LIFTING G-IRREDUCIBLE BUT GL_n-REDUCIBLE GALOIS REPRESENTATIONS

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Abstract. In recent work, the authors proved a general result on lifting G-irreducible odd Galois representations $\mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{F}}_\ell)$, with F a totally real number field and G a reductive group, to geometric ℓ-adic representations. In this note we take G to be a classical group and construct many examples of G-irreducible representations to which these new lifting methods apply, but to which the lifting methods provided by potential automorphy theorems do not.

1. Introduction

Let G be a smooth group scheme over $\mathbb{Z}_\ell$ such that G^0 is a split connected reductive group scheme, and G/G^0 is finite of order prime to ℓ. Let F be a number field with algebraic closure $\overline{F}$ and absolute Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, and let $\bar{\rho}\colon \Gamma_F \to G(\overline{\mathbb{F}}_\ell)$ be a continuous homomorphism. The question of whether $\bar{\rho}$ admits a geometric (or, more ambitiously, automorphic or motivic) lift $\rho\colon \Gamma_F \to G(\overline{\mathbb{Z}}_\ell)$ has attracted a great deal of interest at least since Serre formulated his modularity conjecture in the case $F = \mathbb{Q}$, $G = \mathrm{GL}_2$, and $\bar{\rho}$ absolutely irreducible and odd (see Definition 1.1 below). There are essentially two types of methods, first studied for $G = \mathrm{GL}_2$ and F totally real, for proving such lifting theorems, a purely Galois-theoretic approach developed by Ramakrishna ([Ram99], [Ram02]), and an approach developed in [KW09] and [Kha06] making crucial use of potential automorphy ([Tay02]). Much further work on both of these methods (and their descendants) has led to the papers [FKP19], for the Galois-theoretic method, and [BLGGT14], for the automorphic methods, which more or less represent the state-of-the-art (see also [CEG18] for a refinement of the local hypotheses in [BLGGT14]). Both of these methods crucially rely on the following oddness hypothesis:

Definition 1.1. We say $\bar{\rho}\colon \Gamma_F \to G(\overline{\mathbb{F}}_\ell)$ is odd if for all $v \mid \infty$,

$$h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = \dim(\mathrm{Flag}_{G^0}),$$

where $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ is the Lie algebra of the derived group G^{der} of G^0, equipped with the action of Γ_F via the composite $\mathrm{Ad} \circ \bar{\rho}$, and Flag_{G^0} is the flag variety of G^0.

The main distinction between the output of the two methods is that the Galois-theoretic methods of [FKP19] are applicable to any group G, but yield weaker results, answering only the question of existence of geometric lifts. By contrast, the deeper automorphic methods yield potentially automorphic lifts, which can consequently be put in compatible systems of geometric representations; these methods, however, at present only apply to classical groups.

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In [FKP19, §7] we gave a number of examples of $\bar{\rho}$ valued in an exceptional group G that we showed admitted geometric lifts with Zariski-dense image in $G(\overline{\mathbb{Q}}_\ell)$.

The purpose of this note is to highlight one other distinction between the range of application of [FKP19] and [BLGGT14], resulting from the difference in the irreducibility requirements on the residual representations. Recall that $\bar{\rho}$ is G-irreducible if its image is contained in no proper parabolic subgroup of $G(\overline{\mathbb{F}}_\ell)$. The main theorem of [FKP19] requires that $\bar{\rho}|_{\Gamma_{F(\zeta_\ell)}}$ be G-irreducible (for $\ell \gg_G 0$), whereas the lifting theorem of [BLGGT14] requires that $r \circ \bar{\rho}|_{\Gamma_{F(\zeta_\ell)}}$ be GL_n-irreducible for a representation $r \colon G \to \mathrm{GL}_n$ (for $\ell \gg_n 0$). We will take $G \subset \mathrm{GL}_n$ to be a classical group with its standard representation, and we will construct a series of examples of representations $\bar{\rho} \colon \Gamma_F \to G(\mathbb{F}_\ell)$, for F a suitable totally real field, that are absolutely irreducible as G-representations but reducible as GL_n-representations, and that can be lifted to $G(\overline{\mathbb{Z}}_\ell)$-representations (with Zariski-dense image) by the methods of [FKP19] but not by those of [BLGGT14].

To apply our lifting results, we have to check a few hypotheses, so we now recall a weakening of the main theorem of [FKP19] (the full result yields more precise conclusions about the local restrictions and the image of the lift). Roughly speaking, we need $\bar{\rho}$ with suitable global image and also satisfying some modest local ramification properties:

Theorem 1.2 (See Theorem A of [FKP19])**.** *Let $\ell \gg_G 0$ be a prime. Let F be a totally real field, and let $\bar{\rho} \colon \Gamma_F \to G(\overline{\mathbb{F}}_\ell)$ be a continuous representation unramified outside a finite set of finite places S containing the places above ℓ. Let $\widetilde{F}$ denote the smallest extension of F such that $\bar{\rho}(\Gamma_{\widetilde{F}})$ is contained in $G^0(\overline{\mathbb{F}}_\ell)$, and assume that $[\widetilde{F}(\zeta_\ell) : \widetilde{F}]$ is strictly greater than an integer a_G depending only on the root datum of G (see* [FKP19, Lemma A.6]*). Fix a geometric lift $\mu \colon \Gamma_F \to G/G^{\mathrm{der}}(\overline{\mathbb{Z}}_\ell)$ of $\bar{\mu} := \bar{\rho} \pmod{G^{\mathrm{der}}}$, and assume that $\bar{\rho}$ satisfies the following:*

- *$\bar{\rho}$ is odd.*
- *$\bar{\rho}|_{\Gamma_{\widetilde{F}(\zeta_\ell)}}$ is absolutely irreducible.*
- *For all $v \in S$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a continuous lift $\rho_v \colon \Gamma_{F_v} \to G(\overline{\mathbb{Z}}_\ell)$ of type $\mu|_{\Gamma_{F_v}}$; and that for $v|\ell$ this lift may be chosen to be de Rham and regular in the sense that the associated Hodge–Tate cocharacters are regular.*

Then there is a lift

$$\begin{tikzcd} & G(\overline{\mathbb{Z}}_\ell) \arrow[d] \\ \Gamma_F \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & G(\overline{\mathbb{F}}_\ell) \end{tikzcd}$$

of $\bar{\rho}$ satisfying:

- *The projection of ρ to $G/G^{\mathrm{der}}(\overline{\mathbb{Z}}_\ell)$ equals μ.*
- *ρ is unramified outside a finite set of primes, and the restrictions $\rho|_{\Gamma_{F_v}}$ for $v|\ell$ are de Rham and regular, having the same ℓ-adic Hodge type as ρ_v.*
- *The Zariski-closure of the image $\rho(\Gamma_F)$ contains G^{der}.*

We provide a few kinds of examples. We begin in §2 with an elementary example with $G = \mathrm{GSp}_{2n}$, obtained by appropriately summing irreducible odd two-dimensional representations. Then in §3, we explain a quite general but "soft" approach to constructing examples using Calegari's result on the "potential inverse Galois problem with local conditions" ([Cal12, Proposition 3.2]). This relies on the results of Moret-Bailly and yields examples with no global control over the totally real field F. Finally, we devote the bulk of the paper to a series of more concrete examples relying on Zywina's work ([Zyw19]) on the (actual) inverse Galois problem for orthogonal groups. The representations we construct here have the form $\bar{\rho} = \bar{\theta} \oplus 1$, where N is an even integer, and $\bar{\theta}\colon \Gamma_F \to \mathrm{SO}_N(\mathbb{F}_\ell)$ is an orthogonal representation with large image arising from [Zyw19]. In this case the real work is to compute the action of complex conjugation in Zywina's examples, and we do this in §5. This approach has the advantage that, modulo the purely local question of whether any $\Gamma_{\mathbb{Q}_\ell} \to \mathrm{SO}_{N+1}(\mathbb{F}_\ell)$ admits a Hodge-Tate regular de Rham lift, it would yield examples with $F = \mathbb{Q}$. Recently the techniques for producing such lifts have been greatly advanced: forthcoming work of Emerton-Gee will address this question with GL_{N+1} in place of SO_{N+1}, and others are at work extending the methods of Emerton-Gee to other group. Thus we may be optimistic that in the not-too-distant future these local obstacles will vanish. In any case, we hope that the various methods presented here are all of some interest.

2. An elementary example

In this brief section, we consider a relatively simple example where $G = \mathrm{GSp}_{2n}$, defined with respect to the symplectic form $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, where 1_n denotes the $n \times n$ identity matrix. Let $\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_n \colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be n Galois representations satisfying

- The $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ are irreducible.
- The determinants $\det(\bar{\rho}_i)$ are independent of i.
- The $\bar{\rho}_i$ are all odd, i.e. $\det(\bar{\rho}_i(c)) = -1$ for all i.

Writing $\bar{\rho}_i(g) = \begin{pmatrix} a_i(g) & b_i(g) \\ c_i(g) & d_i(g) \end{pmatrix}$, define

$$\bar{\rho}(g) = \left(\begin{array}{cccc|cccc} a_1(g) & 0 & \cdots & 0 & b_1(g) & 0 & \cdots & 0 \\ 0 & a_2(g) & 0 & \cdots & 0 & b_2(g) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n(g) & 0 & \cdots & 0 & b_n(g) \\ \hline c_1(g) & 0 & \cdots & 0 & d_1(g) & 0 & \cdots & 0 \\ 0 & c_2(g) & 0 & \cdots & 0 & d_2(g) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & c_n(g) & 0 & \cdots & 0 & d_n(g) \end{array}\right).$$

The condition that $\det(\bar{\rho}_i)$ be independent of i guarantees that $\bar{\rho}$ is valued in GSp_{2n} and has symplectic multiplier $\det(\bar{\rho}_i)$. Oddness of each $\bar{\rho}_i$ therefore implies oddness of $\bar{\rho}$. Since the $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ are irreducible, non-isotropic, and pair trivially with one another, we see that $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ leaves no proper isotropic subspace invariant. Thus $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is GSp_{2n}-irreducible, while obviously not GL_{2n}-irreducible.

Proposition 2.1. *Assume $\ell \gg_n 0$. With notation as above, $\bar{\rho}$ admits a Hodge-Tate regular de Rham lift $\rho\colon \Gamma_{\mathbb{Q}} \to \mathrm{GSp}_{2n}(\overline{\mathbb{Z}}_\ell)$ with Zariski-dense image.*

Proof. From the above discussion, this will now follow from Theorem 1.2 provided we can check the local hypotheses in that theorem. For this, we note that by Lemma 2.2 below each $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}_\ell}}$ admits a potentially crystalline lift $\rho_{i,\ell}$ satisfying:

- $\det(\rho_{i,\ell})$ is independent of i.
- The Hodge-Tate weights of the $\rho_{i,\ell}$ are distinct.

Let $\mu\colon \Gamma_{\mathbb{Q}} \to \overline{\mathbb{Z}}_\ell^\times$ be a global character such that $\mu|_{\Gamma_{\mathbb{Q}_\ell}} = \det(\rho_{i,\ell})$. Such a μ exists either by the local and global Kronecker-Weber theorems or by noting that $\det(\rho_{i,\ell})$ is an integer power of the cyclotomic character multiplied by a finite-order character; this reduces to the case of finite-order characters, where we can realize any finite extension of $\mathbb{Q}_\ell$ as a completion of a finite extension of $\mathbb{Q}$. For such a μ, the local hypothesis at ℓ of Theorem 1.2 is satisfied by the choice of similitude character μ and the local lift ρ_ℓ constructed by summing the $\rho_{i,\ell}$ in the same way as $\bar{\rho}$ is defined in terms of the $\bar{\rho}_i$. For primes $p \neq \ell$ where $\bar{\rho}$ ramifies, [Boo18, Theorem 1.1] implies that $\bar{\rho}|_{\Gamma_{\mathbb{Q}_p}}$ admits a lift $\Gamma_{\mathbb{Q}_p} \to \mathrm{GSp}_{2n}(\overline{\mathbb{Z}}_\ell)$ with similitude character μ. We now conclude by Theorem 1.2. □

Here is the local lemma used in the proof:

Lemma 2.2. *Let $\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_n\colon \Gamma_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be continuous representations with the same determinant $\bar{\tau} = \det(\bar{\rho}_i)$, $i = 1, \dots, n$. Then there exist potentially crystalline lifts $\rho_i\colon \Gamma_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$ such that the union of the Hodge-Tate weights of the ρ_i is a set with $2n$ distinct elements, and $\det(\rho_i)$ is independent of i.*

Proof. First we note that it suffices to produce potentially crystalline lifts ρ_i with (all taken together) distinct Hodge-Tate weights and $\det(\rho_i)$ having the same (single) Hodge-Tate weight for all i. Indeed, then each quotient $\det(\rho_i)/\det(\rho_1)$ is a finite-order character valued in a pro-ℓ group (since the reduction mod ℓ is trivial), and so we can extract a square root and twist ρ_i to have the same determinant as ρ_1 (and this finite-order twist does not affect the property of being potentially crystalline). To finish the proof, we apply [Mul13, Théorème 2.5.3, Theorem 2.5.4], which show the following:

- If $\bar{\rho}_i$ is irreducible, then for any choice of Hodge-Tate weights $\{m_{i,1}, m_{i,2}\}$, there exists a potentially crystalline lift ρ_i of $\bar{\rho}_i$ with these weights. We choose these in such a way that $m_{i,1} + m_{i,2}$ is independent of i, but the multi-set $\{m_{i,j}\}_{i,j}$ is in fact a set.
- If $\bar{\rho}_i = \begin{pmatrix} \bar{\chi}_{i,1} & * \\ 0 & \bar{\chi}_{i,2} \end{pmatrix}$ is an extension of characters, then for each i there are potentially crystalline lifts $\chi_{i,j}$ of $\bar{\chi}_{i,j}$ with Hodge-Tate weights summing to any pre-specified value and a potentially crystalline lift of $\bar{\rho}_i$ that is an extension of $\chi_{i,2}$ by $\chi_{i,1}$. (To see this claim requires inspection of the proof of [Mul13, Théorème 2.5.4].)

□

3. Approach via the potential IGP with local conditions

The general approach of the present section is quite flexible and will certainly apply to other Galois images and target groups than those used here; we do not strive for maximal generality. Let $N = 2n$ be an even integer, and consider the standard embedding $\mathrm{SO}_N \to \mathrm{SO}_{N+1}$ of special orthogonal groups over $\mathbb{Z}_\ell$ (defined, for definiteness, with respect to the symmetric pairings given by the identity matrices[1]). Let $\Gamma = \mathrm{SO}_N(\mathbb{F}_\ell)$, and fix the following order two element c_∞ of Γ:

[1]This group over $\mathbb{Z}_\ell$ may not be split, but we will only apply the lifting results of [FKP19] after making a finite extension of $\mathbb{Z}_\ell$. Many variants on the present construction are possible; in particular, we could take different forms of SO_N.

- If $N \equiv 0 \pmod 4$, then $c_\infty = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ (note that $\det(c_\infty) = (-1)^n = 1$).
- If $N \equiv 2 \pmod 4$, then $c_\infty = \left(\begin{array}{cc|cc} 0_{n-1} & 0 & 1_{n-1} & 0 \\ 0 & -1 & 0 & 0 \\ \hline 1_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$ (note that $\det(c_\infty) = (-1)^{n-1} = 1$).

We now circumvent the inverse Galois problem by applying Calegari's solution ([Cal12, Proposition 3.2]) to the potential inverse Galois problem with local conditions, and we thus produce the examples of this section:

Proposition 3.1. *There exists a totally real field $F/\mathbb{Q}$ and a Galois extension K/F satisfying:*

1. *There is an isomorphism $\bar{\theta}\colon \mathrm{Gal}(K/F) \xrightarrow{\sim} \Gamma$.*
2. *$K/\mathbb{Q}$ is linearly disjoint from $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$.*
3. *With respect to the isomorphism $\bar{\theta}$, for all $v|\infty$, complex conjugation $c_v \in \mathrm{Gal}(K/F)$ is conjugate to c_∞.*
4. *The prime ℓ splits completely in $F/\mathbb{Q}$, and for all places $v|\ell$ of F, and all $w|v$ of K, $K_w = F_v$ is the trivial extension.*

Assume now $\ell \gg_N 0$, and define $\bar{\rho} = \bar{\theta} \oplus 1$. Then $\bar{\rho}\colon \Gamma_F \to \mathrm{SO}_{N+1}(\mathbb{F}_\ell)$ has a Hodge-Tate regular geometric lift $\rho\colon \Gamma_F \to \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ with image Zariski-dense in SO_{N+1}.

Proof. Existence of K, F, and $\bar{\theta}$ follow immediately from [Cal12, Proposition 3.2]. To finish the proof, we must verify that the hypotheses of Theorem 1.2 are satisfied for $\bar{\rho} = \bar{\theta} \oplus 1$. A maximal (proper) parabolic subgroup of SO_{N+1} is the stabilizer of an isotropic subspace $W \subset \overline{\mathbb{F}}_\ell^{N+1}$. Since the image of $\bar{\theta}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is $\mathrm{SO}_N(\mathbb{F}_\ell)$, which acts absolutely irreducibly in its standard N-dimensional representation for $\ell \gg_N 0$, $\bar{\rho}$ stabilizes exactly two proper subspaces of $\overline{\mathbb{F}}_\ell^{N+1}$, namely the space of $\bar{\theta}$ and the complementary line. Clearly neither of these subspaces is isotropic, so in all cases $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is SO_{N+1}-absolutely irreducible.

Booher's result ([Boo18, Theorem 1.1]) shows that for v not above ℓ at which $\bar{\rho}$ is ramified, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift to $\Gamma_{F_v} \to \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$. (Note that Booher's result shows a GO_{N+1}-deformation ring with fixed orthogonal multiplier is formally smooth of suitably large dimension; we fix the multiplier to be trivial to produce an $\mathrm{O}_{N+1} = \mathrm{SO}_{N+1} \times \{\pm 1\}$ lift and then project to SO_{N+1}.) Since the extensions K_w/F_v are trivial for $v|\ell$, $\bar{\rho}|_{\Gamma_{F_v}}$ clearly has a Hodge-Tate regular crystalline lift, simply by taking an appropriate sum of powers of the cyclotomic character. To conclude, we check that $\bar{\rho}$ is odd. Indeed, since the adjoint representation of SO_{N+1} is isomorphic to the second exterior power of the standard representation, we find

$$\mathrm{Tr}(\bar{\rho}(c)|_{\mathfrak{so}_{N+1}}) = \frac{\mathrm{Tr}(\bar{\rho}(c))^2 - \mathrm{Tr}(\bar{\rho}(c^2))}{2} = \frac{1-(N+1)}{2} = -\mathrm{rk}(\mathrm{SO}_{N+1}).$$

This is equivalent to the oddness condition $\dim(\mathfrak{so}_{N+1}^{\mathrm{Ad}(\bar{\rho}(c))=1}) = \dim \mathrm{Flag}_{\mathrm{SO}_{N+1}}$ assumed in Theorem 1.2. $\square$

Remark 3.2. Perhaps the most interesting case here is when $N \equiv 2 \pmod 4$. Then the representations $\bar{\theta}$ are not odd, and indeed cannot be odd since -1 does not belong to the

Weyl group of SO_N for any $N \equiv 2 \pmod 4$).[2] Existing lifting techniques cannot lift them to geometric SO_N-representations, and we expect that no such lifts that are Hodge-Tate regular as GL_N-representations can exist. See Proposition 3.3 below for a proof of this in some cases, but we now explain a general heuristic. Such a lift (necessarily GL_N-irreducible) would conjecturally arise from an automorphic representation π of (split) $\mathrm{SO}_N/\mathbb{Q}$ with cuspidal transfer to GL_N. The archimedean L-parameter $\mathrm{rec}_{\pi_\infty} \colon W_{\mathbb{R}} \to \mathrm{SO}_N(\mathbb{C})$ would then (by archimedean purity for GL_N) restrict to $\mathbb{C}^\times \subset W_{\mathbb{R}}$ as $\mathrm{rec}_{\pi_\infty}(z) = z^\mu \cdot \bar{z}^{-\mu}$ for some cocharacter μ of the diagonal torus $T \subset \mathrm{SO}_N$. Let $\{e_i^*\}_{i=1}^n$ be the standard basis of $X_\bullet(T)$. Then the most regular situation that can arise has $\mathrm{rec}_{\pi_\infty}(j)$ equal to the element c_∞ above, and $\mu = \sum_{i=1}^n p_i e_i^*$ with $p_1, \dots, p_{n-1}$ distinct, and $p_n = 0$ (using the Weil group relation $jzj^{-1} = \bar{z}$). Such an L-parameter is in fact SO_N-regular, but it is clearly not GL_N-regular. We note that these "most regular" lifts that might be possible would be the ℓ-adic representations associated to cuspidal automorphic representations on (suitable forms of) SO_N that are non-degenerate limits of discrete series at archimedean places. See [GK19] for important recent progress on the problem of associating Galois representations to automorphic representations that are non-degenerate limits of discrete series at infinity.

We can unconditionally rule out "many" candidates for regular lifts of $\bar{\theta}$ using potential automorphy theorems:

Proposition 3.3. *Let $\rho \colon \Gamma_{\mathbb{Q}} \to \mathrm{O}_N(\overline{\mathbb{Z}}_\ell)$ for $\ell > 2(2N+1)$ be a continuous representation satisfying:*

(1) $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is GL_N-irreducible.
(2) $\rho|_{\Gamma_{\mathbb{Q}_\ell}}$ is as GL_N-representation potentially diagonalizable in the sense of [BLGGT14, §1.4] and has distinct Hodge-Tate weights.

Then $\mathrm{Tr}(\rho(c)) = 0$. In particular, when $N \equiv 2 \pmod 4$, ρ cannot factor through SO_N.

Proof. In this proof we freely use the terminology and notation of [BLGGT14]; we argue as in, for instance, [BLGGT14, Proposition 3.3.1], making use of Harris's tensor product trick. Choose a quadratic imaginary field $K/\mathbb{Q}$ linearly disjoint from $\mathbb{Q}(\bar{\rho}, \mu_\ell)/\mathbb{Q}$, and let $\psi \colon \Gamma_K \to \overline{\mathbb{Z}}_\ell^\times$ be a geometric (hence potentially crystalline and potentially diagonalizable) character such that $r := \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\psi)$ satisfies:

- $r^* \cong r \otimes \mu$, where $\mu(c) = -1$ (this imposes no condition on ψ).
- $r|_{\Gamma_{\mathbb{Q}_\ell}}$ is Hodge-Tate regular, with Hodge numbers $\{m_1, m_2\}$ having the property that no two Hodge numbers of ρ differ by $|m_1 - m_2|$.
- $(\bar{\rho} \otimes \bar{r})|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is irreducible (a simple way to arrange this is to choose ψ such that at an auxiliary prime q split in $K/\mathbb{Q}$ and unramified in $\bar{\rho}$, $\bar{\psi}$ is ramified at one prime above q but not at the conjugate prime above q) .

To produce such a ψ, we can apply [BLGGT14, Lemma A.2.5], similarly to its use in [BLGGT14, Proposition 3.3.1]. Consider then the representation $\rho' := \rho \otimes r \colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_{2N}(\overline{\mathbb{Z}}_\ell)$. Since ρ is orthogonal, and r is symplectic, $\rho' \colon \Gamma_{\mathbb{Q}} \to \mathrm{GSp}_{2N}(\overline{\mathbb{Z}}_\ell)$ is symplectic. Moreover, $(\rho')^* \cong \rho' \otimes \mu$, and thus we see that ρ' is totally odd polarizable. Moreover, $\rho'|_{\Gamma_{\mathbb{Q}_\ell}}$ is still

[2] We learned from Wushi Goldring the observation that an SO_N-representation that is not odd can transfer to an odd SO_{N+1}-representation.

potentially diagonalizable with distinct Hodge-Tate weights. Thus we can apply the potential automorphy theorem of [BLGGT14, Corollary 4.5.2] to conclude that (ρ', μ) is potentially automorphic. Now we apply [BLGGT14, Lemma 2.2.4]: in the notation of *loc. cit.*, $(\rho' = \rho \otimes r \cong \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\rho|_{\Gamma_K} \otimes \psi), \mu)$ is potentially automorphic and polarized, so $(\rho|_{\Gamma_K} \otimes \psi, \mu)$ is also potentially automorphic and polarized. By [BLGGT14, Lemma 2.2.1, Lemma 2.2.2], $(\rho, 1)$ is potentially automorphic, i.e. there exists a regular algebraic, polarized cuspidal automorphic representation (π, χ) of $\mathrm{GL}_N(\mathbb{A}_{\mathbb{Q}})$ such that $(\rho, 1) \cong (r_{\ell,\iota}(\pi), \kappa_\ell^{1-N} r_{\ell,\iota}(\chi))$, where κ_ℓ denotes the ℓ-adic cyclotomic character, and $r_{\ell,\iota}(\pi)$ is the automorphic Galois representation associated to π (see [BLGGT14, Theorem 2.1.1]; note that in their normalization, $r_{\ell,\iota}(\pi)$ is the Galois representation whose local restrictions correspond under local Langlands to $\pi \otimes |\cdot|^{\frac{1-N}{2}}$). It follows that $\chi = |\cdot|^{N-1}$, and $\pi \otimes |\cdot|^{\frac{1-N}{2}}$ is the (on the nose) self-dual regular L-algebraic cuspidal automorphic representation corresponding to ρ under the local Langlands correspondence. Applying [Taï16, Theorem A] to $\pi \otimes |\cdot|^{\frac{1-N}{2}}$, we find that $|\mathrm{Tr}(\rho(c))| \leq 1$, and thus ($N$ is even) $\mathrm{Tr}(\rho(c)) = 0$. $\square$

Remark 3.4. The main limitation in this result is the potential diagonalizability assumption. But in a compatible system of Hodge-Tate regular automorphic Galois representations, almost all members will be potentially diagonalizable by the theory of Fontaine-Laffaille. Calegari ([Cal12]) has proven a stronger result that a geometric and Hodge-Tate regular $\Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$ must be odd, without a potential diagonalizability hypothesis.

4. Review of Zywina's work

In this section we recall the construction of Zywina ([Zyw19]) that in many cases realizes the simple groups of orthogonal type over $\mathbb{F}_\ell$ as Galois groups of regular extensions of $\mathbb{Q}(t)$, and in particular as Galois groups over $\mathbb{Q}$ (in infinitely many ways). Some of the finer points–namely, the simplicity itself–of this construction will not matter for our present purposes: rather, we want to use this construction as a source of Galois groups over $\mathbb{Q}$ isomorphic to large subgroups of even orthogonal groups, and we recall only what is necessary for our purposes.

Fix an even integer $N \geq 6$. Let $R = \mathbb{Z}[S^{-1}]$, for a finite set of primes S that may be enlarged as the construction proceeds. Zywina artfully chooses polynomials $a_2(t), a_4(t), a_6(t) \in R[t]$, such that the discriminant $\Delta(t)$ of the Weierstrass equation $y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$ is non-zero (and the j-invariant is non-constant) and considers a family of quadratic twists of this Weierstrass equation. This leads to the following construction of a rank N orthogonal local system on an open subset of $\mathbb{P}^1_R$, which we rapidly summarize:

- Let $A = R[u, \Delta(u)^{-1}])$ and set $M = \mathrm{Spec}(A)$.
- Let $j \colon U \to \mathbb{P}^1_M$ be the inclusion of the defined by $U = \mathrm{Spec}(A[t, (t-u)^{-1}, \Delta(t)^{-1}])$.
- Let $E \to U$ be the elliptic curve defined by the Weierstrass equation
$$(t-u)y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$
- Let $\pi \colon \mathbb{P}^1_M \to M$ be the structure morphism, and define
$$\mathcal{G} = R^1\pi_*(j_*E[\ell]),$$
where $E[\ell]$ is the local system (of $\mathbb{F}_\ell$-modules) on U defined by the ℓ-torsion subgroup scheme of E. The sheaf $\mathcal{G}$ on M is clearly constructible, and Zywina shows ([Zyw19, Lemma 3.3]) that it is lisse of rank N (this depends on the careful choice of the $a_i(t)$ and is verified case-by-case in [Zyw19, §6]). Poincaré duality provides an orthogonal

pairing $\mathcal{G} \times \mathcal{G} \to \mathbb{F}_\ell$, and so given a geometric generic point $\bar{\chi}$ of M we obtain a representation

$$\theta_\ell \colon \pi_1(M, \bar{\xi}) \to \mathrm{O}(\mathcal{G}_{\bar{\xi}}).$$

In fact Zywina must, in order to realize the simple groups as Galois groups, work with the pull-back $h^*(\mathcal{G})$ of this local system along a suitable finite étale cover $h \colon W \to M$, where W is again an open subscheme of $\mathbb{P}^1_R$ (see the proof of [Zyw19, Theorem 4.1]). He denotes by

$$\vartheta_\ell \colon \pi_1(W) \to \mathrm{O}(V_\ell)$$

the representation associated to $h^*(\mathcal{G})$ (omitting reference to the choice of compatible base-point).

Recall (see [Zyw19, §1.1]) that the spinor norm is a homomorphism $\mathrm{sp} \colon \mathrm{O}(V_\ell) \to \mathbb{F}_\ell^\times/(\mathbb{F}_\ell^\times)^2$, and let $\Omega(V_\ell) \subset \mathrm{SO}(V_\ell)$ be the subgroup of elements with trivial spinor norm. When the discriminant $\mathrm{disc}(V_\ell) := \mathrm{sp}(-1)$ is the trivial coset, -1 belongs to $\Omega(V_\ell)$. We will use the following consequence of Zywina's main theorem and arguments:

Proposition 4.1. *Let $N \geq 6$ be an even integer, and let $\ell \geq 5$ be a prime. Then the elliptic curve $E \to U$ may be chosen to ensure that the quadratic space V_ℓ has discriminant $(\mathbb{F}_\ell^\times)^2$, there are infinitely many $w_i \in \mathbb{Q}$ such that the specializations ϑ_{ℓ,w_i} (defined up to conjugation)*

$$\Gamma_{\mathbb{Q}} \xrightarrow{w_i} \pi_1(W) \xrightarrow{\vartheta_\ell} \mathrm{O}(V_\ell)$$

are non-isomorphic and satisfy $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}}) = \Omega(V_\ell)$.

When $N \equiv 6 \pmod 8$, we also consider a second family $E \to U$, where the above holds except with the conclusion that $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}})$ contains $\Omega(V_\ell)$ with index 2, and $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}(i)})$ equals $\Omega(V_\ell)$. We will refer to the two examples when $N \equiv 6 \pmod 8$ as Case 6_Ω and Case 6_{O}.

Proof. When $N \equiv 0, 2, 4 \pmod 8$, or in Case 6_Ω, the existence of specializations w_i with $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}}) = \Omega(V_\ell)$ is immediate from the proof of [Zyw19, Theorem 1.1] and the Hilbert irreducibility theorem ([Ser08, §3.3-3.4]), which produces w_i such that the fixed fields of the ϑ_{ℓ,w_i} are linearly disjoint over $\mathbb{Q}$. Case 6_{O} arises from an earlier version of [Zyw14], where the root number calculation of [Zyw14, §6.4] shows that the root number $\varepsilon_{E_{h(w)}}$ at a specialization $w \in W(\mathbb{F}_p)$ (for any sufficiently large prime p) is $\left(\frac{-1}{p}\right)$ (not 1 as claimed). Consequently $\det(\vartheta_\ell)(\mathrm{Frob}_w) = \left(\frac{-1}{p}\right)$ for all $p \gg 0$, $w \in W(\mathbb{F}_p)$, and so $\det(\vartheta_\ell) \colon \pi_1(W) \to \mathbb{F}_\ell^\times$ factors through the non-trivial quadratic character $\pi_1(W) \to \Gamma_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \to \mathbb{F}_\ell^\times$ (with the first map induced by the structure morphism). The proof of [Zyw19, Theorem 4.1] still shows that the geometric monodromy group $\vartheta_\ell(\pi_1(W_{\overline{\mathbb{Q}}}))$ contains $\Omega(V_\ell)$, and that $\vartheta_\ell(\pi_1(W_{\mathbb{Q}(i)}))$ is contained in $\Omega(V_\ell)$ (since it is contained in $\mathrm{SO}(V_\ell)$, and the rest of the relevant calculation only uses information about the Euler characteristic χ and the Tamagawa number $c_{E_{h(w)}}$, in the notation of *loc. cit.*). Thus $\pi_1(W_{\overline{\mathbb{Q}}}) = \pi_1(W_{\mathbb{Q}(i)}) = \Omega(V_\ell)$, and $\pi_1(W) \subset \mathrm{O}(V_\ell)$ contains $\Omega(V_\ell)$ with index 2. We can again invoke Hilbert irreducibility to produce the desired u_i: there is a thin set $T \subset W(\mathbb{Q})$ such that for all $w \in W(\mathbb{Q}) \setminus t$, $\vartheta_{\ell,w}$ cuts out a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\vartheta_\ell(\pi_1(W))$. Since $W(\mathbb{Q}) \setminus T \subset \mathbb{Q}(i)$ is not a thin subset (see [Ser08, Proposition 3.2.1]), applying Hilbert irreducibility to $\vartheta_\ell|_{\pi_1(W_{\mathbb{Q}(i)})}$ shows that there are infinitely many $w_i \in W(\mathbb{Q}) \setminus T$ such that $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}(i)})$ equals $\Omega(V_\ell)$. The resulting extensions are linearly disjoint over $\mathbb{Q}(i)$. ∎

5. Complex conjugation

In our application of the lifting theorem of [FKP19], we most importantly have to understand the action of complex conjugation in the Galois representations $\vartheta_{\ell,w}$, $w \in W(\mathbb{Q})$. The Betti-étale comparison isomorphism reduces this to a transcendental calculation, which we perform in this section. Recall that for any smooth variety $X/\mathbb{R}$ there is a "transport of structure" isomorphism $\mathrm{F}_\infty \colon H^*(X(\mathbb{C}), \mathbb{Q}) \to H^*(X(\mathbb{C}), \mathbb{Q})$ induced by the action of complex conjugation on the points of the manifold $X(\mathbb{C})$; and that functoriality of the Betti-étale comparison isomorphism

$$H^*(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^*_{\acute{e}t}(X_{\mathbb{C}}, \mathbb{Q}_\ell),$$

implies that the automorphism F_∞ corresponds to the action of complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the étale cohomology. Moreover, in the Hodge decomposition $H^r(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=r} H^{p,q}(X(\mathbb{C}))$, F_∞ exchanges $H^{p,q}(X(\mathbb{C}))$ and $H^{q,p}(X(\mathbb{C}))$. It follows that when r is odd, $\mathrm{Tr}(c|H^r_{\acute{e}t}(X_{\mathbb{C}}, \mathbb{Q}_\ell)) = 0$. For r even, however, the contribution of $H^{\frac{r}{2},\frac{r}{2}}(X(\mathbb{C}))$ terms can make it a subtle matter to compute the trace. In this section, we address this problem for the middle cohomology of certain elliptic surfaces over $\mathbb{R}$. We begin with a general lemma:[3]

Lemma 5.1. *Let M be a compact differentiable manifold and $f : M \to M$ a diffeomorphism such that $f^n = Id_M$ for some positive integer n. Let F be the fixed point locus of f, i.e., the set of points $x \in M$ such that $f(x) = x$. Then*

$$\mathrm{Tr}(f^*|H^*(M, \mathbb{Q})) = \chi(F)$$

where the LHS is the alternating sum of the traces and χ denotes the topological Euler characteristic, i.e., the trace of the identity map.

Proof. By averaging any Riemannian metric on M with respect to the subgroup of $\mathrm{Diff}(M)$ generated by f, we see that M has an f-invariant Riemannian metric g. Using the exponential map with respect to this metric at any point of F, we see that F is a closed submanifold of M (with connected components possibly of varying dimension). For $0 < \epsilon' < \epsilon \ll 0$, let T_ϵ (resp. $T_{\epsilon'}$) be the tubular neighbourhood of F in M of radius ϵ (resp. ϵ') constructed as in [MS74, Theorem 11.1] using the metric g. Since g is f-invariant, so are these tubular neighbourhoods.

Let A_ϵ be the closure of T_ϵ in M and $B_{\epsilon'} = M \backslash T_{\epsilon'}$. Both of these sets are f-invariant compact manifolds with boundary. Since $B_{\epsilon'} \cap F = \emptyset$, f has no fixed points on $B_{\epsilon'}$. Thus, by the "no fixed points" version of the Lefschetz fixed point theorem, $\mathrm{Tr}(f^*|H^*(B_{\epsilon'}, \mathbb{Q})) = 0$ and $\mathrm{Tr}(f^*|H^*(A_\epsilon \cap B_{\epsilon'}, \mathbb{Q})) = 0$.

We now consider the Mayer–Vietoris sequence for the cover of M given by A_ϵ and $B_{\epsilon'}$. Since f preserves these sets, it induces an endomorphism of this sequence, and exactness implies that the alternating sums of the traces of f^* on the terms of this sequence must be 0. Since the inclusion of F in A_ϵ is a homotopy equivalence, we get that

$$\mathrm{Tr}(f^*|H^*(M, \mathbb{Q})) = \mathrm{Tr}(f^*|H^*(A_\epsilon, \mathbb{Q})) = \chi(F)$$

as claimed. ∎

[3] We thank Prakash Belkale for help in simplifying our original proof of this Lemma.

Lemma 5.2. *Let X be a smooth projective variety over $\mathbb{R}$, as before denoting by F_∞ the involution of $X(\mathbb{C})$ and its cohomology induced by complex conjugation. Then*

$$\mathrm{Tr}(\mathrm{F}_\infty | H^*(X(\mathbb{C}), \mathbb{Q})) = \chi(X(\mathbb{R}))$$

Proof. This follows immediately from Lemma 5.1 since the fixed point locus of F_∞ is precisely $X(\mathbb{R})$. □

We now turn to the study of real elliptic surfaces. The following lemma is standard for elliptic surfaces over $\mathbb{C}$.

Lemma 5.3. *Let $\pi : E \to C$ be an elliptic surface over $\mathbb{R}$. Then*

$$\chi(E(\mathbb{R})) = \sum_{x \in S} \chi(E_x(\mathbb{R})),$$

where $S \subset C(\mathbb{R})$ is the set of real points over which π is not smooth and E_x is the fibre of π over x.

Proof. For a topological space T we denote by $\chi_c(T)$ the Euler characteristic with compact supports. If $U \subset T$ is open and $Z = T \backslash U$, then the long exact sequence of cohomology with compact supports implies that $\chi_c(T) = \chi_c(U) + \chi_c(Z)$ (as long as all the cohomology groups are finite dimensional).

Let $U' = C(\mathbb{R}) \backslash S$. The map $\pi_{\mathbb{R}}$ induced by π from $E(\mathbb{R}) \to C(\mathbb{R})$ is a proper fibre bundle over each connected component of U' with fibre homeomorphic to a circle, two disjoint circles, or the empty set. The Leray spectral sequence (with compact support) then shows that $\chi_c(\pi^{-1}(U')) = 0$. The lemma follows from this, the additivity of χ_c, and the fact that $E(\mathbb{R})$ is compact. □

The following lemma gives the Euler characteristic of the real points of some types of singular fibres (in the Kodaira classification) of elliptic fibrations with a section over $\mathbb{R}$. We note that in general this depends on the real structure of the fibre, not just the Kodaira symbol.

Lemma 5.4. *In the notation of the Kodaira classification (see, e.g., [Sil94, IV.8]), and allowing all $n \geq 0$ in the I_n^* case, we have the following Euler characteristic calculations:*

$$\chi(I_1) = \begin{cases} -1 & \text{if the fibre is split,} \\ 1 & \text{if the fibre is non-split;} \end{cases} \tag{5.1}$$

$$\chi(I_2) = \begin{cases} -2 & \text{if the fibre is split,} \\ 0 & \text{if the fibre is non-split;} \end{cases} \tag{5.2}$$

$$\chi(II) = 0; \tag{5.3}$$

$$\chi(III) = -1; \tag{5.4}$$

$$\chi(I_n^*) = \begin{cases} -n-4 & \text{if all components are defined over } \mathbb{R}, \\ -n-2 & \text{if all but two components are defined over } \mathbb{R}. \end{cases} \tag{5.5}$$

$$\chi(III^*) = -7. \tag{5.6}$$

Proof. We will describe the real points explicitly in all cases, the formulae for χ being immediate from this. For a description of the complex curves corresponding to the Kodaira symbols, the reader may consult [Sil94, §IV.8-IV.9].

For I_1, the complex curve is an irreducible rational curve with a single node. The node must be defined over $\mathbb{R}$ and is the image of two real points in the normalisation in the split case or a pair of complex conjugate points in the non-split case. Thus, since $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to a circle, the real points of such a curve form a "figure 8" in the split case or it is a disjoint union of a circle and a point in the non-split case.

For I_2, the complex curve is a union of two smooth rational curves intersecting transversally in two points. Both components must be defined over $\mathbb{R}$ since at least one is. In the split case the intersection points are defined over $\mathbb{R}$, so the real points are homeomorphic to a union of two circles meeting in two distinct points. In the non-split case, the intersection points are not defined over $\mathbb{R}$ so the real points are homeomorphic to a circle or a disjoint union of two circles.

For II, the complex curve is an irreducible rational curve with a single cusp. The real points must therefore be homeomorphic to a circle.

For III, the complex curve is a union of two smooth rational curves intersecting tangentially in a single point, which must therefore be defined over $\mathbb{R}$. The real points therefore form a "figure 8".

For I_n^* (including $n = 0$), we consider the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on the dual graph of the fibre. The (smooth) component of the identity section is defined over $\mathbb{R}$ and corresponds to an end vertex of the dual graph, and the only possible graph automorphisms fixing this point are the identity and the automorphism that swaps the two opposite end vertices. Thus all components of the fibre except possibly those corresponding to these last two vertices are defined over $\mathbb{R}$. Since all components intersect in a single point, those components defined over $\mathbb{R}$ must intersect over $\mathbb{R}$, so the real picture is homotopy-equivalent either to a bouquet of $n + 5$ circles or to a bouquet of $n + 3$ circles.

For type III^*, we argue similarly: the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on the dual graph of the fibre must fix one end vertex, and the only possible graph automorphism with this property is the identity. Thus all components are defined over $\mathbb{R}$, and the real points are homotopy-equivalent to a bouquet of 8 circles. □

Let $W \subset H^2(E(\mathbb{C}), \mathbb{Q})$ be the subspace spanned by the fundamental classes of all irreducible components of all the (complex) fibres of π as well as the fundamental class of a section C of π (defined over $\mathbb{R}$).

$$W = \mathbb{Q}[C] \oplus \mathbb{Q}[E_{sm}] \oplus \bigoplus_{x \in \mathbb{P}^1(\mathbb{C})} \frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]}, \tag{5.7}$$

where E_x is the scheme theoretic fibre over x, the $E_{x,i}$ are the reduced irreducible components of E_x, E_{sm} is any smooth fibre and $[D]$, for an algebraic 1-cycle D, denotes the fundamental class in $H^2(E(\mathbb{C}), \mathbb{Q})$. The summand corresponding to any irreducible fibre E_x is zero, so the sum is actually finite.

If a complex fibre is not defined over $\mathbb{R}$, complex conjugation maps it to a distinct fibre, so the trace of F_∞ corresponding to the subspace of W spanned by all the irreducible components of all these fibres is 0. Thus, as far as computing the trace goes, we only need consider E_x for $x \in C(\mathbb{R})$. In this case, the corresponding summand $\frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]}$ is preserved by F_∞ so it suffices to consider the trace on each such fibre separately. We now make a list of the possibilities in terms of the Kodaira type and the real structure as in Lemma 5.4.

Lemma 5.5. *For a Kodaira symbol $*$ we denote by* $\mathrm{Tr}(*)$, *the trace of* F_∞ *acting on* $\frac{\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]}$, *where* E_x *is a real fibre of type* $*$.

$$\mathrm{Tr}(I_1) = 0 \tag{5.8}$$

$$\mathrm{Tr}(I_2) = -1 \tag{5.9}$$

$$\mathrm{Tr}(II) = 0; \tag{5.10}$$

$$\mathrm{Tr}(III) = -1; \tag{5.11}$$

$$\mathrm{Tr}(I_n^*) = \begin{cases} -n-4 & \text{if all components are defined over } \mathbb{R}, \\ -n-2 & \text{if all but two components are defined over } \mathbb{R}. \end{cases} \tag{5.12}$$

$$\mathrm{Tr}(III^*) = -7. \tag{5.13}$$

Proof. The formulae follow from the following elementary facts and the description of the possible real structures as in the proof of Lemma 5.3:

- If $E_{x,i}$ is a $\mathbb{C}$-irreducible component of E_x defined over $\mathbb{R}$ then F_∞ acts on $\mathbb{Q}[E_{x,i}]$ by -1 since complex conjugation reverses orientation.
- If $E_{x,i}$ and $E_{x,j}$ are two distinct $\mathbb{C}$-irreducible components of E_x which are conjugate, then the trace of F_∞ on $\mathbb{Q}[E_{x,i}] \oplus \mathbb{Q}[E_{x,j}]$ is 0.
- F_∞ acts on $\mathbb{Q}[E_x]$ by -1.

□

The fact that the numbers associated to I_n^* in Lemmas 5.3 and 5.5 are the same greatly simplifies the computations to follow.

5.1. Let U be the maximal open subset of $C_{\mathbb{C}}$ over which the map $\pi : E_{\mathbb{C}} \to C_{\mathbb{C}}$ is smooth and let $j : U(\mathbb{C}) \to C(\mathbb{C})$ be the inclusion. Let $\mathcal{E}^i$ be the local system on $U(\mathbb{C})$ given by $R^i\pi_*(\mathbb{Q}_{\pi^{-1}(U)})$ and let $\mathcal{F}$ be the constructible sheaf $j_*(\mathcal{E}^1)$. Let $V = H^1(C(\mathbb{C}), \mathcal{F})$. By the decomposition theorem ([BBD82]) and the description of intermediate extension on a smooth curve, we see (noting that the fibres of π are connected) that

$$R\pi_*(\mathbb{Q}[2]) \cong j_*\mathcal{E}^0[2] \oplus j_*\mathcal{E}^1[1] \oplus j_*\mathcal{E}^2[0] \oplus \mathcal{P}_{C\setminus U} \tag{5.14}$$

$$\cong \mathbb{Q}[2] \oplus j_*\mathcal{E}^1[1] \oplus \mathbb{Q}[0](-1) \oplus \mathcal{P}_{C\setminus U}, \tag{5.15}$$

where $\mathcal{P}_{C\setminus U}$ is a sheaf supported on $C \setminus U$: it is clear from the decomposition theorem that $\mathcal{P}_{C\setminus U}$ must have punctual support, and to see that it is indeed a sheaf placed in degree zero we note that it must be Verdier self-dual. Its stalks are easily computed using the proper base-change theorem, and we obtain

$$H^2(E(\mathbb{C}), \mathbb{Q}) = H^2(C(\mathbb{C}), \mathbb{Q}) \oplus H^1(C(\mathbb{C}), j_*\mathcal{E}^1) \oplus H^0(C(\mathbb{C}), \mathbb{Q}(-1)) \oplus \bigoplus_{x \in (C\setminus U)(\mathbb{C})} \frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]},$$

i.e. a direct sum decomposition

$$H^2(E(\mathbb{C}), \mathbb{Q}) \cong V \oplus W.$$

(Properties of the cycle class map imply that the cycle classes of the identity section and the smooth fibre, respectively, account for the terms corresponding to $\mathcal{E}^0$ and $\mathcal{E}^2$.) Furthermore, since U is defined over $\mathbb{R}$, complex conjugation induces an involution on V (which we also denote by F_∞) and the isomorphism above is equivariant for this action.

Since $\mathrm{Tr}(\mathrm{F}_\infty | H^r(E(\mathbb{C}), \mathbb{Q})) = 0$ when r is odd (in fact, for our elliptic surface over $\mathbb{P}^1$ with non-constant j-invariant these odd cohomology groups are zero), it is clear that using Lemma 5.2 we may compute the trace of F_∞ on V if we know $\chi(E(\mathbb{R}))$ and the trace of F_∞ on W, since the traces on both $H^0(E(\mathbb{C}), \mathbb{Q})$ and $H^4(E(\mathbb{C}), \mathbb{Q})$ are equal to 1.

Proposition 5.6. *Let $\pi : E \to C$ be an elliptic fibration over $\mathbb{R}$ with a section, and assume the singular fibres of π, defined over $\mathbb{R}$, are of the following forms:*

- *(1) There is one singular fibre of type I_1 that is not split over $\mathbb{R}$, a singular fibre of type I_2 that is not split over $\mathbb{R}$, a singular fibre of type III_3, and all other singular fibres defined over $\mathbb{R}$ are of type I_0^*.*
- *(2) There is one singular fibre of type I_1 that is not split over $\mathbb{R}$, a singular fibre of type I_1 that is split over $\mathbb{R}$, two singular fibres of type II, and all other singular fibres defined over $\mathbb{R}$ are of type I_0^*.*
- *(3_{O}) There is one singular fibre of type I_2 that is split over $\mathbb{R}$, one singular fibre of type I_2 that is not split over $\mathbb{R}$, two singular fibres of type I_4^*, and all other singular fibres defined over $\mathbb{R}$ are of type I_0^*.*
- *(3_Ω) There are two singular fibres of type I_2 that are split over $\mathbb{R}$, two singular fibres of type I_4^*, and all other singular fibres defined over $\mathbb{R}$ are of type I_0^*.*
- *(4) There is one singular fibre of type I_1 that is not split over $\mathbb{R}$, one singular fibre of type I_2 that is split over $\mathbb{R}$, one singular fibre of type III^*, and all other singular fibres defined over $\mathbb{R}$ are of type I_0^*.*

Then

$$\mathrm{Tr}(\mathrm{F}_\infty|V) = \begin{cases} 2 & \text{in Case (1);} \\ 0 & \text{in Case (2);} \\ 0 & \text{in Case } (3_{\mathrm{O}}); \\ -2 & \text{in Case } (3_\Omega); \\ 0 & \text{in Case (4).} \end{cases}$$

Proof. By Lemma 5.2 and the preceding discussion,

$$\mathrm{Tr}(\mathrm{F}_\infty|V) = \chi(E(\mathbb{R})) - (2 + \mathrm{Tr}(\mathrm{F}_\infty|W)). \tag{5.16}$$

We now compute the RHS in each case.

Using Lemma 5.3 and the list of singular fibres, we see that in Case (1)

$$\chi(E(\mathbb{R})) = 1 + 0 - 1 - 4a_1 - 2a_2,$$

where a_1 (resp. a_2) is the number of fibres of type I_0^* of the first (resp. second) type. On the other hand, using Lemma 5.5 and the list of singular fibres

$$\mathrm{Tr}(\mathrm{F}_\infty|W) = -2 + 0 - 1 - 1 - 4a_1 - 2a_2,$$

where the first -2 corresponds to the sum of the traces on a smooth fibre and the section. Inserting these numbers in (5.16) we get that the LHS is 2 as claimed.

By the same method, in Case (2) we have

$$\chi(E(\mathbb{R})) = 1 - 1 + 0 - 4a_1 - 2a_2,$$

and

$$\mathrm{Tr}(\mathrm{F}_\infty|W) = -2 + 0 + 0 + 0 - 4a_1 - 2a_2,$$

so in this case $\mathrm{Tr}(\mathrm{F}_\infty|V) = 0$ as claimed.

In Case (3_{O}), we again have cancellation of the contribution from I_0^* and I_n^* fibres, and we find

$$\mathrm{Tr}(\mathrm{F}_\infty|V) = -2 + 0 - (2 - 1 - 1 - 1 - 1) = 0.$$

In Case (3_Ω), computing similarly we find $\mathrm{Tr}(\mathrm{F}_\infty|V) = -2$.

In Case (4), we likewise find $\mathrm{Tr}(\mathrm{F}_\infty|V) = 0$. □

Finally, we return to the special case where $C = \mathbb{P}^1$ and note that the cases considered in the previous Proposition do indeed correspond to the cases considered in [Zyw19, §6] and (for Case (3_{O})) [Zyw14, §6.4]. We return to the setting and notation of §4, and we let $w \in W(\mathbb{R})$ be any real point, giving rise (fixing a $\mathbb{C}$-valued geometric point $\bar{w}$ over w) to the stalk $h^*(\mathcal{G})_{\bar{w}} \cong \mathcal{G}_{h(\bar{w})} \cong H^1(\mathbb{P}^1_{\mathbb{C}}, j_* E_{h(w)}[\ell])$ (we omit indicating the $h(w)$-specialization in the notation for this $\mathbb{P}^1$ and j so as not to burden the notation). A minimal proper regular model $\pi\colon E_{h(w)} \to \mathbb{P}^1_{\mathbb{R}}$ of the Weierstrass equation (over $\mathbb{R}(t)$) defines our elliptic surface. The map π is smooth over an open subset $U_{h(w)} \subset \mathbb{P}^1$ that depends on w.

Lemma 5.7. *The Cases (1)-(4) of Proposition 5.6 correspond to the singular fibres, and their configurations over $\mathbb{R}$, of the elliptic surfaces considered in [Zyw19, §6] as follows: Case (1) describes the fibres when $N \equiv 2 \pmod 8$; Case (2) when $N \equiv 4 \pmod 8$; Case (3_{O}) when $N \equiv 6 \pmod 8$, and we are in Case 6_{O}, and Case (3_Ω) when we are in Case 6_Ω; and Case (4) when $N \equiv 0 \pmod 8$. Moreover, assuming ℓ is large enough as in [Zyw19, §2.5], the traces recorded in Proposition 5.6 via analysis of the cohomology groups $H^1(\mathbb{P}^1_{\mathbb{C}}, j_* R^1\pi_*\mathbb{Q}_{\pi^{-1}(U_{h(w)})})$ are the negatives of the traces of $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the $\mathbb{F}_\ell$-vector spaces $H^1(\mathbb{P}^1_{\mathbb{C}}, j_* E_{h(w)}[\ell])$.*

Proof. The claim about the structure of the bad fibres follows almost immediately from the descriptions in [Zyw19, §6]. The one point to note is that Tate's algorithm shows that Zywina's description of when the I_n fibres are split or non-split is also valid over $\mathbb{R}$ (e.g., for $N \equiv 2 \pmod 8$, the I_1 fibre at ∞ is split if and only if -3 is a square in $\mathbb{R}$, hence it is non-split).

For the claim about the traces, note that [Zyw19, Equation (2.1)] shows that there is a short exact sequence[4]

$$0 \to H^1(\mathbb{P}^1_{\mathbb{C}}, j_*T_\ell(E)) \xrightarrow{\cdot \ell} H^1(\mathbb{P}^1_{\mathbb{C}}, j_*T_\ell(E)) \to H^1(\mathbb{P}^1_{\mathbb{C}}, j_*E[\ell]) \to 0,$$

so $H^1(\mathbb{P}^1_{\mathbb{C}}, j_*T_\ell(E))$ is a free $\mathbb{Z}_\ell$-module of rank equal to the $\mathbb{F}_\ell$-dimension of $H^1(\mathbb{P}^1_{\mathbb{C}}, j_*E[\ell])$. Since $\ell \neq 2$, the eigenvalues of $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ are the same on each space, and we can therefore compute $\mathrm{Tr}(c|H^1(\mathbb{P}^1_{\mathbb{C}}, j_*E[\ell]))$ by computing $\mathrm{Tr}(c|H^1(\mathbb{P}^1_{\mathbb{C}}, j_*T_\ell(E) \otimes \mathbb{Q}_\ell))$. This Galois module is isomorphic to $H^1(\mathbb{P}^1_{\mathbb{C}}, j_*R^1\pi_*\mathbb{Q}_{\ell,\pi^{-1}(U_{h(w)})})^\vee(-1)$; dualizing leaves $\mathrm{Tr}(c)$ unchanged, and the -1 Tate twist multiplies $\mathrm{Tr}(c)$ by -1, concluding the proof of the Lemma. □

6. Main theorem

It is now a simple matter to prove our main result:

Theorem 6.1. *Let $N \geq 6$ be an even integer, and let $\ell \gg_N 0$ be a sufficiently large prime. There is a totally real field F and infinitely many non-isomorphic Galois representations*

$$\bar{\rho}\colon \Gamma_F \to \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$$

[4] This uses the assumption on ℓ: exactness of $0 \to j_*T_\ell(E) \to j_*T_\ell(E) \to j_*E[\ell] \to 0$ is equivalent to there being no ℓ-torsion in $(R^1 j_* T_\ell(E))_x = T_\ell(E)/(\gamma_x - 1)T_\ell(E)$, where γ_x is the local monodromy at the point of bad reduction x. This requires $\ell \neq 2, 3$ and that ℓ does not divide $\mathrm{ord}_x(j(E))$ when this valuation is negative, i.e. in the multiplicative reduction case.

such that $\bar{\rho}$ is irreducible as an SO_{N+1}-valued representation, but reducible as a GL_{N+1}-valued representation, and $\bar{\rho}$ admits a geometric lift $\rho\colon \Gamma_F \to \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ with Zariski-dense image.

Proof. Consider one of the specializations $\vartheta_{\ell,w_i}\colon \Gamma_{\mathbb{Q}} \to \mathrm{O}(V_\ell)$ from Proposition 4.1, where V_ℓ is a quadratic space over $\mathbb{F}_\ell$ of rank N and trivial discriminant. We have the following four possibilities for these representations, and we define in each case $\bar{\rho}'_{w_i}\colon \Gamma_{\mathbb{Q}} \to \mathrm{O}_{N+1}(\overline{\mathbb{F}}_\ell) = \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)\times\{\pm 1\}$ (note that $N+1$ is odd) as follows, letting $\delta_{K/\mathbb{Q}}$ be the non-trivial quadratic character of an imaginary quadratic field linearly disjoint from the fixed field of ϑ_{ℓ,w_i}:

$N \pmod 8$	$\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}})$	$\mathrm{Tr}(\vartheta_{\ell,w_i}(c))$	$\bar{\rho}'_{w_i}$
0	$\Omega(V_\ell)$	0	$\vartheta_{\ell,w_i} \oplus 1$
2	$\Omega(V_\ell)$	-2	$\vartheta_{\ell,w_i} \oplus 1$
4	$\Omega(V_\ell)$	0	$\vartheta_{\ell,w_i} \oplus 1$
6, Case 6_Ω	$\Omega(V_\ell)$	2	$(\delta_{K/\mathbb{Q}} \otimes \vartheta_{\ell,w_i}) \oplus 1$
6, Case 6_{O}	$\Omega(V_\ell) \subsetneq \vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}}) \subsetneq \mathrm{O}(V_\ell)$	0	$\vartheta_{\ell,w_i} \oplus 1$

We let $\bar{\rho}_{w_i}$ be the projection to the SO_{N+1}-component; of course, $\bar{\rho}_{w_i} = \bar{\rho}'_{w_i}$ except in Case 6_{O}. Each $\bar{\rho}_{w_i}$ is clearly reducible as $\mathrm{GL}_{N+1}(\overline{\mathbb{F}}_\ell)$-representation, and we claim that even after restriction to $\Gamma_{\mathbb{Q}(i,\zeta_\ell)}$ each $\bar{\rho}_{w_i}$ is irreducible as $\mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$-representation. First note that since $\Omega(V_\ell)/\{\pm 1\}$ is a non-abelian simple group, $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}(i,\zeta_\ell)})$ also equals $\Omega(V_\ell)$. A maximal (proper) parabolic subgroup of SO_{N+1} is the stabilizer of an isotropic subspace $W \subset \overline{\mathbb{F}}_\ell^{N+1}$. Since in all cases the image $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}(i,\zeta_\ell)})$ is $\Omega(V_\ell)$, $\bar{\rho}_{w_i}$ stabilizes exactly two proper subspaces of $\overline{\mathbb{F}}_\ell^{N+1}$, namely V_ℓ itself and the complementary line (the standard representation $\Omega(V_\ell) \to \mathrm{SO}(V_\ell \otimes \overline{\mathbb{F}}_\ell)$ is irreducible for $\ell \gg_N 0$). Clearly neither of these subspaces is isotropic, so in all cases $\bar{\rho}|_{\Gamma_{\mathbb{Q}(i,\zeta_\ell)}}$ is absolutely irreducible. (Note that in Case 6_{O}, $\mathbb{Q}(i)$ is the field denoted $\widetilde{F}$ in [FKP19].)

Each $\bar{\rho}_{w_i}$ is odd by the same calculation as in Proposition 3.1: this is what demands in Case 6_Ω incorporating the twist by $\delta_{K/\mathbb{Q}}$.

Finally, to apply Theorem 1.2, we have to check a local lifting hypothesis on $\bar{\rho}_{w_i}$; here is where we will replace $\mathbb{Q}$ by a suitable totally real field. Namely, we do not know at present that any $\bar{\rho}_{w_i}|_{\Gamma_{\mathbb{Q}_\ell}}$ admits a Hodge-Tate regular de Rham lift to $\mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$, so we circumvent this problem by passing to a finite extension. For each w_i, $\bar{\rho}_{w_i}|_{\Gamma_{\mathbb{Q}_\ell}}$ cuts out a finite extension of $\mathbb{Q}_\ell$, and as w_i varies these extensions have bounded degree, so their composite $L/\mathbb{Q}_\ell$ is still finite. There is a solvable totally real extension $F/\mathbb{Q}$, linearly disjoint from $\mathbb{Q}(i,\zeta_\ell)$, such that for all primes v of F above ℓ, the extension $F_v/\mathbb{Q}_\ell$ is isomorphic to $L/\mathbb{Q}_\ell$. As $\Omega(V_\ell)$ has no proper abelian quotient, it follows easily that such an F is linearly disjoint from $\mathbb{Q}(\bar{\rho}_{w_i})$ for all w_i.[5] The required irreducibility (by linear disjointness) and oddness still hold for $\bar{\rho}_{w_i}|_{\Gamma_F}$, but now for all places $v|\ell$ of F, $\bar{\rho}_{w_i}|_{\Gamma_{F_v}}$ is trivial, and so it is easy to see that $\bar{\rho}_{w_i}|_{\Gamma_{F_v}}$ admits a Hodge-Tate regular crystalline lift $\Gamma_{F_v} \to \mathrm{SO}_{N+1}(\mathbb{Z}_\ell)$, simply by taking a suitable sum of powers of the cyclotomic character. At all places v not above ℓ at which $\bar{\rho}_{w_i}$ is ramified, there exists a lift $\Gamma_{F_v} \to \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ by [Boo18, Theorem 1.1]. We have therefore satisfied all of the hypotheses of Theorem 1.2, and so for all w_i there are Hodge-Tate regular geometric lifts $\rho_{w_i}\colon \Gamma_F \to \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ of $\bar{\rho}_{w_i}$. ∎

[5] A slightly simpler version of this argument would choose the extension F depending on w_i. This would also enable us to avoid invoking Booher's theorem below, by also choosing $F = F(w_i)$ to trivialize $\bar{\rho}_{w_i}$ at all primes of ramification.

Remark 6.2. As in Remark 3.2, perhaps the most interesting cases here are when $N \equiv 2 \pmod 8$ and Case 6_Ω when $N \equiv 6 \pmod 8$, since then we begin with SO_N-valued representations that are not odd. Again, existing lifting techniques cannot lift these SO_N-representations to Hodge-Tate regular geometric SO_N-representations; of course, replacing $E[\ell]$ by $T_\ell(E)$ provides some geometric lift, but it has only three distinct Hodge-Tate weights, with high multiplicities.

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