}_2(\mathbb{F}_{13})$ acting on $\mathrm{Lie}(F_4)$ are for certain irreducible constituents greater than 1, so these examples use the full generality of our methods: compare Remark 5.9 and the discussion in the introduction. We also note that every non-trivial irreducible representation of F_4 has some multiplicity greater than 1 in its formal character, so we cannot apply potential automorphy theorems as in [BCE+18] to lift our $\bar{\rho}_i$. Moreover, the actions of the subgroups A_6 and $\mathrm{PSL}_2(\mathbb{F}_{13})$ on U (the irreducible 26-dimensional representation of F_4) are reducible, further precluding the use of potential automorphy theorems.

We cannot resist two more examples, which likewise cannot be treated by other methods:

Proposition 7.10. *For all sufficiently large primes p, there are representations $\bar{\rho}_1\colon \Gamma_{\mathbb{Q}} \to E_8(\overline{\mathbb{F}}_p)$ and $\bar{\rho}_2\colon \Gamma_{\mathbb{Q}} \to E_8(\overline{\mathbb{F}}_p)$ having images $\mathrm{im}(\bar{\rho}_1) \cong \mathrm{PSL}_2(\mathbb{F}_{41})$ and $\mathrm{im}(\bar{\rho}_2) \cong \mathrm{PSL}_2(\mathbb{F}_{49})$, and that admit geometric deformations $\rho_1, \rho_2\colon \Gamma_{\mathbb{Q}} \to E_8(\overline{\mathbb{Z}}_p)$ with Zariski-dense images.*

Proof. The same method of proof applies: $\mathrm{PSL}_2(\mathbb{F}_{41})$ and $\mathrm{PSL}_2(\mathbb{F}_{49})$ embed into the complex Lie group $E_8(\mathbb{C})$ ([GR01]), and the necessary oddness follows from [GR01, Table 1, Table 2]. Shih's theorem ([Ser08, Theorem 5.1.1]) still applies to construct $\mathrm{PSL}_2(\mathbb{F}_{41})$ as a Galois group over $\mathbb{Q}$ (note $\left(\frac{3}{41}\right) = -1$), and the paper [DV00] establishes the realization of $\mathrm{PSL}_2(\mathbb{F}_{49})$ as a Galois group over $\mathbb{Q}$ (via modular forms, so that complex conjugation is non-trivial). □

Appendix A. Some group theory: irreducible $G(k)$-representations for $p \gg_G 0$

We prove a few group-theoretic lemmas showing that the image hypotheses of §5.1 (namely, Assumption 5.1) in fact follow from the seemingly simpler assumption that $\bar{\rho}$ is "absolutely irreducible," as long as p is sufficiently large. We note that the explicit bounds extracted here depend on the classification of finite simple groups. Recall that a subgroup $\Gamma \subset G^0(k)$ is absolutely irreducible if Γ is not contained in any proper parabolic subgroup of $G^0_{\bar{k}}$.

Lemma A.1. *Let $\Gamma \subset G^0(k)$ be an absolutely irreducible finite subgroup. Assume $p > 2(\dim_k(\mathfrak{g}^{\mathrm{der}})+1)$. Then:*

1. *$\mathfrak{g}^{\mathrm{der}}$ is a semisimple $k[\Gamma]$-module.*
2. *$H^1(\Gamma, \mathfrak{g}^{\mathrm{der}}) = 0$, and the same holds if the action of Γ on $\mathfrak{g}^{\mathrm{der}}$ is twisted by a character of Γ.*

Proof. Let h_G be the maximum of the Coxeter numbers of the simple factors of G^0. By [Ser05, Corollaire 5.5], for $p > 2h_G - 2$, $\mathfrak{g}$, and hence its summand $\mathfrak{g}^{\mathrm{der}}$, is a semisimple Γ-module. We claim then that Γ contains no non-trivial normal subgroup of p-power order. Indeed, suppose there were such a subgroup $H \trianglelefteq \Gamma$. Consider any irreducible $\bar{k}[\Gamma]$-summand U of $\mathfrak{g}_{\bar{k}}$. The $\bar{k}$-vector space of invariants U^H is non-trivial (since H is a p-group) and is stabilized by Γ, hence must equal all of U. This holds for all U, so $\mathfrak{g}$ is a trivial H-module, and therefore H is contained in the center $Z_{G^0}(k)$; but the latter clearly has order prime to p, a contradiction. Thus Γ has no non-trivial normal subgroup of order p, and by [Gur99, Theorem A], $H^1(\Gamma, \mathfrak{g}^{\mathrm{der}}) = 0$ for $p > 2(\dim_k(\mathfrak{g}^{\mathrm{der}}) + 1)$ (to be precise, apply this result to $\Gamma/\Gamma \cap Z_{G^0}(k)$ acting on $\mathfrak{g}^{\mathrm{der}}$). □

The following lemma, with a different proof, also appears in [BHKT16, Lemma 5.1]:

Lemma A.2. *Let G be a connected reductive group over $\bar{k}$. Assume $p > 5$, and that $p \nmid n+1$ for any simple factor of G^{ad} of Dynkin type A_n. Let $\Gamma \subset G(\bar{k})$ be absolutely irreducible. Then* $H^0(\Gamma, \mathfrak{g}^{\mathrm{der}}) = 0$.

Proof. By our characteristic assumptions (which imply that G^{der} and G^{ad} have isomorphic Lie algebras), we may and do assume $G = G^0$ is an adjoint group, and by considering each simple factor of G^0 we may and do further assume that G is simple. Let X be an element of $\mathfrak{g}^{\Gamma}$. We have the Jordan decomposition $X = X_s + X_n$ into semisimple and nilpotent parts in $\mathfrak{g}$, and uniqueness of Jordan decomposition implies that both X_s and X_n are Γ-invariant. Since Γ is then contained in the intersection $C_G(X_s) \cap C_G(X_n)$, it suffices to show that $C_G(X)$ is contained in a proper parabolic when X is either semisimple or nilpotent. In either case, as long as $p > 5$ (for G not of type A_n) or $p \nmid n+1$ (for G of type A_n), $C_G(X)$ is smooth (by a theorem of Richardson: see [Jan04, 2.5 Theorem]). Assume X is a non-zero nilpotent. Then [Jan04, 5.9 Proposition] implies that $C_G(X)$ is contained in a proper parabolic subgroup. Now assume X is a non-zero semisimple element. There is a maximal torus T of G such that X belongs to $\mathfrak{t} = \mathrm{Lie}(T)$ ([Bor91, 11.8]). As usual, we can diagonalize the T-action on $\mathfrak{g}$ to obtain a root system (in the real vector space $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{R}$). The subgroup $C_G(X)$ is a connected reductive group containing T: for the connectedness, we use that $p > 5$ (ensuring p is not a "torsion prime") so that we can invoke [Ste75, Theorem 3.14]. By [BT65, 3.4 Proposition], $C_G(X)$ is determined by the root subgroups it contains (since it contains a maximal torus of G). For a root $\alpha \in \Phi(G^0, T)$, let $u_\alpha \colon \mathbf{G}_a \to G$ be the corresponding root subgroup. For $t \in T$, the relation

$$u_\alpha(y) t u_\alpha(y)^{-1} = t \cdot u_\alpha((\alpha(t)^{-1} - 1)y)$$

lets us compute that (passing to the Lie algebra) $C_G(X)$ precisely contains those $U_\alpha = \mathrm{im}(u_\alpha)$ drawn from the subset

$$\Phi' = \{\alpha \in \Phi(G^0, T) : d\alpha(X) = 0\}$$

of $\Phi = \Phi(G^0, T)$. We claim that the semisimple rank of $C_G(X)$ is strictly less than that of G. Temporarily granting this, we have that the roots Φ' span a proper subspace $\mathbb{R}\Phi' \subset \mathbb{R}\Phi = X^\bullet(T)_{\mathbb{R}}$. By [Bou68, VI.1.7 Proposition 23], Φ' is a root system in the real vector space $\mathbb{R}\Phi'$, and we can also consider it as a subsystem of the root system $\Phi'' = \mathbb{R}\Phi' \cap \Phi$. The latter, by [Bou68, VI.1.7 Proposition 24] has a basis I that extends to a basis of Φ; and since $\mathbb{R}\Phi'$ is strictly contained in $\mathbb{R}\Phi$ this basis of Φ'' is a proper subset of the extended basis of Φ. It follows that $C_G(X)$ is contained in the (proper) Levi subgroup of G associated to I, and therefore that Γ is reducible.

To complete the proof, we establish the postponed claim that the inclusion $\mathbb{R}\Phi' \subseteq \mathbb{R}\Phi$ is proper. It suffices to show that $C_G(X)$ is not semisimple, i.e. has positive-dimensional center. Suppose it were semisimple. Its root system is a (not necessarily simple) subsystem of that of G, and so there are only finitely many possibilities for the root systems of the simple factors H of $C_G(X)^{\mathrm{ad}}$. Under our assumptions on p, each of these simple factors satisfies the following two properties:

- $H^{\mathrm{sc}} \to H^{\mathrm{ad}}$ induces an isomorphism on Lie algebras.
- $\mathrm{Lie}(H)$ has trivial center.

Indeed, note that $\mathrm{Lie}(H)$ has non-trivial center only when $p \leq 3$ or H is of type A_n and $p \mid n+1$: see the discussion of [Sel67, pp. 47-48] (which ensures that $\mathrm{Lie}(H)$ has a nonsingular trace form), and then apply [Sel67, Theorem I.7.2]. Thus under our assumptions on p, $\mathrm{Lie}(C_G(X)) = C_{\mathfrak{g}}(X)$ must have trivial center. But X visibly lies in the center, and we have therefore contradicted the supposed semisimplicity of $C_G(X)$.

∎

In the main theorem, we will use the next three lemmas (Lemma A.6, specifically) to show that $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ have no common subquotient.

Lemma A.3. *Given integers $n, c_1 > 0$, there exists an integer $c_2 > 0$ (depending only on n and c_1) such that if $\Gamma \subset \mathrm{GL}_n(k)$ is a finite subgroup admitting a cyclic quotient of order c_2 and not containing any normal subgroup of order p^a with $a > 0$, then the centre of Γ contains a cyclic subgroup of order prime to p and $\geq c_1$.*

Proof. By Theorem 0.2 of [LP11], for any finite subgroup $\Gamma \subset \mathrm{GL}_n(k)$ there exist normal subgroups $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1 \subset \Gamma$ such that Γ_3 is a p-group, Γ_2/Γ_3 is an abelian group of order prime to p, Γ_1/Γ_2 is a product of finite simple groups of Lie type and Γ/Γ_1 has order bounded by a constant depending only on n. Our assumptions imply that Γ_3 is trivial. From the proof of the theorem [LP11, p. 1156] this imples that Γ_2 is in the centre of Γ_1, so the conjugation action of Γ on Γ_2 factors through Γ/Γ_1.

Let $\Gamma' = (\Gamma_1)^{\mathrm{der}}$. Clearly Γ' lies in the kernel of any homomorphism from Γ to an abelian group and Γ_2 surjects onto Γ_1/Γ'. Furthermore, $\Gamma' \cap \Gamma_2$ has order bounded by a constant depending only on n: this again follows from the construction of Γ_1 and Γ_2 in [LP11, p. 1156] (note particularly the construction of the group denoted G_2 in loc. cit.). Since the order of Γ/Γ_1 is bounded, if Γ has a large cyclic quotient, the coinvariants of the action of Γ/Γ_1 on Γ_1/Γ' must also have a large cyclic quotient, and so also a large cyclic subgroup. The lemma follows since if A is any abelian group with an action of a finite group Δ, the kernel of the averaging map from A_Δ to A^Δ is killed by the order of Δ. □

Remark A.4. The constant c_2 can be effectively bounded by invoking an explicit bound on the index $[\Gamma : \Gamma_1]$ obtained by Collins ([Col08]) using (unlike [LP11]) the classification of finite simple groups.

Lemma A.5. *For G any (split) connected reductive group over k there exists a constant n_G, depending only on the root datum of G, such that for any semisimple element $s \in G(k)$ the centralizer of s^n in G is a (not necessarily proper) Levi subgroup of G for some n dividing n_G.*

Proof. Let T be a maximal torus of G containing s and let t be any element of $T(\bar{k})$. By the theorem in §2.2 of [Hum95], $C_G(t)$ is generated by T, the root subgroups U_α for which $\alpha(t) = 1$ and representatives (in $N(T)$) of the subgroup $W(t)$ of the Weyl group $W(G,T)$ fixing t. Let $\Phi(t)$ be the subset of $\Phi(G^0, T)$ consisting of all roots which are trivial on t. Let $T^{W(t)}$ be the subgroup of T fixed pointwise by $W(t)$ and let $T^{\Phi(t)} = \cap_{\alpha\in\Phi(t)} \mathrm{Ker}(\alpha)$. Let n'_G be the lcm of the orders of the torsion subgroups of all the character groups of the groups of multiplicative type $T^{W(t)} \cap T^{\Phi(t)}$ for all $t \in T(\bar{k})$; there are only finitely many distinct such subgroups since both $W(G,T)$ and $\Phi(G^0, T)$ are finite sets. Then n'_G depends only on the root datum of G, and the order of the component group of any subgroup $T^{W(t)} \cap T^{\Phi(t)}$ divides n'_G.

It follows that $s_1 := s^{n'_G}$ is contained in a torus T_1 such that $T_1 \subset T^{W(s)} \cap T^{\Phi(s)}$. We clearly have $W(s) \subset W(s_1)$ and $\Phi(s) \subset \Phi(s_1)$. If both inclusions are equalities then $C_G(s)$ equals $C_G(s_1)$. Since $C_G(s_1) \supset C_G(T_1) \supset C_G(s)$ by construction, it would follow that $C_G(s)$ is equal to the centralizer of a torus, hence (by [BT65, 4.15 Théorème]) a Levi subgroup. If either of the inclusions is strict, we repeat the procedure after replacing s by s_1. Since $W(G,T)$ and $\Phi(G^0,T)$ are both finite, after at most $m_G := |W(G,T)| + |\Phi(G^0,T)|$ steps we must have equality. Thus, we may take n_G to be $(n'_G)^{m_G}$. □

Lemma A.6. *For G any split semisimple group over k there exists a constant a_G depending only on the root datum of G such that if $\Gamma \subset G(k)$ is an absolutely irreducible subgroup then Γ has no cyclic quotient of order $\geq a_G$.*

Proof. We may clearly assume that G is of adjoint type. If Γ contains a nontrivial normal subgroup U of order a power of p then U is inside a p-Sylow of $G(k)$, i.e., the unipotent radical of a Borel. By a theorem of Borel–Tits [BT71, 3.1 Proposition], there is a parabolic $P \subset G$ containing $N_G(U)$ whose unipotent radical contains U. Since G is reductive, P is a proper parabolic if U is nontrivial. Since U is normal in Γ, this implies Γ is in a proper parabolic of G, contradicting irreducibility.

By embedding G in GL_n for some n, we may now apply Lemma A.3 with $c_1 - 1$ equal to the number n_G obtained from Lemma A.5, to get c_2 such that if Γ has a cyclic quotient of order $\geq c_2$ then the centre of Γ contains a cyclic subgroup Z of order at least c_1 and of order prime to p. By Lemma A.5 there exists an integer $n < c_1$ so that $C_G(s^n)$ is a Levi subgroup, where s is any generator of Z. By construction, s^n is not the identity and since G is adjoint, it is also not central, so $C_G(s^n)$ is a proper Levi subgroup of G. But $\Gamma \subset C_G(s^n)$ and this contradicts irreducibility once again. □

Putting together the results of this section, we deduce:

Corollary A.7. *Let $p \gg_G 0$ be a prime, and let $\bar{\rho}\colon \Gamma_{F,S} \to G(k)$ be a representation such that $\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$ is absolutely irreducible. Assume $[\widetilde{F}(\zeta_p) : \widetilde{F}]$ is greater than the constant a_G of Lemma A.6. Then all of the conditions in Assumption 5.1 hold for $\bar{\rho}$.*

Proof. Under our assumptions on p and absolute irreducibility of $\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_p)}}$, Lemmas A.1 and A.2 imply most of the conclusion. Moreover, Lemma A.6 implies that $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*$ have no common $\mathbb{F}_p[\Gamma_{\widetilde{F}}]$-subquotient. To see this, first note that the lemma implies that the fixed field $\widetilde{F}(\bar{\rho}(\mathfrak{g}^{\mathrm{der}}))$ cannot contain $\widetilde{F}(\zeta_p)$ (for then the adjoint image of $\bar{\rho}(\Gamma_{\widetilde{F}})$ would have a large cyclic quotient, as we take $[\widetilde{F}(\zeta_p) : \widetilde{F}] > a_G$). Letting $\{W_i\}_{i \in I}$ be the simple $\mathbb{F}_p[\Gamma_{\widetilde{F}}]$-module constituents of $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$, if $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ and $\bar{\rho}(\mathfrak{g}^{\mathrm{der}}) \cong \bar{\rho}(\mathfrak{g}^{\mathrm{der}})(1)$ had a common constituent, there would be an isomorphism $W_i \cong W_j(1)$ for some $i, j \in I$. We can choose $\sigma \in \Gamma_{\widetilde{F}}$ acting trivially on W_i and W_j but non-trivially on $\widetilde{F}(\zeta_p)$, contradicting the equivalence $W_i \cong W_j(1)$. Thus, all the conditions of Assumption 5.1 hold. □

Appendix B. Application of results of Lazard

The following lemma and its corollary, the latter being crucial for us, is deduced from results of of [Laz65].

Lemma B.1. *Let G be a compact p-adic Lie group such that its Lie algebra $\mathfrak{g}$ is semisimple. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Z}_p$ and let M be a finitely generated free $\mathcal{O}$-module on which G acts continuously and $\mathcal{O}$-linearly. If $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ does not contain the trivial representation of $\mathfrak{g}$,[11] then there exists an integer $n \geq 0$ such that for ϖ a uniformizer of $\mathcal{O}$, $H^i(G, M/\varpi^m M)$ is killed by ϖ^n for all $i > 0$ and $m \geq 0$. If M contains the trivial representation then the same holds for $i = 1$.*

[11] The Lie algebra $\mathfrak{g}$ acts on $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by [Laz65], V, Lemma (2.4.4).

Proof. We first assume that G is a pro p-group which is p-valued ([Laz65], III §2). By ([Laz65],V, (2.2.3.1)), there exists a ring A (the completed group algebra of G over $\mathbb{Z}_p$) such that

$$H^i(G, M) = \mathrm{Ext}^i_A(\mathbb{Z}_p, M).$$

Moreover, by ([Laz65], V, (2.2.2.3)), $\mathbb{Z}_p$ has a finite resolution by free A modules of finite rank. It follows that $H^*(G, M)$ is computed by a finite complex of $\mathcal{O}$-modules, each term of which is a finite direct sum of copies of M. Furthermore, $H^*(G, M/\varpi^m M)$ is computed by tensoring this complex with $\mathcal{O}/\varpi^m\mathcal{O}$.

By ([Laz65], V, Theorem (2.4.10)), the semisimplicity of $\mathfrak{g}$, and Theorems 21.1 and 24.1 of [CE48], it follows that $H^i(G, M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ for all $i > 0$ if M does not contain the trivial representation and for $i = 1, 2$ otherwise. Since M is assumed to be finitely generated, the previous paragraph shows that $H^i(G, M)$ is also finitely generated, so $H^i(G, M)$ is killed by ϖ^n for some $n \geq 0$, for all $i > 0$ if M does not contain the trivial representation and for $i = 1, 2$ otherwise. The statement for $M/\varpi^m M$ follows from this by applying the universal coefficient theorem to the complex computing the cohomology:

$$H^i(G, M/\varpi^m M) \cong (H^i(G, M) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^n\mathcal{O}) \oplus \mathrm{Tor}_1^{\mathcal{O}}(H^{i+1}(G, M), \mathcal{O}/\varpi^n\mathcal{O}).$$

For a general compact analytic G, by (3.1.3) and (3.1.7.4) of ([Laz65], III) there exists a normal subgroup of finite index which is a p-valued pro p-group. The result then follows from the above special case by using the Hochschild–Serre spectral sequence. □

We owe the deduction of the following corollary to D. Prasad.

Lemma B.2. *Keep the assumptions of Lemma B.1 and also assume that $H^0(G, M/\varpi M) = 0$. Then for any $m \geq 0$ there exists an integer $N(m)$ so that the map*

$$H^1(G, M/\varpi^N M) \to H^1(G, M/\varpi^m M)$$

is the zero map if $N \geq N(m)$.

Proof. The maps $M/\varpi^N M \to M/\varpi^N M$ given by multiplication by ϖ^n induce the zero map on H^1 by Lemma B.1. Also, since $H^0(G, M/\varpi M) = 0$, $H^0(G, M/\varpi^r M) = 0$ for any $r \geq 0$, which implies that all the maps $H^i(G, M/\varpi^s M) \to H^1(G, M/\varpi^N M)$ (induced by inclusions of $M/\varpi^s M$ in $M/\varpi^N M$) are injective. The lemma follows immediately from these two statements.

□

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