MATH 7890, UTAH SPRING 2017: INTRODUCTION TO THE GEOMETRIC SATAKE CORRESPONDENCE

Contents

1. Overview (notes by Michael Zhao)

1.1. **Next Few Weeks.** Over the next few weeks, we will cover

1. compact groups,
2. basics of algebraic groups,
3. Flag varieties and Bruhat decomposition,
4. Highest weight theory and Borel-Weil theorem,
5. Tannakian categories.

The next unit will start with the smooth representations of reductive groups over local fields.

1.2. **Course Overview.** Our goal is to get to the geometric Satake equivalence. Let k be an algebraically closed field. Let G be a connected reductive algebraic group over k (e.g. $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, G_2). The geometric Satake equivalence says there is the equivalence of abelian tensor (!) categories

$$\mathrm{Perv}_{\text{“}G(k[[t]])\text{”}}\left(\text{“}G(k((t)))\Big/G(k[[t]])\text{”}\right) \simeq \mathrm{Rep}(G^\vee).$$

The left-hand side is the category of perverse sheaves on Gr_G, the affine Grassmannian of G, and the right-hand the category of finite dimensional algebraic representations of the

Langlands dual group of G (e.g if $G = \mathrm{GL}_n(k)$, $G^\vee = G$, and if $G = \mathrm{Sp}_{2n}(k)$, $G^\vee = \mathrm{SO}_{2n+1}(k)$).

1.2.1. *Part I of the Class.* Understand $\mathrm{Rep}(G^\vee)$.

1.2.2. *Part II of the Class.* The classical Satake isomorphism and smooth representations of reductive groups over local fields. It is the isomorphism

$$\mathcal{C}^\infty_{c,\text{“}G(k[[t]])\text{”}}\left(\text{“}G(k((t)))\Big/G(k[[t]])\text{”}\right) \xrightarrow{\sim} R(G^\vee),$$

obtained by taking the Grothendieck group of both sides (applying K_0) of the geometric Satake equivalence. The left-hand side is called the spherical Hecke algebra of G, which arises from studying the smooth representations of $G(k((t)))$ or $G(\mathbb{Q}_p)$, and is made into an algebra by convolution of functions. The right-hand side is the representation ring of the Langlands dual of G.

1.2.3. *Part III of the Class.* Understand the Perv of the geometric Satake equivalence. This entails understanding the six operations (e.g. f_*, f^*) on derived categories of (constructible) sheaves on algebraic varieties, and the full subcategory of perverse sheaves.

1.2.4. *Part IV of the Class.* This will be distributed throughout the course, but the goal is to understand what Gr_G is.

1.2.5. *Part V of the Class.* The geometric Satake equivalence.

2. Representation and Structure Theory of Compact Lie Groups (notes by Michael Zhao)

Definition 2.1. A **Lie group** is a group object in the category of smooth manifolds, i.e. $m : G \times G \to G,\ \mathrm{inv} : G \to G,\ \mathrm{id} : 1 \to G$, satisfying the expected commutative diagrams.

Example 2.2. The following are Lie groups.

- $\mathrm{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$
- $\mathrm{Sp}_{2n}(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) \mid g \text{ preserves a non-degenerate alternating form}\}$
- More generally, closed subgroups $G \hookrightarrow \mathrm{GL}_n(\mathbb{R})$.
- Unipotent matrices, invertible upper triangular matrices (Borel subgroup).
- $\mathrm{SL}_2(\mathbb{R})$. $\pi_1(\mathrm{SL}_2(\mathbb{R})) \simeq \mathbb{Z}$. This can be established by looking at the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$. The covering space corresponding to $2\mathbb{Z} \subset \pi_1(\mathrm{SL}_2(\mathbb{R}))$ is denoted by $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, and called the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$. It is a Lie group but can't be embedded into $\mathrm{GL}_n(\mathbb{R})$.

The metaplectic group is not relevant for us.

2.1. **Lie Algebras.** The Lie algebra $\mathfrak{g}$ of G is given by the tangent space at the identity. It is a priori a vector space, but the Lie bracket gives it an algebra structure, which we will describe briefly.

There is a map $\Psi : G \to \mathrm{Aut}(G)$ since G acts on G by conjugation. Since $\Psi(g)$ preserves the identity, we can look at the map

$$\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}),$$

given by

$$g \to d\Psi(g)\big|_1.$$

By differentiating at the identity, we obtain the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}).$$

Then we can define the Lie bracket as $[X, Y] := \mathrm{ad}(X)(Y)$.

Example 2.3. For $G = \mathrm{GL}_n(\mathbb{R})$, $\mathfrak{g} = M_n(\mathbb{R})$, and $[X, Y] = XY - YX$.

2.2. **Exponential Map.** Here we will explain the role of Lie algebras, by constructing a map $\exp : \mathfrak{g} \to G$. For $G = \mathrm{GL}_n(k)$, $\exp(X) = \sum_n X^n/n!$.

2.2.1. exp *Facts.* Before giving the construction, note several facts.

1. exp is a local diffeomorphism near 0, by the implicit function theorem.
2. exp is not a homomorphism once G is non-abelian.

2.2.2. exp *Construction.* Take $X \in \mathfrak{g}$. Use left-multiplication to propagate X to a left-invariant vector field ξ_X on G, i.e. $\xi_X(g) = (m_g)_*(X)$.

Construct an integral curve φ_X for this vector field by the following procedure. Solve the differential equation $\psi(0) = 1, \psi'_X(t) = \xi_X(\psi_X(t))$. This gives a function $\psi_X : (-\varepsilon, \varepsilon) \to G$.

Note that ψ_X is a homomorphism. Fix s and let t vary. Let $\alpha(t) = \psi_X(s)\psi_X(t)$ and let $\beta(t) = \psi_X(s + t)$. Then $\alpha(0) = \beta(0)$, and $\alpha'(t) = \xi_X(\alpha(t))$ and $\beta'(t) = \xi_X(\beta(t))$ by left-invariance of ξ_X. Thus, $\alpha = \beta$.

This can be extended to a homomorphism $\varphi_X : \mathbb{R} \to G$, by setting $\varphi_X(t) = \phi_X(t/N)^N$, where N is an integer such that $t/N < \varepsilon/2$. This does not depend on the choice of N; if M was another such integer, we would have

$$\psi_X\left(\frac{t}{MN}\right)^N = \psi_X\left(\frac{t}{M}\right), \psi_X\left(\frac{t}{MN}\right)^M = \psi_X\left(\frac{t}{N}\right).$$

Hence

$$\psi_X\left(\frac{t}{M}\right)^M = \phi_X\left(\frac{t}{MN}\right)^{MN} = \phi_X\left(\frac{t}{N}\right)^N.$$

Using this definition and the fact that φ_X is a homomorphism in $(-\varepsilon, \varepsilon)$, it can be shown φ_X is a homomorphism $\mathbb{R} \to G$.

Definition 2.4. $\exp_G(X) := \varphi_X(1)$.

Remark 2.5. For G a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, $\exp_G : \mathfrak{g} \to G$ is surjective. The basic idea of the proof is to use the Killing form on a Lie algebra. Use multiplication to transport it to a Riemannian metric. The integral curves are geodesics, and any two points are connected by a geodesic.

Exercise 2.6. Find a G where $\exp_G$ is not surjective.

We do not need to look far for such an example. If $G = \mathrm{GL}_2(\mathbb{R})$, and λ_1, λ_2 are two *distinct* negative real numbers, consider

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

If there is an $A \in M_n(\mathbb{R})$ with $\exp(A) = B$, and A has eigenvalues α_1, α_2, then $e^{\alpha_i} = \lambda_i$. Since λ_i are negative, α_1 and α_2 are complex conjugates. Thus λ_1, λ_2 have the same absolute value, a contradiction.

2.3. Representation of Compact Lie Groups.

Definition 2.7. A representation of G a compact Lie group is a continuous (hence smooth) homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

The motto here is to study representations of G by restricting to a maximal tori.

Definition 2.8. A (compact) torus is a (compact) connected abelian Lie group.

Example 2.9. S^1 is a torus, and all compact examples are isomorphic to $(S^1)^n$.

Definition 2.10. A **character** (or **weight**) of T is a continuous homomorphism $T \to S^1$. Write $X^\bullet(T)$ for the abelian group of characters of T.

Exercise 2.11. Prove the following lemma.

Lemma 2.12. *If $T \simeq \mathbb{R}^n/\mathbb{Z}^n$, then all characters of T have the form*

$$(x_1, ..., x_n) \to \prod_{r=1}^{n} e^{2\pi i a_r x_r},$$

where all $a_r \in \mathbb{Z}$.

Proof. We just need to know what homomorphisms there are from $\mathbb{R}/\mathbb{Z}$ to S^1 (viewed as the unit circle in $\mathbb{C}$). This is the same as knowing the homomorphisms from $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, and any such φ lifts to a homomorphism ϕ from $\mathbb{R}$ to $\mathbb{R}$, by using twice the fact that a universal covering space covers any connected cover.

This construction shows $\phi(1) \equiv 0 \pmod 1$, so for some integer n, $\phi(1) = n$, $\phi(a) = na$, $b\phi(a/b) = \phi(a) = na$. By continuity, $\phi(x) = nx$ for any x. Then $\varphi(x) = \phi(x) = nx \pmod 1$. As $x \to e^{2\pi i x}$ is a homeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$, any homomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ is given by $x \to e^{2\pi i n x}$, for some n. □

Lemma 2.13. *Any compact torus has a (dense set of) topological generators, i.e. there is $t \in T$ with $T = \overline{\langle t \rangle}$.*

Proof. Assume $T \simeq \mathbb{R}^n/\mathbb{Z}^n$, and take $T \ni t = (t_1, ..., t_n)$. Let $H := \overline{\langle t \rangle}$. Then use the

Lemma 2.14. *$H = T$ if and only if $1, t_1, ..., t_n$ are $\mathbb{Q}$-linearly independent.*

If $H \neq T$, then T/H, a compact torus, has a non-trivial character χ, and

$$\chi(x_1, ..., x_n) = \prod_r e^{2\pi i a_r x_r},$$

for some integers a_r not all zero. Then $\chi\big|_H = 1$ if and only if $\chi(t) = 1$ if and only if $\sum a_r t_r \in \mathbb{Z}$, which implies $1, t_1, ..., t_n$ are $\mathbb{Q}$-linearly dependent. □

Theorem 2.15. *Let G be a compact connected Lie group. Fix a maximal torus $T \subset G$.*

(1) Every $g \in G$ is conjugate to an element of T.

(2) Any other maximal torus $T' \subset G$ is G-conjugate to T.

(3) $C_G(T) = T$.

Exercise 2.16. Find a counter-example for non-compact groups. We already know (1) for $U(n)$. Why?

An example of a non-compact Lie group without any tori is $\mathbb{R}^n$. A maximal torus of $U(n)$ is the subgroup of all diagonal matrices. The spectral theorem tells us that any unitary matrix can be diagonalized by another unitary matrix gives (1) for $U(n)$.

Proof. Idea of the proof:

(1) follows from surjectivity of $\exp_G$.

(2) is immediate from (1).

(3) uses surjectivity of $\exp_G$.

□

2.4. **Onto Representation Theory.** Let G be a compact connected group as before.

Proposition 2.17. *Any finite dimensional representation of G is isomorphic to a direct sum of irreducible representations.*

Proof. The proof is the same as in the finite case. Let $V \in \text{Rep}(G)$. Fix a hermitian inner product $(\ ,\ )$ on V. Produce a G-invariant inner product $\langle\ ,\ \rangle$ by using the Haar measure that exists on G:

$$\langle v, w\rangle = \int_G (g \cdot v, g \cdot w)\ dg.$$

If W is a G-subrepresentation of V, then V decomposes as $V = W \oplus W^{\perp}$ ($W^{\perp}$ is G-stable since $\langle\ ,\ \rangle$ is G-invariant. □

The following is due to Schur.

Corollary 2.18. *If V is irreducible, then* $\text{End}_{\mathbb{C}[G]}(V) = \mathbb{C}$.

Proof. Let $T \in \mathrm{End}_{\mathbb{C}[G]}(V)$. Since V is finite-dimensional, T has an eigenvalue λ. Then if T is non-constant, $T - \lambda \cdot 1$ acts on V and satisfies $0 \neq \ker(T - \lambda \cdot 1) \subset V$. Since V is irreducible, $\ker(T - \lambda \cdot 1) = V$, so $T = \lambda$. □

Definition 2.19. Let $V \in \mathrm{Rep}(G)$. The **character** χ_V of V is a homomorphism $G \to \mathbb{C}$ given by $g \to \mathrm{tr}(g\big|_V)$.

Some basic facts about characters.

1. χ_V only depends on the isomorphism class of V.
2. χ_V only depends on the conjugacy class of its argument.
3. $\chi_{V \oplus W} = \chi_V + \chi_W$.
4. $\chi_{V \otimes W} = \chi_V \chi_W$.
5. $\chi_{V^*} = \overline{\chi_V}$, where $V = \mathrm{Hom}(V, \mathbb{C})$.

The first four properties give us a homomorphism $K_0(\mathrm{Rep}(G)) = R(G) \to C(G)$. Viewing χ_V in $L^2(G)$ with an inner product

$$\langle f_1, f_2 \rangle = \int_G \overline{f_1(g)} f_2(g)\ dg,$$

we have the following basic calculation.

Proposition 2.20. *(1) For $V, W \in \mathrm{Rep}(G)$, $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$.*

(2) Characters of irreducible representations are orthnormal (Schur).

(3) $\chi_V = \chi_W$ if and only if $V \simeq W$.

Proof. Prior to describing a sketch of the proof, we will need the following two lemmas (**left as exercises**).

Lemma 2.21. *For all $V \in \mathrm{Rep}(G)$, $\dim V^G = \displaystyle\int_G \chi_V(g)\ dg$.*

Proof. Let ρ be the map $G \to \mathrm{GL}_n(V)$ associated to V. Recall that there exists a unique linear transformation A such that for any $f \in V^*$,

$$\int_G f(\rho(g)v)\ dg = f(Av),$$

and that the action of $\int_G \rho(g)\ dg$ is defined to be the action of A. We will show that $\mathrm{im}\, A = V^G$ and that A is a projection operator. Since the trace of a projection operator and its rank coincide, this will prove the lemma.

Due to left-invariance of Haar measure and this fact, we can prove

$$\rho(h) \int_G \rho(g)\ dg = \int_G \rho(h)\rho(g)\ dg = \int_G \rho(hg)\ dg = \int_G \rho(g)\ dg,$$

when viewed as operators on V. For instance, the first equality follows from the fact that for any linear form f, $f \circ \rho(h)$ is also a linear form. This lets us prove that $\rho(h)Av = Av$ for $A = \int_G \rho(g)\ dg$. Hence $\mathrm{im}\, A \subset V^G$. Now suppose we have v such that $\rho(g) \cdot v = v$ for every

$g \in G$. Then for any $f \in V^*$, $f(\rho(g) \cdot v) = f(v)$. Taking the integral over G, $f(Av) = f(v)$. On V^G, then, A acts as the identity. From this, surjectivity and idempotency follow. □

Lemma 2.22. *For all V, the natural map*

$$\bigoplus_W W \otimes \mathrm{Hom}_G(W, V) \to V,$$

where the sum is taken over all irreducible representations W, given by

$$w \otimes \varphi \to \varphi(w),$$

is an isomorphism of G-representations (the action of G on $\mathrm{Hom}_G(W, V)$ is trivial).

Proof. Since W is irreducible, any element of $\mathrm{Hom}_G(W, V)$ is an embedding. Since $\mathrm{Rep}(G)$ is semi-simple, the dimension of $\mathrm{Hom}_G(W, V)$ counts the multiplicity of W in V. Then it suffices to show that for every irreducible W,

$$W \otimes \mathrm{Hom}_G(W, V) \simeq W^d,$$

where $d := \dim \mathrm{Hom}_G(W, V)$. Taking a basis of $\mathrm{Hom}_G(W, V)$ gives an identification with $\mathbb{C}^d$, and provides the isomorphism. □

Then we can calculate

$$\begin{aligned} \dim \mathrm{Hom}_G(V, W) &= \dim(V^* \otimes W)^G \\ &\overset{2.21}{=} \int_G \chi_{V^* \otimes W}(g)\, dg \\ &= \int_G \overline{\chi_V(g)} \cdot \chi_W(g)\, dg \\ &= \langle \chi_V, \chi_W \rangle, \end{aligned}$$

which proves (1). Part (2) is proven by using (1) and Schur's Lemma. For part (3), use lemma 2.22, with part (1). If $\chi_{V_1} = \chi_{V_2}$, then for all irreducible W, $\dim \mathrm{Hom}_G(W, V_i)$ is the same for $i = 1, 2$. Then apply the lemma. □

Proposition 2.23. *Let $T < G$ be a maximal torus. Then $V, W \in \mathrm{Rep}(G)$ are isomorphic if and only if $V\big|_T \simeq W\big|_T$.*

Proof. If $V\big|_T \simeq W\big|_T$, then $\chi_V\big|_T = \chi_W\big|_T$. Since characters are invariant under conjugation, and the union of the conjugates of T is G, $\chi_V = \chi_W$ on all of G, hence $V \simeq W$. □

This justifies the motto we had earlier, that to study representations of G we can restrict our attention to a maximal torus.

2.5. **The Representation Ring.**

We can rephrase proposition 2.23; recall that

$$R(G) = K_0(\mathrm{Rep}(G)).$$

Proposition 2.24. *The restriction map* res : $R(G) \to R(T)$ *given by*

$$R(G) \ni V \to V\big|_T = \bigoplus_{\lambda \in X^\bullet(T)} V_\lambda \; (= \{v \in V | t \cdot v = \lambda(t)v, \; \forall t \in T\})$$

is injective.

Thus to classify representations of G, we can just identify the image. Moreover, there is a symmetry present in the image of this map.

2.5.1. *Weyl Group.* That symmetry is related to the Weyl group.

Definition 2.25. Let $N_G(T)$ be the normalizer of T in G. The **Weyl group** $W(G,T) := N_G(T)/T$. Often we will just write W, as this group doesn't depend on the choice of T, up to isomorphism.

Remark 2.26. Note that W is a finite group. Note that $N_G(T)$ is closed, hence a compact subgroup. Also, $\exp_T$ induces an isomorphism of T with $\mathrm{Lie}(T)$ modulo some lattice Λ. W acts faithfully on this lattice, i.e. we have an embedding

$$W \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(\Lambda),$$

so W is both compact and discrete, hence finite.

Example 2.27. If $G = \mathrm{SU}(n)$, then $W \simeq S_n$ (think permutation matrices).

The action of W on T induces an action of W on $X^\bullet(T)$, given by

$${}^\omega\chi(t) = \chi(\omega^{-1} \cdot t).$$

Hence $\mathrm{res}(R(G))$ lands in $\mathbb{Z}[X^\bullet(T)]^W \subset \mathbb{Z}[X^\bullet(T)]$, because characters are conjugacy invariant.

Theorem (of the Highest Weight)**.**

$$\mathrm{res} : R(G) \simeq \mathbb{Z}[X^\bullet(T)]^W$$

Later, we'll prove this in the setting of reductive algebraic groups.

Example 2.28. Let $G = SU(2) = \{g \in \mathrm{SL}_2(\mathbb{C}) \mid \; {}^t\overline{g} \cdot g = 1\}$ and take

$$T := \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mid t \in S^1 \right\}.$$

To prove surjectivity of res (onto $\mathbb{Z}[X^\bullet(T)]^W$), we just need to write down enough representations of G. Let V be the standard two-dimensional representation. Then for all $k \geq 0$, $\mathrm{Sym}^k(V)$ is an irreducible representation of G of dimension $k+1$.

Let e_1, e_2 be a basis of V. Then a basis of $\mathrm{Sym}^k(V)$ is given by $e_1^k, e_1^{k-1}e_2, ..., e_2^k$, and the action on these basis elements is $t^k, t^{k-1}t^{-1} = t^{k-2}, ..., t^{-k}$. Let χ_n be the irreducible character

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \to t^n$$

of T. Then

$$\mathrm{Sym}^k(V)\big|_T = \chi_k \oplus \chi_{k-2} \cdots \oplus \chi_{-k}.$$

Note that if $\omega \in W \setminus \{1\}$, then ${}^{\omega}\chi_k = \chi_{-k}$, because

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}.$$

These $\mathrm{Sym}^k(V)\big|_T$ for all $k \geq 0$ visibly span $\mathbb{Z}[X^\bullet(T)]^W$.

3. Algebraic Groups (Adam Brown)

Aim: Classify algebraic representations of a connected reductive algebraic group in characteristic 0.

Definition 3.1. Let k be a field. An algebraic group G over k is a group object in the category of algebraic varieties over k. Recall that a group object has the following maps

$$m : G \times G \to G, \ \mathrm{inv} : G \to G, \ \mathrm{id} : \mathrm{Spec}(k) \to G$$

satisfying the expected commutative diagrams.

3.1. Functorial Perspective.

We can think of G as a group valued functor

$$\begin{aligned} k\text{-alg} &\to \mathrm{Grps} \\ R &\mapsto G(R) := \mathrm{Hom}(\mathrm{Spec}R, G) \end{aligned}$$

This is a sufficient perspective by Yoneda's lemma.

Lemma 3.2. *If $\mathcal{C}$ is any (locally small) category then*

$$\begin{aligned} \mathcal{C} &\to \mathit{Fun}(\mathcal{C}^{op}, \mathit{Set}) \\ X &\mapsto h_X : Y \mapsto \mathit{hom}_{\mathcal{C}}(Y, X) \end{aligned}$$

is a fully faithful embedding.

Note: A category is locally small if each class of morphisms $\hom(X, Y)$ is a set, rather than a proper class.

Example 3.3. We can view $G = \mathrm{SL}_n$ as the functor

$$R \mapsto \mathrm{SL}_n(R) = R \text{ linear automorphisms of } R^n \text{ with determinant equal to } 1$$

. The coordinate ring for G is

$$k[G] = \mathrm{Spec}\ k[X_{ij}]/(\det X_{ij} - 1)$$

Example 3.4. The coordinate ring of $G = \mathrm{GL}_n$ is

$$\mathrm{Spec}\ k[X_{ij}, d]/(d \cdot \det X_{ij} - 1)$$

Example 3.5. Let E/k be an elliptic curve. (Note that this example is not affine)

Example 3.6. Upper triangular matrices with 1's on the diagonal:

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Upper triangular matrices:

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}.$$

Proposition 3.7. *If G is an affine algebraic group over k then G is a closed subgroup of some GL_n.*

3.2. Representations of Algebraic Groups.

Definition 3.8. Let G be an algebraic group over k, V be a vector space over k (possibly infinite dimensional). A representation of G on V is a morphism of group valued functors $G \to GL_V$. Precisely, for all $\phi : R \to S$ we have the following commutative diagram

$$\begin{tikzcd} G(R) \arrow[r, "f_R"] \arrow[d] & \mathrm{GL}_V(R) \arrow[d] \\ G(S) \arrow[r, "f_S"] & \mathrm{GL}_V(S) \end{tikzcd}$$

Note that if $\dim V < \infty$ then GL_V is an algebraic group.

Example 3.9. Let us consider an infinite dimensional representation. Let $k[G]$ be the coordinate ring of G. Then $k[G]$ is a $G \times G$ representation where

$$(g_1, g_2) \cdot f(x) = f(g_1^{-1} x g_2)$$

where $g_1, g_2, x \in G$ and $f \in k[G]$. To frame this group action in our functorial definition of a representation, we notice that

$$f \in \mathrm{Hom}_{k-\mathrm{alg}}(k[t], k[G]) = \mathrm{Hom}(G, \mathbb{A}^1)$$

The groups of primary interest are the following:

Lemma 3.10. *Let G be an algebraic group over k. The following are equivalent:*

1. *Finite dimensional representations of G are semisimple*

2. *Every representation of G is semisimple*

3. *$k[G]$ is semisimple under either the right or left regular representation*

4. *There exists a faithful finite dimensional representation $G \to GL_V$ so that for all $n \geq 1$, $V^{\otimes n}$ is semisimple.*

Such a group is called linearly reductive.

Some ideas of proof:
$1 \to 2$: Every representation is a filtered union of finite dimensional subrepresentations

$2 \to 3$: This is clear.
$3 \to 1$:

Lemma 3.11. *Every finite dimensional G representation W is contained in $k[G]^{\oplus N}$ for some $N \in \mathbb{Z}$.*

Proof. Let W be a finite dimensional G representation. Let $\lambda_1, \cdots, \lambda_n$ be a basis of W^*. Define

$$\begin{aligned} W &\to k[G]^n \\ w &\mapsto (\langle \lambda_i, g_i w\rangle)_{i=1,\cdots,n} \end{aligned}$$

This map is G equivariant and injective. □

$4 \to 1$: This follows from the following proposition (given without proof).

Proposition 3.12. *Let $G \hookrightarrow GL_V$ be a faithful finite dimensional representation. Then every finite dimensional representation of G is isomorphic to a subquotient of some $\bigoplus(V \oplus V^*)^{\otimes m}$.*

Back to general G:
If W is any irreducible G representation, consider

$$\begin{aligned} \iota_W : W^* \otimes W &\to k[G] \\ \lambda \otimes w &\mapsto g \mapsto \lambda(gw) \end{aligned}$$

Lemma 3.13. *The image of ι_W is the W isotypic component in $k[G]$ under the right regular action.*

Proof. Let $j : W \to k[G]$ be any G equivariant map. Claim: $j(W) \subset \iota_W(W^* \otimes W)$. Let $\lambda \in W^*$ be defined by $\lambda(w) = j(w)(1)$. Then $j(w)(g) = j(gw)(1) = \lambda(gw) = \iota_W(\lambda \otimes w)(g)$. □

Corollary 3.14. *If G is linearly reductive, then*

$$k[G] \cong \bigoplus_{W_i \text{ fd irr rep}} W_i^* \otimes W_i$$

where the sum is countable (finite dimensional irreducible representations form a countable set).

Proof. G linearly reductive implies that $k[G]$ is semisimple. □

Corollary 3.15. *If G and H are linearly reductive, then $G \times H$ is linearly reductive.*

Proof.

$$k[G\times H] \cong k[G]\otimes k[H] \cong (\bigoplus_{\text{fd irr of } G} W_i^*\otimes W_i)\otimes(\bigoplus_{\text{fd irr of } H} V_j^*\otimes V_j) \cong \bigoplus (W_i^*\otimes V_j^*)\otimes(W_i\otimes V_j)$$

Since irreducible $G \times H$ representations are irreducible G representations tensored with irreducible H representations, we have that $k[G \times H]$ is a semisimple $G \times H$ representation. So $G \times H$ is linearly reductive. □

3.3. **Maximal Tori.** We will begin this section with a short review and some supplemental material from previous lectures:
Fact: W is a finite group.

Proof. $N_G(T)$ is closed, so $N_G(T)$ is compact.

$$\exp_T : \mathfrak{t}/\text{lattice } \Lambda \to T$$

is an isomorphism and W acts faithfully on Λ. So

$$W \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}\Lambda$$

So W is compact and discrete, therefore finite. Example: If $G = SU(n)$ then $W = S_n$. □

Definition 3.16. a k group scheme is a group object in the category of k-schemes. We say it is algebraic if it is finite type over k.

Our groups of interest will be smooth affine algebraic groups (which we will further refer to as groups over k). Fact: If char$k = 0$, then an affine algebraic group is smooth.

Definition 3.17. A linearly reductive group G over k is a smooth affine algebraic group such that all representations are semisimple.

Definition 3.18. A group T over k is a torus if

$$T_{\bar{k}} \cong \mathbb{G}_m^n$$

for some $n \in \mathbb{Z}$, where $\mathbb{G}_m = GL_1$. We say T is split if $T \cong \mathbb{G}_m^n$ over k.

Example 3.19. Let $k = \mathbb{R}$. $\mathbb{R}^\times$ is a split torus. $S^1 = \mathbb{S}^1(\mathbb{R})$ is a non split torus. $\mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ is a non split torus.

$$\mathbb{S} : \mathbb{R}\text{-alg} \to \text{Grp}$$
$$R \mapsto \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} R)$$

Exercise: Show $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$.

3.4. **Representations of Tori.**

Lemma 3.20. *(1) Tori are linearly reductive.*

(2) The simple representations of split torus $\mathbb{G}_m^n$ are given by characters

$$\chi_{m_1,\cdots,m_n} : (t_1, \cdots, t_n) \mapsto \prod t_i^{m_i}$$

Proof. Assume T is split, so $T = \mathbb{G}_m^n$. Then

$$k[T] = k[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] = \bigoplus_{\mathbb{Z}^n} kX_1^{m_1} \cdots X_n^{m_n}$$

which has summands that are stable under the right regular representation, and have character $\chi_{m_1,\cdots,m_n}$. Since $k[T]$ is semisimple, T is linearly reductive. Since all simples appear in $k[T]$, $\chi_{m_1,\cdots,m_n}$ exhaust characters of simple representations. □

Exercise: In characteristic 0, SL_n is linearly reductive. In characteristic $p > 0$, SL_n is not linearly reductive.
There is a different notion of a reductive group:

Definition 3.21. A group G over k is

(1) semisimple if the radical $R(G)$ of $G_{\bar{k}}$ is trivial

(2) reductive if the unipotent radical $R_u(G_{\bar{k}})$ is trivial

where $R(G)$ is the maximal connected solvable smooth normal subgroup, and $R_u(G)$ is the maximal connected unipotent smooth normal subgroup.

Example 3.22. (1)

$$\begin{bmatrix} * & \cdots & & * \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & * \end{bmatrix} \subset GL_n \quad \text{is a solvable group}$$

(2)

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \text{is a unipotent group}$$

(3) (Exercise)

$$R(GL_n) = \begin{bmatrix} z & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z \end{bmatrix} = \text{the center of } GL_n$$

(4) (Exercise)

$$R_u(GL_n) = \{1\} \Rightarrow \; GL_n \text{ is reductive not semisimple}$$

(5) $R(SL_n) = \{1\}$ because $(\mu_n)^\circ = \{1\}$ if char k does not divide n, and is not smooth if char k divides n.

$$Z(SL_n) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \quad \text{such that } z \in \mu_n$$

Fact: in characteristic 0, linearly reductive is equivalent to reductive.
When studying representation theory we will focus on connected linearly reductive groups over $k = \bar{k}$ with characteristic 0.

3.5. Structure Theory of Reductive Groups over $k = \bar{k}$.

Definition 3.23. The Lie algebra of G is defined as

$$\begin{aligned} \mathfrak{g} &= \text{tangent space at } 1 \in G \\ &= \ker(G(k[\epsilon]) \to G(k)) \end{aligned}$$

where $\epsilon^2 = 0$.

Example 3.24. (1) If $G = GL_n$ then $\mathfrak{g} = \{1 + \epsilon A : A \in M_n(k)\}$

(2) If $G = SL_n$ then $\mathfrak{g} = \{1 + \epsilon A : \det(1 + \epsilon A) = 1\} = \{A \in M_n(k) : \mathrm{trace}(A) = 0\}$

(3) If $G = Sp_{2n} = \{g \in GL_{2n} : g^t J g = J\}$ where J is an alternating pairing, then $\mathfrak{g} = \{1 + \epsilon A \in GL_{2n}(k[\epsilon]) : (1 + \epsilon A)^t J (1 + \epsilon A) = J \quad (\text{equivalently } A^t J + JA = 0)\}$

G acts on $\mathfrak{g}$ by conjugation:

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$
$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$
$$\mathrm{ad}(x)(y) := [x, y]$$

Now we will decompose $\mathfrak{g}$ under Ad_T. Take G to be a connected reductive group with maximal (split) torus T. We know T is linearly reductive so there is a semisimple decomposition of any representation of T.

$$V \cong \bigoplus_{\lambda \in X^\bullet(T)} V_\lambda$$

For $V = \mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha$$

where $\Phi(G, T) = \{0 \neq \lambda \in X^\bullet(T) : \mathfrak{g}_\lambda \neq 0\}$ denotes the set of roots of G with respect to T.

Example 3.25. Let $GL_3 \supset T =$ diagonal matrices

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\alpha \in \{e_1 - e_2, e_2 - e_3, e_1 - e_3, -(e_1 - e_2), \cdots\}$.

$$X^\bullet(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$$

where

$$e_i \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} = t_i$$

As in GL_3, it is always the case that

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} : \mathrm{Ad}(t)(X) = X \text{ for all } t \in T\} = \mathfrak{t}$$

because the centralizer in G of T is T.

Key facts about maximal tori that generalize from the compact setting:
Let G be a connected reductive algebraic group over $k = \bar{k}$, with T a maximal torus.

(1) $C_G(T) = T$, $W := N(T)/C(T)$

(2) All maximal tori are conjugate (but not all g are conjugate to $t \in T$. Example: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL_2$)

(3) $\bigcup_{g\in G(k)} gTg^{-1}$ is Zariski dense in G.
Note: (2) is not true for $k \neq \bar{k}$. Example: Consider $G = SL_2(\mathbb{R})$

$$T_1 = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} \subset SL_2(\mathbb{R}) \quad T_2 = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \subset SL_2(\mathbb{R})$$

T_1 and T_2 are two non-conjugate maximal tori.

As before we have a map:

$$R(G) \hookrightarrow R(T)^W$$

Goal: We want to show that this is an isomorphism. First we need more structure theory (this will give us a definition of the Langlands dual group $G^\vee$). Immediate aim: classification of connected reductive algebraic groups over $k = \bar{k}$ in terms of linear algebraic data.

Definition 3.26. A Borel subgroup in G is a maximal connected solvable subgroup.

Fact: All Borel subgroups in G are conjugate (if $k = \bar{k}$).
A choice of Borel $T \subset B \subset G$ gives us a decomposition of roots $\Phi = \Phi(G,T)$ into "positive" and "negative" roots $\Phi^+ = \{\alpha \in \Phi : \mathfrak{g}_\alpha \subset \mathfrak{b}\}$ and $\Phi^- = -\Phi^+$.

Example 3.27.

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \subset GL_3$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\Phi} \mathfrak{g}_\alpha \quad \Phi^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$$

Fact: There is a set $\Delta = \{\alpha_1, \cdots, \alpha_r\} = \Delta(G,B,T) \subset \Phi^+(G,B,T)$ of simple roots such that any positive root $\alpha \in \Phi^+$ is uniquely expressible as $\alpha = \sum c_i\alpha_i$ with $c_i \in \mathbb{Z}_{\geq 0}$.
Our classification theorem for connected reductive groups over $k = \bar{k}$ rests on the following refinement of root systems.

Definition 3.28. A root datum is a 4-tuple $(X, \Phi, X^\vee, \Phi^\vee)$ where

(1) X and $X^\vee$ are finite free $\mathbb{Z}$-modules with perfect duality

$$\langle , \rangle : X \times X^\vee \to \mathbb{Z}$$

(2) $\Phi \subset X$, $\Phi^\vee \subset X^\vee$ are finite subsets with fixed bijection $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$

(3) $s_\alpha(\Phi) = \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$

Where s_α ($s_{\alpha^\vee}$) is a linear endomorphism of X ($X^\vee$) defined by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$
$$s_{\alpha^\vee}(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle \alpha^\vee$$

Structure Theorem:

Theorem. *Connected reductive algebraic groups (up to isomorphism) over $k = \bar{k}$ are in bijection with root data (up to isomorphism).*

We now show how to associate a root datum with a connected reductive algebraic group. Let G be a connected reductive algebraic group with maximal torus T.

(a) $X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m)$

(b) $\Phi = \Phi(G,T) = \{\alpha \in X^\bullet(T) \setminus 0 : \mathfrak{g}_\alpha \neq 0\}$

(c) $X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T)$

(d) $\Phi^\vee = \Phi^\vee(G,T)$ finite subset of $X_\bullet(T)$ will be defined later

For (a)-(d) to be a root datum, we need

(1) $\langle , \rangle : X^\bullet \times X_\bullet \to \mathbb{Z}$

(2) $\Phi \leftrightarrow \Phi^\vee$, $\alpha \leftrightarrow \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$

(3) $s_\alpha \circlearrowright X^\bullet$ and $s_{\alpha^\vee} \circlearrowright X_\bullet$ preserve Φ and $\Phi^\vee$.

Source of (1):

$$\begin{aligned} X^\bullet \times X_\bullet &\to \mathbb{Z} \\ \chi, \lambda &\mapsto \chi \circ \lambda \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z} \end{aligned}$$

Examples: SL_2, PGL_2. First, we'll typically use the following notation for (co)characters of the diagonal torus:

$$\begin{aligned} e_i &: \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} \mapsto t_i \\ e_i^* &: t \mapsto \begin{bmatrix} 1 & & \\ & t_i & \\ & & 1 \end{bmatrix} \text{ in the i,i}^{\text{th}} \text{ spot} \end{aligned}$$

(1)

$$\begin{aligned} SL_2 \supset \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} &= T \\ X^\bullet &= \frac{\mathbb{Z}e_1 \oplus \mathbb{Z}e_2}{\mathbb{Z}(e_1 + e_2)} \\ \Phi &= \{e_1 - e_2, e_2 - e_1\} \\ X_\bullet &= \{a_1 e_1^* + a_2 e_2^* : a_1 + a_2 = 0\} \\ \Phi^\vee &= \{\pm(e_1^* - e_2^*)\} \end{aligned}$$

(2)

$$\begin{aligned} PGL_2 \supset \begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix} / \mathbb{G}_m &= T^\vee \\ X^\bullet(T^\vee) &= \{a_1e_1 + a_2e_2 | a_1 + a_2 = 0\} \\ \Phi &= \{\pm(e_1 - e_2)\} \\ X_\bullet(T^\vee) &= \frac{\mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*}{\mathbb{Z}(e_1^* + e_2^*} \\ \Phi^\vee &= \{\pm(e_1^* - e_2^*)\} \end{aligned}$$

Note: the root data of SL_2 and PGL_2 are not isomorphic. But the root datum of SL_2(respectively PGL_2) is obtained by taking the root datum $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$ of PGL_2 (respectively SL_2) and froming the dual root datum $(X_\bullet, \Phi^\vee, X^\bullet, \Phi)$.

Definition 3.29. If G is a connected reductive group with root datum $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$, then $G^\vee$ (Langlands dual group of G) is the connected reductive group with root datum $(X_\bullet, \Phi^\vee, X^\bullet, \Phi)$. Note that we are only able to define $G^\vee$ in this way because of the previous structure theorem.

To complete the description of the root datum associated to (G, T) we need to define coroots $\Phi^\vee \subset X_\bullet(T)$. For $\alpha \in \Phi$, $\alpha^\vee \in \Phi^\vee$ is defined as follows:
Basic structure theory of semisimple Lie algebras: For all $\alpha \in \Phi$ there exists $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ identified with $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This lifts to a group homomorphism

$$\Psi_\alpha : SL_2 \to G$$

Then $\alpha^\vee$ is by definition the composition

$$\mathbb{G}_m \to SL_2 \to G$$

where

$$t \mapsto \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \mapsto \Psi_\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

Summary: To G and a choice T of maximal torus we associate $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$.

Proposition 3.30. *This is a root datum.*

Theorem. *(1) Two reductive groups are isomorphic if they have isomorphic root data*

(2) Given any root datum there exists a reductive group G over $k = \bar{k}$ with given root datum (up to isomorphism)

Exercise: Identify Langlads dual group of GL_n and Sp_{2n}. (Hint: $\mathrm{GL}_n \cong GL_n^\vee$ and $Sp_{2n}^\vee \cong SO_{2n+1}$).
Statement of Theorem of highest weight: Let G be a connected reductive group over $k = \bar{k}$ with characteristic zero. Fix $B \supset T$, and the corresponding $\Phi = \Phi^+ \sqcup \Phi^-$.

Definition 3.31. Given B, T, the dominant weights $X^\bullet(T)_+ = \{\lambda \in X^\bullet(T) | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$

Define a partial order on $X^\bullet(T)$ by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a non negative integer linear combination of $\alpha \in \Phi^+$.

Example 3.32. SL_2 and weights e_1, and $e_1 - e_2 \equiv 2e_1$ have no order relation

Definition 3.33. A representation V of G has highest weight λ (necessarily dominant) if $V_\lambda \neq 0$ and $\lambda \geq \mu$ for all $\mu \in X^\bullet(T)$ such that $V_\mu \neq 0$.

Theorem. *Every irreducible representation V of G has a highest weight λ (necessarily dominant) and the map*

$$\{ \textit{ irr rep of } G\} \to X^\bullet(T)_+ \quad \textit{is a bijection}$$
$$V \mapsto \lambda = hw(V)$$

Moreover, $dimV_\lambda = 1$.

Corollary 3.34. *The map*

$$R(G) \to R(T)^W$$

is an isomorphism

Proof. In every W orbit there is a unique B dominant weight. So a basis of right hand side is given by

$$s_\lambda = \sum_W [w\lambda] \text{ for } \lambda \in X^\bullet(T)_+$$

We can complete the proof by using an upper triangular argument to show that $\{\mathrm{ch}(V)\}_{V\in\mathrm{Irr}}$ have a span containing all the s_λ for $\lambda \in X^\bullet(T)_+$. ☐

Example 3.35.

$$B = \text{upper triangular matrices} \subset SL_3 \quad X^\bullet(T) = \frac{\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3}{\mathbb{Z}(e_1 + e_2 + e_2)}$$

$\alpha_1 + \alpha_2$ is the highest weight of the Adjoint representation. $\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ is the highest weight of the standard representation. Exercise: $\frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$ is a highest weight of what representation?

4. Flag Varieties (Notes by Anna Romanova)

For our first example, consider $V = \mathbb{C}^n$. Let $\{e_1, \dots e_n\}$ be the standard basis of $\mathbb{C}^n$, and fix the full flag

$$\mathcal{F}_\bullet = 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_{n-1} \rangle \subset V$$

Note that any other flag can be obtained from this one by acting with an element of $GL_n(\mathbb{C})$. Specifically, any flag $V_\bullet = 0 \subset V_1 \subset \cdots \subset V_n = V$ in $\mathbb{C}^n$ has the form

$$V_\bullet = 0 \subset \langle ge_1 \rangle \subset \langle ge_1, ge_2 \rangle \subset \cdots \subset \langle ge_1, \dots, ge_n \rangle = V$$

for some $g \in GL_n(\mathbb{C})$. From this we see that $GL_n(\mathbb{C})$ acts transitively on the set of full flags in $\mathbb{C}^n$. Under this action, the stabilizer of our standard flag $\mathcal{F}_\bullet$ is the Borel subgroup of upper triangular matrices in $GL_n(\mathbb{C})$. We would like make sense of the statement

$$GL_n(\mathbb{C})/B = \{\text{flags in } \mathbb{C}^n\},$$

geometrically. To do this, we have two tasks:

- Make sense of the quotient $GL_n(\mathbb{C})/B$ as a variety.

- Make sense of the set {flags in $\mathbb{C}^n$} as a variety.

We will examine these tasks from two perspectives. First, we will return to our categorical perspective of algebraic groups as functors of points. Then, we will perform a more geometric construction.

5. The Functor of Points Perspective

Recall that we can apply the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$$

to the category $\mathcal{C}$ of schemes over k and regard a variety or k-scheme X as its functor of points

$$h_X : k\text{-alg} \longrightarrow \mathrm{Set}.$$

We need to determine how we can use this functor of points perspective to think about quotients of algebraic groups. We begin with a cautionary example. A first naive guess as to how to define quotients in this perspective would be to define the following functor:

$$\begin{aligned} GL_n/B : k\text{-alg} &\longrightarrow \mathrm{Set} \\ R &\longmapsto GL_n(R)/B(R) \end{aligned}$$

To determine if this definition is what we're looking for, we must ask ourselves if this functor is the functor of points of a scheme. The following exercise hints that this might not be the case.

Exercise 5.1. Let $n = 2$, and $R = \mathbb{Z}[\sqrt{-5}]$ (notice that this is not a PID). Show that that the map

$$GL_2(\mathbb{Z}[\sqrt{-5}])/B(\mathbb{Z}[\sqrt{-5}]) \longrightarrow \mathbb{P}^1(\mathbb{Z}[\sqrt{-5}])$$

is not surjective.

This exercise illustrates that this functor does not give us the scheme we would expect (namely $\mathbb{P}^1$), so our definition of quotient needs some refinement (see Exercise 5.4 and Theorem 5.5 for a proof that this functor is not representable by a scheme). But first, we will define flag varieties in this categorical setting.

Definition 5.2. Let k be a field and V a vector space over k of dimension n. For each sequence of integers $(d) = d_1 \leq d_2 \leq \cdots \leq d_r = n$, we define the associated **flag variety** to be the functor

$$\begin{aligned} Fl_{(d)} : k\text{-alg} \longrightarrow &\mathrm{Set} \\ R \longmapsto &\left\{ \begin{array}{c} \text{R -submodules } \ V_1 \subset V_2 \subset \cdots \subset V_r = V \otimes_k R \text{ with} \\ \text{each } V_i \text{ an } R\text{-module direct summand of rank } d_i \end{array} \right\} \end{aligned}$$

In the case where $d_i = i$ for $i = 1, \dots, n$, we say $Fl_{(d)}$ is the **full flag variety**.

We can observe that

$$\begin{aligned} GL_n(k)/B(k) &\longrightarrow Fl_{(1,2,\cdots,n)}(k) \\ g &\longmapsto g \cdot \mathcal{F} \end{aligned}$$

where $\mathcal{F}$ is a fixed flag with $B = \mathrm{Stab}_{GL_n}(\mathcal{F})$, is an isomorphism. However, as our initial example reveals, this doesn't hold for general R; i.e.

$$GL_n(R)/B(R) \longrightarrow Fl(R)$$

is not always surjective. The problem here from a functorial perspective is that the naive presheaf quotient

$$\begin{aligned}(\mathrm{Aff}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Set}\\ R &\longmapsto GL_n(R)/B(R)\end{aligned}$$

"is not a sheaf." We explain what this means precisely in the following pages.

We start by recalling the constructions of classical sheaf theory. For a topological space X, a *presheaf* on X is a functor

$$\mathcal{P} : (\mathrm{Open}_X)^{op} \longrightarrow \mathrm{Set},$$

and a *sheaf* on X is a presheaf $\mathcal{P}$ that satisfies a glueing condition. We can define a similar notion of sheaf on the category of topological spaces, Top. A *presheaf* on Top is a function

$$\mathcal{P} : \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

and a *sheaf* on Top is a presheaf $\mathcal{P}$ satisfying glueing conditions with respect to covering families of open immersions.

Notice that in this categorical setting, the notion of intersections no longer makes sense. The equivalent notion that we need is the fiber product; i.e. for open immersions $f_1 : U_2 \to X$ and $f_2 : U_2 \to X$, we define our gluing conditions on the fiber product $U_1 \times U_2$:

$$\begin{tikzcd}
 & U_1 \arrow[r, "f_1"] & X \\
U_1 \times_X U_2 \arrow[ur] \arrow[dr] & & \\
 & U_2 \arrow[r, "f_2"] & X
\end{tikzcd}$$

Examples of sheaves on Top are:

- The presheaf $\mathcal{P} = (X \mapsto \text{Continuous functions on } X)$ is a sheaf.
- For all objects $X \in \mathrm{Top}$, the presheaf

  $$\begin{aligned}h_X : \mathrm{Top}^{\mathrm{op}} &\longrightarrow \mathrm{Set}\\ Y &\longmapsto Hom(Y, X)\end{aligned}$$

  is a sheaf.

Now, our goal is to replicate this structure in algebraic geometry. We repace the category Top with the category Aff/k. In this setting, a *presheaf* is a functor

$$(\mathrm{Aff}/k)^{\mathrm{op}} \longrightarrow \mathrm{Set}.$$

For example, our functor $R \mapsto GL_n(R)/B(R)$ is a presheaf. To replicate the structure described above on Top, we must decide what kind of "covers" and "glueing conditions" we should impose to get a notion of a sheaf. There are a variety of ways we can do this. For instance, étale topology, fppf topology, or fpqc "topology" will give us a notion of sheaves on Aff/k. We will work with the fpqc ("fidelement plat quasi compact") topology.

In this setting, *covers* are jointly surjective familiies

$$\bigsqcup_{i=1}^{d} \mathrm{Spec} R_i \longrightarrow \mathrm{Spec} R$$

with each $R \to R_i$ flat. This allows us to define a notion of a sheaf on the category Aff/k:

Definition 5.3. A *sheaf* for the fpqc topology is a presheaf

$$\mathcal{F} : (\mathrm{Aff}/k)^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

satisfying

(i) (locality) $\mathcal{F}(\prod_{i=1}^n R_i) \simeq \prod_{i=1}^n \mathcal{F}(R_i)$, and

(ii) (gluing) For all faithfully flat ring homomorphisms $R \to R'$,

$$\mathcal{F}(R) \to \mathcal{F}(R') \underset{q}{\overset{p}{\rightrightarrows}} \mathcal{F}(R' \otimes_R R')$$

is an equalizer diagram in the category Set; i.e. $\mathcal{F}(R)$ includes into $\mathcal{F}(R')$ as $\{x \in \mathcal{F}(R') | p(x) = q(x)\}$. Here, p is obtained from the ring homomorphism $- \otimes 1 : R' \to R' \otimes_R R'$ and q from $1 \otimes - : R' \longrightarrow R' \otimes_R R'$.

A follow-up exercise to the one at the beginning of this section is the following:

Exercise 5.4. Show that $R \mapsto GL_n(R)/B(R)$ is not a sheaf for the fpqc topology.

We have the following theorem (Grothendieck's theory of fpqc descent).

Theorem 5.5. *For all objects $X \in \mathrm{Aff}/k$, the representable functor h_X is a sheaf in the fpqc topology.*

This theorem suggests that the correct notion of quotients of algebraic groups in our categorical setting is that GL_n/B should be the "sheafification[1]" of the presheaf $R \mapsto GL_n(R)/B(R)$. And indeed, the results of the following section will indicate that this is the correct construction. This ends our first perspective on the construction of the quotient. The philosophy that one should take away from this is that the correct approach is to "make the naive construction in group theory, then make sure it's a sheaf."

6. The Geometric Perspective

Definition 6.1. Let G be an algebraic group over k. Let $H \hookrightarrow G$ be a Zariski closed subgroup of G. Then a *quotient* $X = G/H$ is a pair $(X, \pi : G \to X)$, where X is a scheme over k, satisfying

(i) π is faithfully flat,

(ii) π is right-H-invariant, and

[1]This is not technically well-defined. There is no sheafification functor for the fpqc topology. However, as long as the presheaves are "small enough," sheafification will be possible, and the presheaves that we are interested are "small enough." For a precise statement of what it means for a presheaf to be "small enough" see William Charles Waterhouses article *Basically Bounded Functors and Flat Sheaves.*

(iii) the map

$$\begin{aligned} G \times H &\longrightarrow G \times_X G \\ (g, h) &\longmapsto (g, gh) \end{aligned}$$

is an isomorphism; i.e. the non-empty fibers of $G(R) \to X(R)$ are just $H(R)$-orbits for all k-algebras R.

A consequence of this definition is that (X, π) is *initial* among H-equivariant maps from G to any k-scheme Y with a trivial H-action. More precisely, for any k-scheme Y with trivial H-action and map $f : G \to Y$ such that $f(gh) = f(g)$, there exists a unique factorization $\bar{f}$ so that the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\pi"'] & Y \\ & G/H \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

commutes. The proof of this universal property uses fpqc descent. Morever, it shows that this geometric construction agrees with the functorial one from the previous section.

Next we explore why such a constuction exists.

Theorem 6.2. *Let G be a smooth affine algebraic group over k, and $H \hookrightarrow G$ a Zariski closed subgroup. Then the quotient $G \to G/H$ exists and is a smooth quasiprojective variety over k. If B is a Borel subgroup of G then G/B is a projective variety.*

Proof. We will use the following theorem of Chevalley.

Theorem 6.3. *(Chevalley) If G is a smooth affine algebraic group and $H \subset G$ a closed subgroup as in the statement of the theorem, then there exists a finite dimensional representation $r : G \to GL(V)$ and a line $L \subset V$ such that $H = Stab_G(L)$. (More precisely, H represents the functor $R \mapsto Stab_{G(R)}(L \otimes_k R)$.)*

Let L, V be the line and vector space given by the theorem. We have an action map:

$$\begin{aligned} G &\longrightarrow \mathbb{P}(V) \\ g &\longmapsto g \cdot L \end{aligned}$$

Let $G \cdot L$ be the orbit of G acting on L; i.e. the set-theoretic image of this action map.

Fact 1. $G \cdot L \subset \mathbb{P}(V)$ is a locally closed subset in the Zariski topology on $\mathbb{P}(V)$.

We can use this fact to endow $G \cdot L$ with a scheme structure: $\overline{G \cdot L}$ inherits the reduced closed subscheme structure, and since $G \cdot L$ is locally closed, $G \cdot L$ is open in $\overline{G \cdot L}$ and inherits a scheme structure as an open subset. Therefore, $G \cdot L$ is a quasiprojective variety and one can check that the map $G \to G \cdot L$ is a quotient map in the sense defined at the beginning of this section. This completes the proof of the first part of the theorem.

Before proving that G/B is projective, we will elaborate on Fact 1. Generally, the image $\mathcal{O}$ of the action map contains some open subset U of its closure $\overline{\mathcal{O}}$. Since G-translates gU are in $\mathcal{O}$, we see that $\mathcal{O}$ is open in $\overline{\mathcal{O}}$. In particular, the complement $\overline{\mathcal{O}} \backslash \mathcal{O}$ is a union of orbits of strictly smaller dimension. We conclude that *orbits of minimal dimension are necessarily closed.*

Now we prove the second part of the theorem. Again, let L, V be the line and finite dimensional vector space guaranteed by Chevalley's theorem, so $B = \mathrm{Stab}_G(L)$ and G/B is constructed as the image of the orbit map $g \mapsto g \cdot L$ as before. Since B stabilizes L, B acts on V/L, and since B is connected and solvable, it stabilizes a full flag $\overline{V_\bullet} = 0 \subset \overline{V_1} \subset \overline{V_2} \subset \cdots \subset \overline{V_{\mathrm{dim} V/L}} = V/L$ in V/L. (This follows from the Lie-Kolchin Theorem.) We can lift this flag to V, and we obtain a B-stable full flag $V = 0 \subset L \subset V_1 \subset V_2 \subset \cdots \subset V$, where V_i is the preimage of $\overline{V_i}$ under the quotient map. Since $B = \mathrm{Stab}_G(L)$, and $V_\bullet$ is B-stable, we have that $B = \mathrm{Stab}_G(V_\bullet)$. This gives us a new action of G on the full flag variety of $GL(V)$,

$$\begin{aligned} G &\to Fl \\ g &\mapsto g \cdot (0 \subset L \subset V_1 \subset \cdots \subset V) \end{aligned}$$

Now, G/B is also the image of this action map. Furthermore, since B is a *maximal* connected solvable subgroup of G, $G \cdot (0 \subset L \subset V_1 \subset \cdots \subset V)$ is an orbit of minimal dimension. Indeed, if $\mathcal{F}_\bullet$ is any flag and $\mathcal{S} = \mathrm{Stab}_G(\mathcal{F}_\bullet)$, then the connected component of the identity $\mathcal{S}^\circ$ is connected and solvable, so it is contained in a maximal connected solvable subgroup of G; i.e. a Borel subgroup B'. Since all Borel subgroups are conjugate, this implies

$$\mathrm{dim}\mathcal{S}^\circ \leq \mathrm{dim}B, \text{ and thus } \mathrm{dim}G/B \leq \mathrm{dim}G/\mathcal{S}^\circ = \mathrm{dim}G/\mathcal{S}.$$

Therefore, G/B is an orbit of minimal dimension, so it is necessarily closed in Fl, which is a projective variety. This completes the proof of the theorem. □

Now we have two ways of thinking about quotients and we have found a family of projective varieties G/B. Our next step is to investigate the structure of these particular flag varieties G/B.

7. The Bruhat Decomposition

We begin this section by recalling the structure of root subspaces of a reductive algebraic group. To a pair (G, T), where G is a connected reductive group and T is a maximal torus, we can associate a collection of roots $\Phi(G, T)$ by decomposing G into simultaneous eigenspaces under the Adjoint action of T, as described previously. Recall one of our key structural results of reductive algebraic groups: For $\alpha \in \Phi(G, T)$, there exists a homomorphism

$$\psi_\alpha : SL_2 \longrightarrow G,$$

obtained from the Lie algebra isomorphism $\mathfrak{sl}_2 \longrightarrow \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. We define *root subgroups* $U_\alpha \subset G$ to be

$$U_\alpha := \psi_\alpha\left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right) = exp(\mathfrak{g}_\alpha).$$

Let s_α be the image of $\psi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ in the Weyl group $W = W(G, T) = N_G(T)/T$.

Fact 2. The set $\{s_\alpha\}_{\alpha \in \Delta(G,T,B)}$ generates the Weyl group.

Now we turn our attention to the Bruhat decomposition. Let G be a connected reductive group over an algebraically closed field $k = \bar{k}$. Let $T \subset B \subset G$ be a choice of a torus and a Borel subgroup of G. Then $B \times B$ acts on G by $(b_1, b_2) \cdot g = b_1 g b_2^{-1}$. The orbits BgB are locally closed subvarieties of G. (Note that these orbits are not always closed. For example, if $g \in B$, then the orbit $BgB = B \hookrightarrow G$ is a closed subvariety. But if $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2$, then BgB is open in SL_2.) For each $w \in W$, let $\dot{w} \in N_G(T)$ be a choice of lift. Set $C(w) = B\dot{w}B$. Any choice of lift will give the same $C(w)$, so this is well defined. Our goal of this section is to study the structure of these $C(w)$.

Let $N = R_u(B)$ be the unipotent radical of B. (For example, if B is the upper triangular Borel subgroup in GL_n, N is the subgroup of upper triangular matrices with 1's on the diagonal.) The following lemma reveals much about the structure of $C(w)$.

Lemma 7.1. *For $w \in W$, let $R(w) = \Phi^+ \cap (w\Phi^-)$ be the collection of positive roots that are sent to negative roots under action by w^{-1}. Let $N_{R(w)}$ be the subgroup of N generated by $\{U_\alpha | \alpha \in R(w)\}$. Then the multiplication map*

$$N_{R(w)}\dot{w} \times B \longrightarrow C(w)$$

is an isomorphism of varieties. Moreover, the multiplication map

$$\prod_{\alpha \in R(w)} U_\alpha \longrightarrow N_{R(w)}$$

is an isomorphism of varieties.[2]

Note that the space $\prod_{\alpha \in R(w)} U_\alpha$ is a product of affine spaces. The main result of this section is the following theorem.

Theorem 7.2. *(Bruhat Decomposition)*

$$G = \bigsqcup_{w \in W} C(w)$$

Exercise 7.3. Let $G = SL_2$ and $W = \left\{1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. Show directly that the theorem holds in this case; i.e. that

$$SL_2 = B \sqcup B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B,$$

where B is the Borel subgroup of upper triangular matrices.

The theorem immediately implies our main structural result about the projective varieties G/B. Let $Y(w) = C(w)/B$. Such $Y(w)$ are referred to as *Bruhat cells*. By lemma 3.1, $Y(w)$ is an affine variety; i.e. $Y(w) \simeq N_{R(w)}\dot{w} \times B/B \simeq \mathbb{A}^{\#R(w)}$.

Corollary 7.4. *The flag variety G/B is the disjoint union of Bruhat cells*

$$G/B = \bigsqcup_{w \in W} Y(w).$$

[2]Note that this is not an isomorphism of groups.

This gives us an affine stratification of the flag variety.

Remark 7.5. The Bruhat cells $Y(w)$ are locally closed subvarieties of G/B. Their closure, $X(w) := \overline{Y(w)}$ is called the *Schubert variety* of w. These are complete, but may be singular. The singularities of Schubert varieties are of great representation-theoretic significance.

Example 7.6. For any G, there exists a unique element $w_0 \in W$ such that $\#R(w_0)$ is maximal. The number $\#R(w)$ is called the *length* of the element $w \in W$,[3] and defines a *length function* $\ell : W \to \mathbb{Z}$ by $\ell(w) = \#R(w)$. This unique element w_0 is often referred to as the *longest element* of the Weyl group since its length is maximal. The associated stratum $Y(w_0)$ is open and dense in G/B. This longest element w_0 has the property that $w_0(\Phi^-) = \Phi^+$, so $R(w_0) = \Phi^+ \cap (w_0\Phi^-) = \Phi^+$, and therefore $N_{R(w_0)} = N$.

Remark 7.7. There is a partial order on W characterized by $v \le w$ if $Y(v) \subset \overline{Y(w)} = X(w)$.[4]

Now we give the idea of the proof of the theorem.

Proof. The proof relies on the following combinatorial result, which we will not prove.

Lemma 7.8. *For all simple reflections s_α, $\alpha \in \Delta$, and all $w \in W$,*

$$C(s_\alpha) \cdot C(w) = \begin{cases} C(s_\alpha w) & \text{if } \ell(s_\alpha w) = \ell(w) + 1 \\ C(s_\alpha w) \cup C(w) & \text{if } \ell(s_\alpha w) = \ell(w) - 1 \end{cases}$$

Exercise 7.9. Check this for SL_2.

Granted this lemma, we can deduce the decomposition $G = \sqcup_{w\in W} C(w)$ as follows:

(1) Let $H = \cup_{w\in W} C(w)$. We wish to show that $H = G$. The lemma implies that $C(s_\alpha) \cdot H \subset H$ for all $\alpha \in \Delta$. But it can be shown that $\langle C(s_\alpha)\rangle_{\alpha\in\Delta} = G$, so $G \cdot H \subset H$. We conclude that $H = G$.

(2) (Disjointness) Let $w, w' \in W$ and assume that $C(w) \cap C(w') \neq \varnothing$. Since both $C(w)$ and $C(w')$ are $B \times B$-orbits, this can only happen if $C(w) = C(w')$. In particular, $\ell(w) = \ell(w')$. We use induction in $\ell(w)$ to show that $w = w'$. If $\ell(w) = \ell(w') = 0$, then $w = 1 = w'$ and we are done, so we assume $\ell(w) > 0$. Then there exists s_α for $\alpha \in \Delta$ such that $\ell(s_\alpha w) = \ell(w) - 1$. By the lemma,

$$C(s_\alpha w) \subset C(s_\alpha) \cdot C(w) = C(s_\alpha) \cdot C(w') \subset C(s_\alpha w') \cup C(w').$$

This implies that $C(s_\alpha w) = C(s_\alpha w')$ or $C(s_\alpha w) = C(w')$. Since $\dim C(w') = \dim C(s_\alpha w) + 1$, the latter is impossible. Therefore, we can imply the induction hypothesis to $C(s_\alpha w) = C(s_\alpha w')$, and we are done.

$\square$

[3] The number $\#R(w)$ can also be described combinatorially by expressing an element $w \in W$ as a product of simple reflections. The minimal number of simple reflections needed to express w is a well-defined invariant and it agrees with the cardinality of $R(w)$. This explains the nomenclature "length." For more information on the combinatorics of Coxeter groups, see James E. Humphreys *Reflection Groups and Coxeter Groups.*

[4] This partial order can also be realized combinatorially in terms of the length.

8. Borel-Weil Theorem and Theorem of highest weight (notes by Allechar Serrano Lopez)

Our main goal is to construct a highest weight λ irreducible representation of G for all $\lambda \in X^{\bullet}(T)_+$ and to show that these are all the irreducible representations of G. Our approach is via the Borel-Weil theorem. Recall that, given a representation V of a connected reductive group G (containing a fixed Borel subgroup B and a fixed maximal torus $T \leq B$) and a choice of a highest weight vector $v \in V$, we get a G-equivariant map

$$G/B \to \mathbb{P}(V)$$

$$v \mapsto (gv).$$

The idea of the Borel-Weil theorem is to run this argument in reverse: we construct G-equivariant line bundles on G/B and study the associated maps (see above) to projective spaces. We begin with some preliminaries on highest weights.

Lemma 8.1. *If V is a highest weight λ representation, then λ is dominant.*

Proof. Recall that $\mathrm{ch}(V)$ is $W-$invariant. So if $V_\lambda \neq 0$, then $V_{s_\alpha(\lambda)} \neq 0$ for all $s_\alpha(\lambda) \in W$. By assumption, $\lambda \geqslant s_\alpha(\lambda)$, i.e., $\langle \lambda, \alpha^\vee \rangle \alpha \in \mathbb{Z}_{\geqslant 0}\Phi^+$, so $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in \Phi^+$. □

Definition 8.2. Let V be a representation of G. Say that $v \in V$ is a highest weight vector of highest weight λ if $Nv = v$ (or $\mathfrak{n}v = 0$, where $\mathfrak{n} = \mathrm{Lie}(N)$) and T acts on V by λ.

Example 8.3. Consider $G = \mathrm{SL}_2$ and $V = \mathrm{Sym}^3\mathbb{C}^2 \oplus \mathrm{Sym}^4\mathbb{C}^2$, where $\langle e_1, e_2 \rangle$ and $\langle f_1, f_2 \rangle$ generate $\mathbb{C}$ in the first and second summands, respectively. We have that $e_1^3 \in \mathrm{Sym}^3\mathbb{C}^2$ and $f_1^4 \in \mathrm{Sym}^4\mathbb{C}^2$. Then e_1^3 and f_1^4 are the highest weight vectors of weights $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^3, t^4$, respectively; but, V is not a highest weight representation in the sense just defined.

Lemma 8.4. *If V is a highest weight representation of weight λ, then any $v \in V_\lambda \smallsetminus 0$ is a highest weight vector of weight λ.*

Proof. We need to show that $\mathfrak{n}v = 0$. More generally, if $\alpha \in \Phi$, and if $X_\alpha \in \mathfrak{g}_\alpha$, then $X_\alpha \cdot V_\lambda \subset V_{\lambda+\alpha} \cup 0$: for all $t \in T$

$$t \cdot X_\alpha v = \mathrm{Ad}(t)X_\alpha \cdot tv = \alpha(t)X_\alpha\lambda(t)v = (\alpha+\lambda)(t)X_\alpha v$$

In particular, in our highest weight λ representation V, $V_{\alpha+\lambda} = 0$ for all $\alpha \in \Phi^+$, so $\mathfrak{n}v = 0$, i.e., $v \in V_\lambda$ is a highest weight vector. □

Lemma 8.5. *If V is any non-zero $G-$representation, the V contains a highest weight vector with dominant weight.*

Proof. Let λ be any weight in V that is maximal for the partial order $\leq$, and let v be any non-zero element of the weight space V_λ. Since λ is maximal, the argument of the previous lemma shows v is a highest weight vector. The argument of Lemma 8.1 (i.e., W-invariance of the character) then shows λ is dominant. □

Combined with semi-simplicity of G-representations, this lemma implies:

Corollary 8.6. *If $\dim V^N \leqslant 1$, then V is irreducible.*

That concludes the preliminaries. Now we turn to the construction of highest weight representations; the Borel-Weil theorem finds these in the cohomology of G-equivariant line bundles on the flag variety G/B, so we start by reviewing a few facts about linear systems.

Recall that for any variety X over k, we have the following bijection:

$$\{\text{maps } X \xrightarrow{f} \mathbb{P}(V)\} \longleftrightarrow \{\text{ base-point free linear system } (\mathcal{L}, V, i : V^* \to \Gamma(X, \mathcal{L})) \ \} / \cong$$
$$f \longmapsto \begin{pmatrix} f^*(\mathcal{O}(1)), i : V^* \to \Gamma(X, \mathcal{L}) \text{ with the property that for all} \\ x \in X \text{ there exists } \alpha \in V^* \text{ such that } i(\alpha)|_x \neq 0 \end{pmatrix}$$

Recall that $\mathbb{P}(V)$ has a tautological line bundle $\mathcal{O}(-1)$ whose fiber over a line l is just the line l; the dual line bundle $\mathcal{O}(1)$ has fiber $l^* = \mathrm{Hom}_k(l, k)$ over l, and $\Gamma(\mathbb{P}(V), \mathcal{O}(1))$ is canonically isomorphic to V^*, via the map sending $\alpha \in V^*$ to the section $l \mapsto \alpha|_l$.

We want to upgrade this correspondence to a $G-$equivariant version:

Corollary 8.7. *If X is a space with $G-$action and V is a $G-$representation, then there is a bijection*

$$\{G - \textit{equivariant maps } X \to \mathbb{P}(V)\} \leftrightarrow$$
$$\{G - \textit{equivariant base point free linear systems } (\mathcal{L}, V, \iota) \textit{ on } X\}/ \cong$$

Here is the precise definition of a G-equivariant linear system:

Definition 8.8. A $G-$equivariant linear system on X is $(\mathcal{L}, V, i : V^* \to \Gamma(X, \mathcal{L}))$ such that

1) $\mathcal{L}$ is a $G-$equivariant line bundle on X. This means that $\mathcal{L}$ is a line bundle on X equipped with a $G-$action compatible with the action on X under the projection $\mathcal{L} \to X$, and for all $x \in X$ and all $g \in G$, the map $g : \mathcal{L}_x \to \mathcal{L}_{g \cdot x}$ is linear.
2) V is a $G-$representation.
3) i is a map of $G-$representations (see below).

In part (3) of the above definition, we define the action of G on $\Gamma(X, \mathcal{L})$ as follows. For a section $s : X \to \mathcal{L}$, and $g \in G$, define $g \cdot s \in \Gamma(X, \mathcal{L})$ by setting $(g \cdot s)(x) = gs(g^{-1}x)$.

Lemma 8.9. *The correspondence*

$$\left\{ \begin{matrix} G-\textit{equivariant line} \\ \textit{bundles on } G/B \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} 1-\textit{dimensional} \\ B-\textit{representations} \end{matrix} \right\}$$
$$\mathcal{L} \longmapsto \mathcal{L}_B$$

induces a bijection

$$\{G - \textit{equivariant linebundles on } G/B\} / \cong \xrightarrow{\sim} Hom(B, \mathbb{G}_m)$$

Proof. We will construct the inverse map. Given $\chi : B \to \mathbb{G}_m$, we form the associated line bundle

$$G \times^B \mathbb{A}^1 = \{(g,x) \in G \times \mathbb{A}^1\}/(gb,x) \sim (g, \chi(b)x) \to G/B$$

where the equivalence relation is defined for all $g \in G, x \in \mathbb{A}^1, b \in B$. Analytically, this is clearly a line bundle. In the Zariski topology, we use the fact that the map $G \to G/B$ Zariski-locally has sections (this follows from the description of the big Bruhat cell $C(w_0) = N \times B$).

□

Lemma 8.10. *$Hom(B, \mathbb{G}_m) = Hom(T, \mathbb{G}_m) = X^\bullet(T)$.*

Proof. Any homomorphism $B \to \mathbb{G}_m$ is trivial on N (we have shown this before; it reduces to showing that $\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m) = 0$), hence factors as a homomorphism

$$T \xrightarrow{\sim} B/N \to \mathbb{G}_m.$$

□

Definition 8.11. For $\lambda \in X^\bullet(T)$, define $L(\lambda) = G \times^B k(-\lambda)$ to be the line bundle on G/B associated to $-\lambda$.

We can now state the main theorem:

Theorem 8.12 (Borel-Weil)**.** *Let G be a connected, reductive group with $B \leq G$ a Borel subgroup and $T \leq B$ a maximal torus. Then:*

1) *If $\lambda \in X^\bullet(T)_+$, then $V(\lambda) := \mathrm{H}^0(G/B, L(\lambda))^*$ is an irreducible G-representation with highest weight λ.*
2) *The $V(\lambda)$ constructed in this way exhaust all the irreducible representations of G up to isomorphism.*
3) *If $\lambda \in X^\bullet(T) \backslash X^\bullet(T)_+$, then $\mathrm{H}^0(G/B, L(\lambda)) = 0$.*
4) *$\dim V(\lambda)_\lambda = 1$.*

Example 8.13. This example will clarify why we defined $L(\lambda)$ via the character $-\lambda$. Let $G = SL_2(k)$, with $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \mathrm{Stab}\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. Then the tautological line bundle

$$\begin{array}{c} \mathcal{O}(-1) \\ \downarrow \\ SL_2(k)/B = \mathbb{P}^1 \end{array}$$

is $SL_2(k)$-equivariant. In our classification, this corresponds to the B-representation on $\mathcal{O}(-1)_B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. We have

$$\begin{pmatrix} t & * \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

but the tautological bundle doesn't give us global sections; we need its inverse. The bundles with sections are $\mathcal{O}(n)$ for $n \geq 0$: the corresponding characters in $X^\bullet(T)$ are

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{-n}.$$

Corollary 8.14. $R(G) \xrightarrow{\sim} R(T)^W$.

9. Proof of the Theorem (notes by Sean McAffee)

We begin the proof of Borel-Weil with the following proposition:

Proposition 9.1.

1) *For any irreducible representation V of G, there exists a $\lambda \in X^\bullet(T)_+$ such that $V \cong V(\lambda)$.*
2) *For any $\lambda \in X^\bullet(T)$, if $V(\lambda) \neq 0$, then $V(\lambda)$ is irreducible.*

Proof. Any irreducible representation V has some highest weight vector v. As discussed at the beginning of this section, v induces a G-equivariant map $f : G/B \to \mathbb{P}(V)$. This in turn induces a nonzero G-equivariant map

$$i : V^* \to \Gamma(G/B, f^*(\mathcal{O}(1)).$$

This must be injective, since V is irreducible. Dualizing, then, we have $V(\lambda) \twoheadrightarrow V$, so 1) will follow from 2).

To prove 2), first recall that a G-representation W is irreducible if $\dim W^N \leq 1$. We will check this for $W = V(\lambda)$.

Lemma 9.2. *We have that $\dim \Gamma(G/B, L(\lambda))^N \leq 1$, and if nonzero, then the T-action is given by the character $-w_0 \cdot \lambda$ (here w_0 denotes the longest element of the Weyl group).*

Proof. The restriction map

$$\Gamma(G/B, L(\lambda))^N \to \Gamma(C(w_0)/B, L(\lambda))^N \to L(\lambda)_{w_0}$$

is injective, since $C(w_0)$ is dense in G/B and $Nw_0 \xrightarrow{\sim} C(w_0)/B$. Thus $\Gamma(G/B, L(\lambda))^N$ injects into a line, i.e. its dimension is ≤ 1.

To compute the T-action, note that this restriction map is T-equivariant and the following claim: for any $w \in W$, and any $\lambda \in X^\bullet(T)$, T acts on $L(\lambda_w)$ by $-w \cdot \lambda$. Indeed, write $L(\lambda)_w = \{(\dot{w}, x) | x \in k\}$, where $\dot{w}$ is a fixed lift of w (an element of $W = N_G(T)/T)$) to $N_G(T)$. Then, t acts by

$$t \cdot (\dot{w}, x) = (t\dot{w}, x) = (\dot{w}\dot{w}^{-1}t\dot{w}, x) \sim (\dot{w}, -\lambda(\dot{w}^{-1}t\dot{w})x) = (-w\lambda)(t) \cdot x.$$

This completes the proof of the lemma, and hence of part 2) of the proposition.

∎

∎

To complete the proof of Borel-Weil, we need to show that $\Gamma(G/B, L(\lambda))$ is nonzero if and only if λ is dominant. First, we show that $\Gamma \neq 0$ implies λ is dominant. We do this by analyzing the possible weights of Γ. We have seen that restriction to $C(w_o)$ gives a T-equivariant injection

$$\Gamma(G/B, L(\lambda)) \hookrightarrow \Gamma(Y(w_0), L(\lambda)).$$

Let λ^* denote $-w_0 \cdot \lambda$, and let $\mathfrak{n}$ denote $\mathrm{Lie}(N)$. We have that T acts on $\mathfrak{n}$ via Ad, hence T acts on the dual $\mathfrak{n}^*$.

Lemma 9.3.

$$\Gamma(Y(w_0), L(\lambda)) \cong_T k(\lambda^*) \otimes \mathrm{Sym}^\bullet(\mathfrak{n}^*).$$

(Here $\cong_T$ denotes isomorphism as T-representations). In particular, this is a highest weight representation with highest weight λ^*.

Proof. Recall the isomorphism

$$\begin{aligned} N &\xrightarrow{\sim} Y(w_0) \\ n &\mapsto n\dot{w}_0. \end{aligned}$$

Under this map, the left T-action on $Y(w_0)$ intertwines $\mathrm{Ad}(T)$ on N:

$$tn\dot{w}_0 = tnt^{-1}t\dot{w}_0 = tnt^{-1}\dot{w}_0 t' \sim \mathrm{Ad}(t)n\dot{w}_0.$$

This extends to an isomorphism between $N \times L(\lambda)_{w_0}$ and $L(\lambda)|_{Y(w_0)}$:

$$(n, s) \longmapsto n \cdot s$$

$$\begin{tikzcd} N \times L(\lambda)_{w_0} \arrow[r, "\sim"] \arrow[d] & L(\lambda)|_{Y(w_0)} \arrow[d] \\ N \arrow[r, "\sim"] & Y(w_0). \end{tikzcd}$$

This isomorphism is T-equivariant; to be precise, the left-hand side is given the adjoint representation (on the N-component) and left T-multiplication (on the fiber $L(\lambda)_{w_0}$), while the right-hand side has its given left T-multiplication. To check this equivariance claim, we let $s = (\dot{w}_0, x) \in \mathrm{L}(\lambda)_{w_0}, n \in N, t \in T$; we have

$$\begin{aligned} t \cdot (n, s) &= t \cdot (n, (\dot{w}_0, x)) \\ &= (t \cdot n, t \cdot (\dot{w}_0, x)) \\ &= (\mathrm{Ad}(t)(n), (t\dot{w}_0, x)) \\ &= (\mathrm{Ad}(t)(n), (\dot{w}_0(\dot{w}_0^{-1}t\dot{w}_0), x) \\ &= (\mathrm{Ad}(t)(n), (\dot{w}_0, -\lambda(\dot{w}_0^{-1}t\dot{w}_0)x). \end{aligned}$$

Via this T-equivariant trivialization of $L(\lambda)|_{Y(w_0)}$, we can compute

$$\Gamma(Y(w_0), L(\lambda)) \cong_T \Gamma(N, N \times L(\lambda)_{w_0}) \cong_T \mathcal{O}(N) \otimes k(\lambda^*) \cong \mathrm{Sym}^\bullet(\mathfrak{n}^*) \otimes k(\lambda^*).$$

This completes the proof of the lemma, and the proposition follows.

□

The main step in the proof of Borel-Weil is to show that for any $\lambda \in X^\bullet(T)_+$ we have $\mathrm{H}^0(G/B, L(\lambda)) \neq 0$. To show this, our strategy will be to construct a highest weight vector of weight λ^*. We will do this in two steps:

Step 1: Construct the highest weight vector over the "big cell" $Y(w_0)$.

Step 2: Show that this section extends to all of G/B.

First, we remark that a section s on the line bundle

$$\begin{tikzcd} G \times_B k(-\lambda) \arrow[d] \\ G/B \end{tikzcd}$$

can be viewed as a map $gB \mapsto (g, f(g))$ for some function $f : G \to k$. Since $gB = gbB$, we must have $(g, f(g)) = (gb, f(gb))$ for all $g \in G, b \in B$. Furthermore, the equivalence relation on $G \times_B k(-\lambda)$ requires

$$(gb, f(gh)) \sim (g, -\lambda(b)f(gb)).$$

Thus, we can think of sections as being determined by a function $f : G \to k$ such that

$$f(gb) = \lambda(b)f(g), \forall g \in G, b \in B.$$

Under this correspondence, the G-action on a section s translates into the left regular action on a function f with the above properties.

We have that $N \times T \times N$ and $C(w_0) = Nw_0B$ are isomorphic as varieties under the map $(n_1, t, n_2) \mapsto n_1w_0tn_2$; we define a function f on $C(w_0)$ by

$$f(n_1w_0tn_2) := \lambda(t).$$

We claim that f is a highest weight vector of weight $\lambda^* := -w_0\lambda$. Indeed, $N \cdot f = f$ since f is left N-invariant by the left regular action, and we have

$$\begin{aligned} (t_1 \cdot f)(n_1\dot{w}_0tn_2) &= f(t_1^{-1}n_1\dot{w}_0tn_2) \\ &= f((t_1^{-1}n_1t_1)t_1^{-1}\dot{w}_0tn_2) \\ &= f(t_1^{-1}\dot{w}_0tn_2) \\ &= f(\dot{w}_0(\dot{w}_0^{-1}t_1^{-1}\dot{w}_0) \cdot tn_2) \\ &= \lambda(\dot{w}_0^{-1}t_1^{-1}\dot{w}_0t) \\ &= (-w_0\lambda)(t_1)\lambda(t). \end{aligned}$$

This completes Step 1. Now, for the second (and final) step, we need to extend the section defined above to all of G/B. To do this, we first extend to a section on an open subset $U \subset G/B$ such that the complement of U has codimension ≥ 2. Then, we use the analytic continuation principle: for a line bundle $\mathcal{L}$ on an irreducible normal variety X, the restriction map $\Gamma(X, \mathcal{L}) \to \Gamma(U, \mathcal{L})$ is surjective when the codimension of $X - U$ is ≥ 2.

Given f as above, we extend f from $C(w_0) = N\dot{w}_0 B$ to

$$C(w_0) \cup \bigcup_{\alpha \in \Delta} s_\alpha N \dot{w}_0 B.$$

Note that each $s_\alpha N \dot{w}_0 B$ is open, but contains $C(s_\alpha \dot{w}_0)$, and in this way the union under consideration contains all Bruhat cells of codimension one.

We leave it as an exercise to check that any element of $s_\alpha N \dot{w}_0 B$ can be written in the form $n_1 s_\alpha u_\alpha(x) \dot{w}_0 t n_2$, where $n_1, n_2 \in N$ and $x \in \mathbb{G}_a$ (here u_α is the root group homomorphism associated to α). We will use the following calculation in SL_2 to show that f is defined on such an element with $x \neq 0 \in k$:

$$\begin{aligned}
s_\alpha u_\alpha(x) &= \psi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-1} & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \\
&= u_\alpha(-x^{-1})\alpha^\vee(-x^{-1})u_{-\alpha}(x^{-1}).
\end{aligned}$$

Also oberve the identity

$$\begin{aligned}
u_{-\alpha}(x^{-1})\dot{w}_0 t &= \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \\
&= \begin{pmatrix} t^{-1} & 0 \\ x^{-1}t^{-1} & t \end{pmatrix} \\
&= \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1}t^{-2} & 1 \end{pmatrix} \\
&= \dot{w}_0 t u_{-\alpha}(x^{-1}t^{-2}).
\end{aligned}$$

Thus, for $x \neq 0$, we have

$$\begin{aligned}
n_1 s_\alpha u_\alpha(x)\dot{w}_0 t n_2 &= n_1 u_\alpha(-x^{-1})\alpha^\vee(-x^{-1})u_{-\alpha}(x^{-1})\dot{w}_0 t n_2 \\
&= n_1 u_\alpha(-x^{-1})\alpha^\vee(-x^{-1})\dot{w}_0 t u_{-\alpha}(x^{-1}t^{-2})n_2 \\
&= \Big[\underbrace{n_1 u_\alpha(-x^{-1})}_{\in N}\Big]\alpha^\vee(-x^{-1})(\dot{w}_0 t)\dot{w}_0\Big[\underbrace{u_\alpha(x^{-1}t^{-2})n_2}_{\in N}\Big]. \\
&= [\in N]\alpha^\vee(-x^{-1})(\dot{w}_0 t)\dot{w}_0[\in N] \\
&= [\in N]\dot{w}_0(\dot{w}_0\alpha^\vee)(-x^{-1})t[\in N]
\end{aligned}$$

We have that f is already defined here for $(\dot{w}_0\alpha^\vee)(-x^{-1})t$ and takes value $\lambda(\dot{w}_0\alpha^\vee(-x^{-1})t)$, thus

$$= (-x^{-1})^{\langle \lambda, \dot{w}_0\alpha^\vee\rangle}\lambda(t) \text{ for } x \neq 0.$$

It is here and only here that we finally use the fact that λ is dominant; this implies that $\langle \lambda, \dot{w}_0\alpha^\vee\rangle \leq 0$. Thus the expression above extends to $x = 0$, since it involves a nonnegative power of x. So, we have shown we can extend f to $s_\alpha N\dot{w}_0 B$ for any $\alpha \in \Delta$. Therefore, by the analytic continuation principle described earlier, we can extend f to all of G/B. This completes the proof of Borel-Weil.

10. Tannakian Categories: definitions and motivation (notes by Christian Klevdal)

Most of the material in this section may be found in Motives by Yves André.

Definition 10.1. A **tensor category** over a commutative ring F is an F-linear category $\mathcal{C}$ that has a tensor structure, which consists of the following data:

(1) A functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$.

(2) An identity object $\mathbf{1}$.

(3) Associativity and commutativity constraints, unit isomorphisms, which are functorial isomorphisms

$$\begin{aligned}
a_{X,Y,Z}\colon X\otimes(Y\otimes Z) &\xrightarrow{\sim} (X\otimes Y)\otimes Z \\
c_{X,Y}\colon X\otimes Y &\xrightarrow{\sim} Y\otimes X \quad \text{such that } c_{X,Y} = c_{Y,X}^{-1} \\
u_X\colon X\otimes\mathbf{1} &\xrightarrow{\sim} X, \quad u'_X\colon \mathbf{1}\otimes X \xrightarrow{\sim} X
\end{aligned}$$

These isomorphisms are required to satisfy compatibilities

- Associativity compatibility: The two isomorphisms
$$X\otimes(Y\otimes(Z\otimes T)) \to ((X\otimes Y)\otimes Z)\otimes T$$
coming from applying associativity isomorphisms in various ways are equal.

- Compatibility of commutativity and associativity: The two isomorphisms
$$X \otimes (Y \otimes Z) \to (Z \otimes X) \otimes Y$$
coming from applying associativity and commutativity isomorphisms in various ways are equal.
- Compatibility of associativity and unit: The two isomorphisms
$$X \otimes (\mathbf{1} \otimes Y) \to X \otimes Y$$
coming from applying associativity and unit isomorphisms in various ways are equal.

Further, the tensor category $(\mathcal{C}, \otimes)$ is said to be **rigid** if:

(4) There exists an autoduality ${}^\vee \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that for each object X, the functor $- \otimes X^\vee$ is left adjoint to $- \otimes X$ and $X^\vee \otimes -$ is right adjoint to $X \otimes -$.

Note that if $(\mathcal{C}, \otimes)$ is a rigid tensor category then the unit and counit of adjunction give maps $\varepsilon \colon X \otimes X^\vee \to \mathbf{1}$ and $\eta \colon \mathbf{1} \to X^\vee \otimes X$ called evaluation and coevaluation respectively. One can now define a **trace map** from $\mathrm{End}(X)$ to the commutative F-algebra $\mathrm{End}(\mathbf{1})$ sending $f \in \mathrm{End}(X)$ to the composition
$$\mathbf{1} \xrightarrow{f} X^\vee \otimes X \xrightarrow{c_{X^\vee, X}} X \otimes X^\vee \xrightarrow{\varepsilon} \mathbf{1}$$
The first map is induced by f using adjunction $\mathrm{Hom}(X, X) \cong \mathrm{Hom}(\mathbf{1}, X^\vee \otimes X)$. We define the **rank** of X to be the the trace of the identity, that is $\varepsilon \circ c_{X^\vee, X} \circ \eta \in \mathrm{End}(\mathbf{1})$.

Example 10.2. If F is a field and G a group then the category of $\mathrm{Rep}_F(G)$ is a rigid tensor category, where $\otimes$ is the usual tensor product,$\mathbf{1}$ is the trivial representation and $V^\vee$ is the dual. Note $\mathrm{End}(\mathbf{1}) = F$ and the rank of an object V is just the dimension of the representation.

Definition 10.3. A **tensor functor** between tensor categories $(\mathcal{C}, \otimes_{\mathcal{C}}), (\mathcal{T}, \otimes_{\mathcal{T}})$ over F is an F linear functor $\omega \colon \mathcal{C} \to \mathcal{T}$ with functorial isomorphisms
$$o_{X,Y} \colon \omega(X \otimes_{\mathcal{C}} Y) \xrightarrow{\sim} \omega(X) \otimes_{\mathcal{T}} \omega(Y) \qquad o_1 \colon \omega(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\sim} \mathbf{1}_{\mathcal{T}}$$
that are compatible with constraints a, c, u, u'.

We can finally define what a (neutral) Tannakian category is.

Definition 10.4. Suppose $(\mathcal{C}, \otimes)$ is a rigid abelian tensor category, i.e. $\mathcal{C}$ is abelian and $\otimes$ is bi-additive. Let $K = \mathrm{End}(\mathbf{1})$. A **fiber functor** for $\mathcal{C}$ is an exact, faithful tensor functor
$$\omega \colon \mathcal{C} \to \mathrm{Vect}_L$$
to the category of finite dimensional L-vector spaces, where $L \supset K$. If such a fiber functor exists, we say that $\mathcal{C}$ is **Tannakian**. If $L = K$ then we say that $\mathcal{C}$ is **neutral Tannakian**.

10.1. **Homological Motives.** The interest in Tannakian categories in algebraic geometry comes from motives, which we quickly review here. Much of the following is imprecise. For reference, one should consult the book Motives by Yves André or the article Classical Motives by A.J. Scholl. For any field k, we can define the category of pure homological motives over k with coefficients in a field F. In order to define motives, we first need to recall facts about algebraic cycles. Given a variety X over k, and a field F denote $Z^r_F(X)$ to be the free F-module generated by irreducible subvarieties of codimension r. An element of $Z_F^r(X)$ is called an algebraic cycle. A Weil cohomology theory is a functor H^* from smooth projective varieties over k to finite dimensional graded commutative F-algebras with a cycle class map $Z_F^r(X) \to H^{2r}(X)$ satisfying Poincare duality and the Kunneth formula (along with certain compatability and non degeneracy conditions). The typical examples are ℓ-adic cohomology, algebraic de Rham cohomology and if $k \subset \mathbb{C}$ singular cohomology. If we fix a Weil cohomology theory, one can define an equivalence relation called homological equivalence on $Z_F^r(X)$ by $Z \sim_H W$ if they have the same image under the cycle class map. Define $A^r(X) = Z_F^r(X)/\sim_H$. Given a map of varieties (which will mean smooth projective from now on) $f\colon X \to Y$ there are pull back maps $f^*\colon A^r(Y) \to A^r(X)$ and pushforward maps $f_*\colon A^r(X) \to A^{r+\dim(Y)-\dim(X)}(Y)$. There is also a product structure $A^r(X) \otimes A^s(X) \to A^{r+s}(X)$, denoted by $Z \otimes W \mapsto Z \cdot W$. For connected varieties X, Y we define the correspondences of degree r to be $\mathrm{Corr}^r(X,Y) = A^{\dim(X)+r}(X \times Y)$. We can define a composition law $\mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^s(Y,Z) \to \mathrm{Corr}^{r+s}(X,Z)$ given on cycles $Z \in \mathrm{Corr}^r(X,Y), W \in \mathrm{Corr}^s(Y,Z)$ by

$$Z \circ W = (p_{XZ})_*(p_{XY}^* Z \cdot p_{YZ}^* W)$$

where the p maps are the various projections from $X \times Y \times Z$. We may finally define the category of pure homological motives $\mathcal{M}_k$ over F. The objects are triples (X, p, r) where X is a (smooth projective) variety over k, $p \in \mathrm{Corr}^0(X,X)$ is an idempotent correspondence and r is an integer. One should think of a motive as a 'chunk of cohomology', corresponding to the image $pH^*(X)(r)$. For example, if X is smooth and projective and $x \in X$ one can check that $\{x\} \times X$ induces projection $H^*(X) \to H^0(X)$ for any Weil cohomology theory. In this case we say that the H^0 projector is algebraic. It is conjectured that all ("K unneth") projectors $H^*(X) \to H^i(X)$ are algebraic.

We still need to define maps between motives. Given another motive (Y, q, s),

$$\mathrm{Hom}_{\mathcal{M}_k}((X,p,r),(Y,q,s)) = p\mathrm{Corr}^{s-r}(X,Y)q$$

Composition is given by composition of correspondences. There is a functor

$$h\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{M}_k, \qquad X \mapsto h(X) = (X, \Delta_X, 0)$$

from the category of smooth projective varieties over k to motives. A map $\varphi\colon X \to Y$ is sent the the correspondence ${}^t\Gamma_\varphi$ the transpose of the graph. We can define the sum of motives by

$$(X,p,r) \oplus (Y,q,r) = (X \sqcup Y, p+q, r)$$

(we let the reader find the definition of sum for $r \neq s$) and the tensor product

$$(X,p,r) \otimes (Y,q,r) = (X \times Y, p \otimes q, r+s)$$

The identity motive is $\mathbf{1} = (\mathrm{Spec} k, \mathrm{id}, 0)$. With the obvious associativity, commutativity and unit isomorphisms, $\mathcal{M}_k$ is a tensor category. If we define the dual of a motive (X,p,r), where X is purely d dimensional, as

$$(X,p,r)^\vee = (X, {}^tp, d-r)$$

then $\mathcal{M}_k$ is a rigid tensor category.

The hope is that $\mathcal{M}_k$ is a neutral Tannakian category. In order to show this, two things must be true, first that $\mathcal{M}_k$ is abelian and second, that there is a fiber functor. Focusing on the latter part, a natural choice of fiber functor would be H^* where H^* is a Weil cohomology theory (above we stated that Weil cohomology theories are functors from the category of smooth projective varieties. The idea behind motives are to give a sort of 'universal cohomology theory', and indeed there is a unique way to extend H^* to $\mathcal{M}_k$). Lets see if H^* is a tensor functor, which boils down to commutativity of the following diagram.

$$\begin{tikzcd} H^*(X) \otimes H^*(Y) \arrow[r, "\sim"] \arrow[d, "c_{\mathrm{Vect}_F}"] & H^*(X \times Y) \arrow[d, "c_{\mathcal{M}_k}"] \\ H^*(Y) \otimes H^*(X) \arrow[r, "\sim"] & H^*(Y \times X) \end{tikzcd}$$

Tracing around the top, a tensor $\alpha \otimes \beta \in H^i(X) \otimes H^j(Y)$ gets sent to $p^*\alpha \cup q^*\beta$ where $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ are the projections. Going around the bottom, $\alpha \otimes \beta$ maps to $q^*\beta \otimes p^*\alpha$. Hence the diagram only commutes up to the sign $(-1)^{ij}$! So in order for H^* to be a fiber functor, one must modify the commutativity constraint in $\mathcal{M}_k$ degree by degree and in order to perform this, one needs to know that *all* projectors $H^*(X) \to H^i(X)$ are algebraic. This is the Kunneth standard conjecture.

Example 10.5. We will show (or at least provide evidence) that motives over finite fields k (with the possible exception of $\mathbb{F}_p \ldots$) with coefficients in $\mathbb{Q}$ is not a *neutral* Tannakian category, even though it is conjecturally Tannakian. Again, this discussion is imprecise and many details still should be checked. Consider E a supersingular elliptic curve such that $\mathrm{End}(E)$ is a quaternion algebra (here we mean endomorphisms of the elliptic curve over k itself–this is why $k = \mathbb{F}_p$ doesn't work in this example). We wish to compute the rank of the motive $h^1(E) = (E, p, 0)$ where $p = \Delta_E - \{0\} \times E - E \times \{0\}$ (this is the H^1 projector). For motives over finite fields the Kunneth standard conjecture is known to be true, so we can modify the commutativity constraint in such a way so that any Weil cohomology theory H^* is a tensor functor. The benefit of this is to aid in computing the rank of the motive $h^1(E)$. Now one can check that a tensor functor between rigid tensor categories will preserve rank so $\mathrm{rank}\, h^1(E) = \mathrm{rank}\, H^*(h^1(E)) = \mathrm{rank}\, H^1(E)$ for any Weil cohomology theory. If we take H^* to be ℓ-adic cohomology then $\mathrm{rank}\, H^1(E, \mathbb{Q}_\ell)$ is the dimension of $H^1(E, \mathbb{Q}_\ell)$ which is just 2. Hence $h^1(E)$ has rank 2. If $\mathcal{M}_k$ is neutral Tannakian, then there exits a fiber functor

$$\omega \colon \mathcal{M}_k \to \mathrm{Vect}_{\mathbb{Q}}$$

From the above discussion, $\omega(h^1(E))$ is a 2 dimensional rational representation. Since ω is faithful, $\mathrm{End}(E)$ embeds into $\mathrm{End}(h^1(E))$. In other words, we have a faithful 2 dimensional rational representation of a quaternion algebra. This is a contradiction since no such representation can exist.

11. The main theorem of neutral Tannakian categories (notes by Shiang Tang)

Throughout this section, G will be an affine group scheme over a field k. We denote by $\mathrm{Rep}(G)$ its category of finite-dimensional representations, and we let Vec_k denote the category of finite-dimensional k-vector spaces. **Basic Question**: How to recognize a category with structure as $\mathrm{Rep}(G)$ for some affine group scheme G?

First, we ask the opposite question: Fix a field k, and fix an affine group scheme G/k, to what extent is G determined by $\mathrm{Rep}(G)$?

Example 11.1. Let G be a finite group, regarded as a group scheme over $\mathbb{C}$. Then $\mathrm{Rep}(G)$ is an abelian category, better yet a semi-simple category. As an abelian category,

$$\mathrm{Rep}(G) \cong \bigoplus_{W \text{ irreducible } G\text{-reps up to isomorphism}} \mathrm{Vec}_{\mathbb{C}}$$

(Note that there are no nonzero maps between different summands.). This equivalence results by observing that for all $V \in \mathrm{Rep}(G)$,

$$V = \bigoplus_{\text{irreducible } G\text{-reps } W} (W\text{-isotypic components of } V)$$

(each component is canonically isomorphic to $\mathrm{Hom}_G(W, V) \otimes W$), and we have

$$\mathrm{Hom}_G(W^{\otimes n}, W^{\otimes m}) \cong \mathrm{M}_{m,n}(\mathbb{C}).$$

Thus we see that the underlying *abelian* category $\mathrm{Rep}(G)$ does not remember G.

Key idea: We can recover G from the *tensor abelian category* $\mathrm{Rep}(G)$.

Exercise 11.2. (1) Find two non-isomorphic finite groups that have "isomorphic character tables"?

(2) Interpret part (1) in light of the key idea.

Proof. First, we make it precise what it means by two groups having isomorphic character tables: Given a finite group G, denote by $\mathrm{Conj}(G)$ the set of conjugacy classes, and denote by $\mathrm{Char}(G)$ the set of irreducible characters. We say two finite groups G, G' have isomorphic character table if there are bijections $\varphi : \mathrm{Conj}(G) \to \mathrm{Conj}(G')$, $\alpha : \mathrm{Char}(G) \to \mathrm{Char}(G')$ such that $\forall c \in \mathrm{Conj}(G), \forall \chi \in \mathrm{Char}(G)$, $\chi(c) = \alpha(\chi)(\phi(c))$. For instance, it is easy to see that D_4 and Q_8 have isomorphic character tables: both of which have four one-dimensional character and a unique irreducible two-dimensional character. However, there difference will be seen immediately once we consider the alternating squares of their two-dimensional representations: one is isomorphic to the trivial character, the other is not. Therefore, $\mathrm{Rep}(D_4)$ is not isomorphic to $\mathrm{Rep}(Q_8)$ as abelian tensor categories, despite that D_4 and Q_8 have isomorphic character tables. $\square$

Definition 11.3. Let $\mathrm{Aut}^{\otimes}(\omega)$ be the functor from the category of k-algebras to the category of groups defined by:

$$R \mapsto \mathrm{Aut}^{\otimes}(\omega)(R),$$

where $\mathrm{Aut}^{\otimes}(\omega)(R)$ consists of elements $\lambda = (\lambda_X)_{X \in \mathrm{Rep}(G)}$ which is a collection of R-linear automorphisms $\lambda_X : \omega(X) \otimes_k R \to \omega(X) \otimes_k R$, for all $X \in \mathrm{Rep}(G)$, satisfying

(1) $\forall \phi : X \to Y$ in $\mathrm{Rep}(G)$, the diagram
$$\begin{tikzcd} \omega(X) \otimes_k R \arrow[r, "\phi"] \arrow[d, "\lambda_X"] & \omega(Y) \otimes_k R \arrow[d, "\lambda_Y"] \\ \omega(X) \otimes_k R \arrow[r, "\phi"] & \omega(Y) \otimes_k R \end{tikzcd}$$
commutes.

(2) For $X =$ trivial rep, $\lambda_X = \mathrm{id}$.

(3) $\lambda_{\omega(X_1)\otimes\omega(X_2)} = \lambda_{\omega(X_1)} \otimes \lambda_{\omega(X_2)}$, i.e., the following diagram commutes:

$$\begin{tikzcd}
(\omega(X_1)\otimes\omega(X_2))\otimes R \arrow[r, "\lambda_{\omega(X_1)\otimes\omega(X_2)}"] \arrow[d, "\cong"] & (\omega(X_1)\otimes\omega(X_2))\otimes R \arrow[d, "\cong"] \\
(\omega(X_1)\otimes R)\otimes(\omega(X_2)\otimes R) \arrow[r, "\lambda_{\omega(X_1)}\otimes\lambda_{\omega(X_2)}"] & (\omega(X_1)\otimes R)\otimes(\omega(X_2)\otimes R)
\end{tikzcd}$$

Theorem 11.4. *Let k be any field, and let G/k be an affine group scheme. Let*

$$\omega : \mathrm{Rep}(G) \to \mathrm{Vec}_k$$

be the forgetful functor where $\mathrm{Rep}(G)$ is the category of finite-dimensional representations of G and Vec_k is the category of finite-dimensional vector spaces over k. Then the natural map

$$G \to \mathrm{Aut}^{\otimes}(\omega)$$

of functors from the category of k-algebras to the category of groups is an isomorphism. Here the natural map is defined as follows: for any k-algebra R,

$$G(R) \to \mathrm{Aut}^{\otimes}(\omega)(R)$$
$$g \mapsto \lambda_g = (X \otimes R \to X \otimes R)_{X\in\mathrm{Rep}(G)}$$

where the linear automorphism $X \otimes R \to X \otimes R$ is induced by the action of g on X.

Corollary 11.5. *Let G, G' be affine group schemes over k. Suppose*

$$F : \mathrm{Rep}(G') \to \mathrm{Rep}(G)$$

is a tensor functor such that

$$\begin{tikzcd}
\mathrm{Rep}(G') \arrow[rr, "F"] \arrow[dr, "\omega_{G'}"] & & \mathrm{Rep}(G) \arrow[dl, "\omega_G"] \\
& \mathrm{Vec}_k &
\end{tikzcd}$$

commutes. Then there exists a unique homomorphism $G \to G'$ inducing F.

Proof. $F : \mathrm{Rep}(G') \to \mathrm{Rep}(G)$ induces a natural transformation $\Phi : \mathrm{Aut}^{\otimes}(\omega_G) \to \mathrm{Aut}^{\otimes}(\omega_{G'})$ given by

$$\Phi(R) : \mathrm{Aut}^{\otimes}(\omega_G)(R) \to \mathrm{Aut}^{\otimes}(\omega_{G'})(R)$$
$$\lambda \mapsto (\lambda_{F(X)})_{X\in\mathrm{Rep}(G')}$$

We need to check that the map is well-defined, i.e., $\lambda' := (\lambda_{F(X)})_{X\in\mathrm{Rep}(G')}$ lands in $\mathrm{Aut}^{\otimes}(\omega_{G'})(R)$. Item (1) of Definition 1.3 holds since $\omega_G(F(X)) = \omega_{G'}(X)$ by assumption; item (2) obviously holds; since F is a tensor functor, $\lambda'_X \otimes \lambda'_Y = \lambda_{F(X)} \otimes \lambda_{F(Y)} = \lambda_{F(X)\otimes F(Y)} = \lambda_{F(X\otimes Y)} = \lambda'_{X\otimes Y}$, Item (3) holds. $\Phi(R)$ is clearly a homomorphism. By Theorem 1.4, we have natural isomorphisms of functors $\alpha : G \to \mathrm{Aut}^{\otimes}(\omega_G)$, $\alpha' : G' \to \mathrm{Aut}^{\otimes}(\omega'_G)$. Therefore, we obtain a homomorphism $\phi := \alpha'^{-1} \circ \Phi \circ \alpha : G \to G'$. □

In a similar spirit, we can write down a dictionary between properties of G and properties of $\mathrm{Rep}(G)$.

Example 11.6. G is algebraic and of finite type over k is equivalent to $\mathrm{Rep}(G)$ has a tensor generator, i.e., there exists an object X in $\mathrm{Rep}(G)$ such that every object of $\mathrm{Rep}(G)$ is isomorphic to a subquotient of some

$$\bigoplus_{\mathrm{m,n}} \left(X^{\otimes m} \otimes (X^{\vee})^{\otimes n}\right)$$

This may fail when G is not algebraic, for instance, there is no such generator when $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group, which is a profinite group, and is not algebraic.

Let us recall that our main task is going other way: having some category $\mathcal{C}$ that we want to recognize as $\mathrm{Rep}(G)$.

Theorem 11.7 (Main Theorem)**.** *Let k be a field and let $(\mathcal{C}, \otimes)$ be a rigid abelian tensor category with* $\mathrm{End}(\mathbf{1}) = k$ *(*$\mathbf{1}$ *is the unit object) and suppose that there exists a k-linear exact faithful tensor functor (called a "fiber functor")*

$$\omega : \mathcal{C} \to \mathrm{Vec}_k$$

Then

(1) The functor $\mathrm{Aut}^{\otimes}(\omega)$ *from the category of k-algebras to the category of groups is representable by an affine group scheme G over k.*

(2) ω induces an equivalence

$$\mathcal{C} \xrightarrow{\sim} \mathrm{Rep}(G)$$

of tensor categories.

Proof. For a proof, see [Deligne-Milne, 2.11-2.16]. For part (2), ω induces a functor to $\mathrm{Rep}(G)$ since for all $X \in \mathcal{C}$, $\omega(X)$ acquires a G-action from the isomorphism $G \cong \mathrm{Aut}^{\otimes}(\omega)$. □

12. p-adic Groups (notes by Kevin Childers)

12.1. **Introduction.** We've seen the finite dimensional algebraic representation theory of compact Lie groups and reductive algebraic groups. A natural next step in representation theory is to study the representation theory of *real* Lie groups (e.g. $\mathrm{SL}_n(\mathbb{R})$) on Hilbert spaces (e.g. $L^2(\mathrm{SL}_n(\mathbb{R}))$ under the regular representation).

In parallel, we could study the representation theory of groups such as $\mathrm{SL}_n(\mathbb{Q}_p)$. Both the real representation theory and the p-adic representation theory arises naturally in number theory in the theory of modular forms.

A (semi-)classical perspective on modular forms, is to study the $\mathrm{SL}_2(\mathbb{R})$-representation $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$. A more modern approach is to package all completions of $\mathbb{Q}$ into the **ring of adeles**:

$$\mathbb{A}_{\mathbb{Q}} := \prod_v^{*} \mathbb{Q}_v = \{(x_v)_v : x_v \in \mathbb{Z}_v \text{ for all but finitely many } v\}.$$

Here v ranges through all places of $\mathbb{Q}$ and "$\prod^*$" denotes a restricted product with respect to $\mathbb{Z}_v \subset \mathbb{Q}_v$ for $v \neq \infty$ (the right hand side is the definition of restricted product in this case).

The ring of adeles is a locally compact group. Now we can study the $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$-representation $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_\mathbb{Q}))$.

12.2. **Representations of** $\mathrm{GL}_n(\mathbb{F}_p)$**.** To get a feel for what type of representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ we will study, we will review the representation theory of $\mathrm{GL}_2(\mathbb{F}_p)$.

There are two types of easy representations of $\mathrm{GL}_2(\mathbb{F}_p)$. The first are the 1-dimensional representations factoring through the determinant map

$$\mathrm{GL}_2(\mathbb{F}_p) \xrightarrow{\det} \mathbb{F}_p^\times \to \mathbb{C}^\times.$$

The second, are the representations induced from the Borel subgroup, which we now review.

Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ and

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \qquad \subset \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad \supset \qquad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

as subgroups of G. Let $\chi : T \to \mathbb{C}^\times$ be a character. Such a χ can be described as a pair of characters $\chi_1, \chi_2 : \mathbb{F}_p^\times \to \mathbb{C}^\times$:

$$\chi : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \chi_1(t_1)\chi_2(t_2).$$

Such a χ extends to a character on the Borel B, and we obtain a G-representation

$$\mathrm{Ind}_B^G \chi = \{f : G \to \mathbb{C} \mid f(bg) = \chi(b)f(g)\ \forall b \in B, g \in G\}.$$

Notice that the dimension of $\mathrm{Ind}_B^G \chi$ is $|G : B| = |\mathbb{P}^1(\mathbb{F}_p)| = p + 1$.

The next question is: how do the representations induced in this way decompose into irreducible representations? Let

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \psi^w \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \psi \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$$

for any character $\psi : T \to \mathbb{C}^\times$. We have the following:

Proposition 12.1. *With G, B, and T as above, let $\chi, \psi : T \to \mathbb{C}^\times$ be characters. Then*

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \psi) = \begin{cases} 0 & \text{if } \chi \neq \psi, \psi^w, \\ 1 & \text{if } \chi = \psi \text{ or } \chi = \psi^w \text{ but } \psi \neq \psi^w, \\ 2 & \text{if } \chi = \psi = \psi^w. \end{cases}$$

In particular, $\mathrm{Ind}_B^G \chi$ *is irreducible if and only if* $\chi \neq \chi^w$*. If* $\chi = \chi^w$*, then* $\mathrm{Ind}_B^G \chi$ *is the direct sum of two distinct irreducible representations.*

Example 12.2. We give an example of the reducible case. The induction of the trivial representation is just

$$\mathrm{Ind}_B^G(1_B) = \{f : B\backslash G \to \mathbb{C}\}.$$

The constant functions give a subrepresentation, isomorphic to the trivial representation of G. We have an exact sequence

$$0 \to 1_G \to \mathrm{Ind}_B^G(1_B) \to St \to 0.$$

The quotient is an irreducible representation called the *Steinberg representation.*

Proof.

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G(\psi)) &= \mathrm{Hom}_B(\mathrm{Ind}_B^G \chi|_B, \psi) \\ &= \mathrm{Hom}_B(\bigoplus_{w' \in \{1,w\}} \mathrm{Ind}_{B\cap B^{w'}}^B \chi^{w'}, \psi) \\ &= \mathrm{Hom}_B(\chi, \psi) \oplus \mathrm{Hom}_B(\mathrm{Ind}_T^B \chi^w, \psi) \\ &= \mathrm{Hom}_T(\chi, \psi) \oplus \mathrm{Hom}_T(\chi^w, \psi).\end{aligned}$$

The first and last equality are Frobenius reciprocity, and the second inequality is the induction-restriction formula. Here we are using the fact that $1, w$ are a set of representatives for $B\backslash G/B$, and $B \cap B^w = T$ (B^w consists of lower triangular matrices). ☐

We will be studying the analogue of these representations on p-adic groups (e.g. $\mathrm{GL}_2(\mathbb{Q}_p)$).

Remark 12.3. We've now classified the easy representations. For the rest, we take a *non-split* maximal torus in G, induce the representations, and decompose the result. These are called the *cuspidal* representations.

Example 12.4. As an example of a non-split maximal torus, as in Remark 12.3, choose a basis of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$. Then we get a map $\mathbb{F}_{p^2} \to M_2(\mathbb{F}_p)$ by mapping α to the matrix representing "multiplication by α" with respect to the chosen basis. Taking units gives a map $\mathbb{F}_{p^2}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{F}_p)$, which is the inclusion of a maximal torus.

12.3. **Locally profinite groups.** Let K denote a non-archimedean local field (i.e. a finite extension of $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$). Our focus will be the representation theory of connected reductive groups G/K. Of course, some of the results hold more generally.

Definition 12.5. A **locally profinite group** is a Hausdorff topological group, such that every open neighborhood of the identity contains a compact open subgroup.

Exercise 12.6. Let G be a Hausdorff topological group. Show that G is profinite if and only if every open neighborhood of the identity contains a compact open *normal* subgroup.

Example 12.7. $G = \mathrm{GL}_n(\mathbb{Q}_p)$ is locally profinite. A basis of compact open subgroups at the identity is given by

$$\{1 + p^N M_n(\mathbb{Z}_p) \mid N \geq 1\}.$$

The basic class of representations of a locally profinite group G is the following.

Definition 12.8.

(1) Let G be a locally profinite group. A **smooth representation** of G is a homomorphism $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ for some $\mathbb{C}$-vector space V such that $V = \bigcup_K V^K$, where K ranges over compact open subgroups K of G. In other words, every vector $v \in V$ has open stabilizer.

(2) A smooth representation is **admissible** if V^K is finite dimensional for all compact open subgroups K of G.

Let $\mathcal{M}(G)$ denote the category of smooth representations of a locally profinite group G.

Exercise 12.9. Show that $\mathcal{M}(G)$ is an abelian category.

Example 12.10. A 1-dimensional smooth representation of a locally profinite group G is a character $\chi : G \to \mathbb{C}^\times$ such that χ is trivial on a compact open subgroup (i.e. $\ker\chi$ is open). For instance, if $G = \mathbb{Q}_p^\times$, characters $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ with $\chi|_{1+p^n\mathbb{Z}_p}$ trivial for some $n \geq 1$ are smooth. If $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is trivial on $\mathbb{Z}_p^\times$, then χ is completely determined by $\chi(p)$. These characters are called **unramified**.

However, the typical examples of interest to us will be infinite dimensional, since G will typically be non-abelian.

Exercise 12.11. What are the smooth representations when G is a compact group, for example $G = \mathrm{GL}_2(\mathbb{Z}_p)$?

Representations $\pi \in \mathcal{M}(G)$ are not necessarily semi-simple. However, we do have the following.

Lemma 12.12. *Let $\pi \in \mathcal{M}(G)$ and let K be a compact open subgroup of G. Then*

(1) $\pi|_K$ is semi-simple, and

(2) $\pi \mapsto \pi^K$ is an exact functor.

Proof of (1). Let $v \in V$. Since V is smooth, exercise 12.6 implies $v \in V^{K'}$ for some compact open normal subgroup K'. By compactness, K' has finite index in K. Thus the K-representation generated by v is finite dimensional (and semisimple, since it is a representation of the finite group K/K'). So $V|_K$ is the sum of irreducible K-representations. A standard Zorn's Lemma argument shows this implies that $V|_K$ is a *direct* sum of irreducible K-representations. ☐

12.4. **The induction functor.** In order to construct representations of a profinite group G, we want an induction functor from representations of a closed subgroup H.

Lemma 12.13 (Frobenius reciprocity)**.** *The restriction functor $\mathcal{M}(G) \to \mathcal{M}(H)$ has a right adjoint, denoted by $\mathrm{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$.*

In finite group theory, the definition of the induction functor for an H-representation π is

$$\mathrm{Ind}_H^G \pi = \{f : G \to \pi \mid f(hg) = \pi(h)f(g)\ \forall h \in H, g \in G\}.$$

The problem with this definition in our setting is that this will not in general produce smooth representations. However, there is a smoothing functor from the category of all G-representations to the category $\mathcal{M}(G)$, which sends a representation V to $\bigcup_K V^K$, where K ranges over compact open subgroups. Applying the smoothing functor to the above definition gives the

Definition 12.14. Let G be a locally profinite group and H a closed subgroup. Let $\pi \in \mathcal{M}(G)$. We define $\mathrm{Ind}_H^G \pi \in \mathcal{M}(H)$ to be

$$\mathrm{Ind}_H^G \pi = \left\{ f : G \to \pi \;\middle|\; \begin{array}{l} f(hg) = \pi(h)f(g)\ \forall h \in H, g \in G, \text{ and} \forall g \in G \exists \text{ a compact open} \\ \text{subgroup } K \text{ such that } f(gk) = f(g)\ \forall k \in K \end{array} \right\}.$$

With this definition, the proof of Frobenius reciprocity is the same as in the finite group case, with an additional check of the topological conditions.

Exercise 12.15. Show that $\mathrm{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ is an exact functor.

Example 12.16. We will now parallel the constructions given in subsection 12.2 on the locally profinite group $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let B and T denote the borel and maximal torus, parallel to the finite example.

For any smooth character $\chi : T \to \mathbb{C}^\times$ we form a smooth G-representation $\mathrm{Ind}_B^G \chi$ (this will be infinite dimensional). A similar argument to Proposition 12.1 shows

(1) If $\chi \neq \chi^w$, $\mathrm{Ind}_B^G \chi$ is irreducible.

(2) If $\chi = \chi^w$, $\mathrm{Ind}_B^G \chi$ fits in an exact sequence with two smooth irreducible representations. For example, we have an exact sequence

$$0 \to 1_G \to \mathrm{Ind}_B^G(1_B) \to St \to 0.$$

The sequence in (2) does *not* split. The problem we run into is that Ind_B^G is *not* a left adjoint to restriction, as opposed to the finite group setting.

12.5. **Representation theory of reductive groups over local fields.** Let F denote a non-archimedean local field, and let $\mathcal{O}$ denote the ring of integers of F. Let $G/\mathcal{O}$ denote a split, connected, reductive group. We will also abusively write G for $G(F)$. Let $\pi \in \mathcal{M}(G)$ be irreducible and admissible. Consider the condition that $\pi^{G(\mathcal{O})} \neq 0$.

Remark 12.17. In the global setting this happens almost everywhere locally. For example, if F is a number field, an **automorphic representation** of GL_1 over F is a continuous character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$. For any such χ, there exists a finite set S of primes such that χ is trivial on $\mathcal{O}_{F_v}^\times$ for all $v \notin S$. The reason for this is that $\prod_{v \nmid \infty} \mathcal{O}_{F_v}^\times$ is a profinite group, and any continuous homomorphism from a profinite group to $\mathbb{C}^\times$ must factor through a finite quotient. There exists an analogous result for automorphic representations on higher rank groups.

Definition 12.18. Back in the setting described before the remark, fix a compact open subgroup K of G. We say $\pi \in \mathcal{M}(G)$ is K**-spherical** if π is irreducible and $\pi^K \neq 0$.

If we let $K = G(\mathcal{O})$, do K-spherical representations exist? The answer is yes, and to construct them we use the Iwasawa decomposition.

Lemma 12.19 (Iwasawa decomposition)**.** *With notation as above, if G is split, then $G(F) = B(F)G(\mathcal{O})$.*

Proof for GL_n. I claim it suffices to show that $G(\mathcal{O})$ surjects onto $G/B(\mathcal{O})$. Recall that G/B is a projective variety, and as such we can "clear denominators" in projective coordinates to get $G/B(\mathcal{O}) = G/B(F)$. Now $G(F) \to G/B(F)$ is surjective with kernel $B(F)$, so surjection of $G(\mathcal{O})$ onto $G/B(\mathcal{O}) = G/B(F)$ will imply $G(F) = B(F)G(\mathcal{O})$. Now a flag in $G/B(\mathcal{O})$ must look like

$$0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = R^n,$$

where L_i is a free R-module of rank i, which is a direct summand of L_n. In particular, such a flag is in the image of the map $G(\mathcal{O}) \to G/B(\mathcal{O})$. $\square$

Consider a character $\chi : T(F)/T(\mathcal{O}) \to \mathbb{C}^\times$. I claim that $\mathrm{Ind}_B^G(\chi)^K \neq 0$. To see this, define $\varphi_\chi \in \mathrm{Ind}_B^G \chi$ by $\varphi_\chi(bk) = \chi(b)$, where $bk \in B(F)G(\mathcal{O}) = G(F)$. In fact, this definition is forced upon us. Note that this is well-defined, since $\chi|_{B\cap K}$ is trivial.

Remark 12.20. This example does not quite produce K-spherical representations, since $\mathrm{Ind}_B^G \chi$ is not necessarily irreducible. But as for the finite group case of subsection 12.2, $\mathrm{Ind}_B^G \chi$ will be irreducible if and only if $\chi \neq \chi^w$. We will not provide a proof of this.

If we pursued this remark further, we would arrive at the

Theorem 12.21 (Borel-Matsumoto-Casselman)**.** *With notation as above,*

1. *Any unramified principal series* $\mathrm{Ind}_B^G \chi$ *with* χ *unramified has a unique* $K = G(\mathcal{O})$*-spherical subquotient.*
2. *Any* K*-spherical representation embeds into an unramified principle series.*

This theorem provides one way to classify K-spherical representations for $K = G(\mathcal{O})$. Later we will give a different classification, which is more directly related to the Satake isomorphism.

13. Studying K-spherical representations via Hecke algebras (Notes by Sabine Lang)

For the time being, we can take G any locally profinite group, and K any compact open subgroup of G.

Recall that in finite group theory, one studies representations of G by viewing them as $\mathbb{C}[G]$-representations. We can identify $\mathbb{C}[G] \simeq \mathrm{Maps}(G, \mathbb{C})$, which is an identification of $\mathbb{C}$-algebras given by

$$\sum_{g\in G} a_g g \to (f : G \to \mathbb{C}, f(g) = a_g).$$

The multiplication on $\mathrm{Maps}(G, \mathbb{C})$ is the convolution of functions, i.e., for $f_1, f_2 : G \to \mathbb{C}$, we have $(f_1 * f_2)(g) = \sum_{x\in G} f_1(x)f_2(x^{-1}g)$. Now we will mimic this for locally profinite groups.

Definition 13.1. The Hecke algebra $\mathcal{H}(G)$ of G is the $\mathbb{C}$-algebra of functions $f : G \to \mathbb{C}$ that are locally constant with compact support. We write $\mathcal{H}(G) = \mathcal{C}_c^\infty(G)$. The multiplication is given by

$$(f_1 * f_2)(g) = \int_G f_1(x)f_2(x^{-1}g)d\mu_G(x), \text{ for all } g \in G, f_1, f_2 \in \mathcal{C}_c^\infty(G),$$

where μ_G is a Haar measure on G (more details on Haar measure will come later in the notes).

Remark 13.2. For any $f \in \mathcal{C}_c^\infty(G)$, there exists a compact open subgroup $K < G$ such that f is K-bi-invariant (i.e., f factors through $K\backslash G/K$).

Proof. Because f is locally constant, for every $x \in G$ there exists a neighborhood U_x of x such that $f\,|_{U_x}= f(x)$. Since G is locally profinite, there exists a compact open subgroup K_x such that $xK_x \subset U_x$ and $K_x x \subset U_x$ (we can take $\widetilde{K_x}$ with $\widetilde{K_x}x \subset U_x$, $\overline{K_x}$ with $x\overline{K_x} \subset U_x$ and we define $K_x = \widetilde{K_x} \cap \overline{K_x})$. Write G as $G = \cup_{x \in G} x K_x = \cup_{x \in G} K_x x$. Since f has compact support, we can extract finite subcovers of the support of f, so $\mathrm{supp}(f) = \cup_{i=1,\dots,n} x_i K_{x_i} \cap \mathrm{supp}(f) = \cup_{j=1,\dots,m} K_{y_j} y_j \cap \mathrm{supp}(f)$. By choosing $K = \cap_{i=1,\dots,n} K_{x_i} \bigcap \cap_{j=1,\dots,m} K_{y_j}$, we observe that f is K-bi-invariant. □

From the remark, we can write $\mathcal{H}(G) = \cup_K \mathcal{H}(G, K)$ where we take the union over all the compact open subgroups K of G. Here $\mathcal{H}(G, K)$ (also written $\mathcal{H}_K$) denotes the subalgebra (check this!) of K-bi-invariant functions in $\mathcal{H}(G)$.

13.1. **Overview of the translation from smooth G-representations to smooth $\mathcal{H}(G)$-modules.** We will prove the two following results (smooth $\mathcal{H}(G)$-modules will be defined later in the notes):

(1) There is an equivalence of categories
$$\mathcal{M}(G) \to \{\text{smooth } \mathcal{H}(G)\text{-modules}\}.$$

(2) For all K, there is a bijection
$$\{\text{Irreducible smooth } G\text{-rep. } \pi \text{ with } \pi^K \neq 0\} /_{\simeq} \to \{\text{Simple } \mathcal{H}_K\text{-modules}\} /_{\simeq}.$$

Now we can make this more precise. First, we will do some preliminaries on integration. Recall that any locally compact (Hausdorff) topological group has a unique (up to scaling) left- (or right-) invariant Haar measure, i.e., a functional $I : \mathcal{C}_c(G) \to \mathbb{R}$ such that

(1) $I(f) \geq 0$ for $f \geq 0$,
(2) $I(\lambda_g \cdot f) = I(f)$ for all $g \in G$ and $f \in \mathcal{C}_c(G)$, where λ_g denotes the left-regular representation of G on $\mathcal{C}_c(G)$.

We can rewrite the second condition as $\int_G f(g^{-1}x) d\mu(x) = \int_G f(x) d\mu(x)$. We also introduce the notation ρ_g for the right-regular representation of G on $\mathcal{C}_c(G)$.

But with G locally profinite, we just use the following bare-hands approach : a left-invariant Haar measure will be such a functional I defined on $\mathcal{C}_c^\infty(G)$ (and not the whole space $\mathcal{C}_c(G)$), and it is constructed as follows:

- Fix $K < G$ a compact open subgroup.
- Declare $I(\mathbb{1}_K) = 1$ for $\mathbb{1}_K$ the characteristic function of K (normalization).
- For any $K' \subset K$, write K as a finite disjoint union $K = \sqcup a_i K'$ and the left-invariance forces $I(\mathbb{1}_{K'}) = \dfrac{1}{[K : K']}$. Likewise, left-invariance forces $I(\mathbb{1}_{gK'}) = \dfrac{1}{[K : K']}$ for all $g \in G$ and for all such K'.
- Extend linearly.

This will force a unique construction of a left-invariant functional $I : \mathcal{C}_c^\infty(G) \to \mathbb{R}$.

13.2. **Relation between left and right Haar measures.** Let I be a left-invariant Haar measure, written as $I(f) = \int_G f(g) d\mu_G(g)$. Then for any $g \in G$, we consider $I(\rho_g \cdot f) = \int_G f(xg) d\mu_G(x)$. Since left and right regular representations commute, the map that sends f to $I(\rho_g \cdot f)$ is also a left-invariant Haar measure. By uniqueness (up to scaling) of a left-invariant Haar measure on G, there exists $\delta_G(g) \in \mathbb{R}_{>0}$ such that $\delta_G(g) I(\rho_g \cdot f) = I(f)$ for all f.

We can check that $\delta_G : G \to \mathbb{R}_{>0}$ is a smooth character : we apply it to $g_1 \cdot g_2$ as follows :

$$\begin{aligned} I(f) &= \delta_G(g_1 \cdot g_2) I(\rho_{g_1 \cdot g_2} f) = \delta_G(g_1 \cdot g_2) I(\rho_{g_1} \rho_{g_2} f) \\ &= \delta_G(g_1 \cdot g_2) \delta_G(g_1)^{-1} I(\rho_{g_2} f) = \delta_G(g_1 \cdot g_2) \delta_G(g_1)^{-1} \delta_G(g_2)^{-1} I(f), \end{aligned}$$

so we obtain $\delta_G(g_1 \cdot g_2) = \delta_G(g_1)\delta_G(g_2)$. To check that δ is smooth, fix any compact open subgroup K. For $k \in K$,

$$I(\mathbb{1}_K) = I(\rho_k \mathbb{1}_K) = \delta_G(k)^{-1} I(\mathbb{1}_K),$$

so δ_G is trivial on K (note that if we take a subgroup K of G which is not compact open, then $\mathbb{1}_K$ is not an element of $\mathcal{C}_c^\infty(G)$, so this argument does not apply). We call δ_G the modulus character of G.

Example 13.3.
- As we have just seen, if G is compact, then δ_G is trivial.
- If $G = G(F)$ is reductive, then δ_G is trivial.
- A typical example with δ_G non-trivial arises when we take G to be a Borel subgroup.

Moreover, δ_G measures the difference between the left- and right-invariant Haar measures. Define $I'(f) = I(\delta_G^{-1} \cdot f)$. Then I' is a right-invariant Haar measure :

$$I'(\rho_g \cdot f) = I(\delta_G^{-1} \cdot \rho_g \cdot f) = I(\delta_G(g) \cdot \rho_g(\delta_G^{-1} \cdot f)) = \delta_G(g) I(\rho_g(\delta_G^{-1} \cdot f)) = I(\delta_G^{-1} \cdot f) = I'(f).$$

Now we return to the setting of interest with F a non-archimedean local field, $\mathcal{O}$ its ring of integers, $G = G(F)$ and $K = G(\mathcal{O})$. One of our aims is to classify the K-spherical representations of G.

How? One way, as in Theorem 12.21, is to understand how to find all K-spherical representations inside $\mathrm{Ind}_B^G(\chi)$ for $\chi : T(F)/T(\mathcal{O}) \to \mathbb{C}^\times$. We will take a less explicit approach to the classification, but one that arises directly from the Satake isomorphism. We first point out one more thing about unramified principal series that was omitted last time.

Remark 13.4. (A subtlety arising from the failure of unimodularity of B.) We saw that $\mathrm{Ind}_B^G(\chi)^K$ is 1-dimensional in this case but $\mathrm{Ind}_B^G(\chi)$ is not necessarily irreducible. For example, for $G = \mathrm{GL}_2(F)$, we have the short exact sequence

$$0 \to \mathbb{1} \to \mathrm{Ind}_B^G(\mathbb{1}) \to \mathrm{St} \to 0,$$

which does not split, and when we dualize it we get

$$0 \to \mathrm{St} \to \mathrm{Ind}_B^G(\mathbb{1})^\vee \to \mathbb{1} \to 0,$$

where St is the Steinberg representation. Here we do NOT have $\mathrm{Ind}_B^G(\mathbb{1})^\vee \simeq \mathrm{Ind}_B^G(\mathbb{1})$. We have to be careful : in general, $\mathrm{Ind}_B^G(\chi)^\vee \simeq \mathrm{Ind}_B^G(\chi^{-1} \cdot \delta_B^{-1})$.

Pursuing this, we get that $\mathrm{Ind}_B^G(\chi)$ is irreducible either when $\chi_1 = \chi_2$ or when $\chi_1 = \chi_2 \cdot |\ |^2$, where $\chi = (\chi_1, \chi_2)$. We have a G-equivariant pairing

$$\mathrm{Ind}_B^G(\chi) \times \ \mathrm{Ind}_B^G(\chi^{-1}\delta_B^{-1}) \to \mathbb{C}$$

given by

$$(f_1, f_2) \mapsto \int_{B\backslash G} f_1(g) f_2(g) d\mu_{B\backslash G}(g),$$

which implies that $\mathrm{Ind}_B^G(\chi)^\vee \simeq \mathrm{Ind}_B^G(\chi^{-1}\cdot\delta_B^{-1})$. The subtelty here is that we can only define a right-invariant measure $\mu_{B\backslash G}$ on the set $\{f : G \to \mathbb{C} \mid f(bg) = \delta_B(b)^{-1} f(g), f$ locally constant with compact support$\}$.

Remark 13.5. Here $(\ )^\vee$ means the smooth dual, i.e., the smooth vectors in the algebraic dual.

Now, we need to learn how to integrate over B for B a Borel subgroup. (Note that the following discussion works equally well for a parabolic $P = M \ltimes N$ instead of B.)

We write B as the semidirect product $B = T \ltimes N$, which is isomorphic to $T \times N$ as a topological space. But T and N are both unimodular (i.e., δ_T and δ_N are trivial). Fix both left- and right-invariant Haar measures I_N and I_T on $\mathcal{C}_c^\infty(N)$ and $\mathcal{C}_c^\infty(T)$. We use them to define a left-invariant Haar measure in B by $I_B(f) =$"$\int_B f(b)db = \int_T(\int_N f(tn)dn)dt$". The following commutative diagram makes sense of this notation :

$$\begin{tikzcd}
\mathcal{C}_c^\infty(T)\otimes\mathcal{C}_c^\infty(N) \arrow[r, "I_N"] \arrow[d, "\simeq"] & \mathcal{C}_c^\infty(T) \arrow[d, "I_T"] \\
\mathcal{C}_c^\infty(B) \arrow[r, "I_B"] & \mathbb{C}
\end{tikzcd}$$

and the isomorphism

$$\mathcal{C}_c^\infty(T)\otimes\mathcal{C}_c^\infty(N) \to \mathcal{C}_c^\infty(B)$$

is given by

$$\phi\otimes\psi \mapsto (b = tn \mapsto \phi(t)\psi(n)).$$

We observe that I_B is left-invariant. It suffices to show that $I_B(\lambda_{(t_0,n_0)^{-1}}\phi\otimes\psi) = I_B(\phi\otimes\psi)$, for all $\phi\in\mathcal{C}_c^\infty(T)$ and $\psi\in\mathcal{C}_c^\infty(N)$. Analyzing $\lambda_{(t_0,n_0)^{-1}}\phi\otimes\psi$, we get

$$(\lambda_{(t_0,n_0)^{-1}}\phi\otimes\psi)(t,n) = (\phi\otimes\psi)((t_0,n_0)\cdot(t,n)) = (\phi\otimes\psi)(t_0t, t^{-1}n_0tn) = \phi(t_0t)\psi(t^{-1}n_0tn).$$

Using the fact that $t^{-1}n_0t$ is an element of N, and the invariance of both I_N and I_T, we obtain $I_B(\lambda_{(t_0,n_0)^{-1}}\phi\otimes\psi) = I_B(\phi\otimes\psi)$.

We want to use this definition of the left-invariant measure I_B to calculate the corresponding modulus character δ_B. Recall that it is defined by $\delta_B(b)I_B(\rho_b\cdot f) = I_B(f)$. Fix now $b_0 = t_0n_0$. We compute

$$I_B(\rho_{b_0}\cdot f) = \int_T\int_N f(tnt_0n_0)dndt = \int_T\int_N f(tt_0t_0^{-1}nt_0n_0)dndt,$$

and writing $f = \phi \otimes \psi$ we get

$$I_B(\rho_{b_0} \cdot f) = \int_T \phi(tt_0)dt \int_N \psi(t_0^{-1}nt_0n_0)dn = I_T(\phi) \int_N \psi(t_0^{-1}nt_0)dn,$$

using the fact that both I_T and I_N are unimodular. Note that we cannot use the invariance of I_N to simplify $\int_N \psi(t_0^{-1}nt_0)dn$ since we are multiplying by elements of T, and not elements of N.

Remark 13.6. For a fixed $t_0 \in T$, the functional sending ψ to $\int_N \psi(t_0^{-1}nt_0)dn$ is a Haar measure on N. By unicity (up to scaling) of a Haar measure, we obtain $\int_N \psi(t_0^{-1}nt_0)dn = cst(t_0)I_N(\psi)$. Here $cst(t_0)$ does not depend on ψ, so we can choose any ψ we like to compute this quantity.

We observe that $I_B(\rho_{b_0} \cdot f) = I_T(\phi)cst(t_0)I_N(\psi) = cst(t_0)I_B(f)$, so $\delta_B(b_0) = cst(t_0)^{-1}$.

Example 13.7. Take $G = \mathrm{GL}_2(F)$, $t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ and $\psi = \mathbb{1}_{N(\mathcal{O})}$. In this case, $N(\mathcal{O}) = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathcal{O}\}$. We may assume the normalization $I_N(\mathbb{1}_{N(\mathcal{O})}) = 1$. We want to compute $\int_{G(F)} \mathbb{1}_{N(\mathcal{O})}(\begin{pmatrix} 1 & x\frac{t_2}{t_1} \\ 0 & 1 \end{pmatrix})dx = \int_F \mathbb{1}_{\frac{t_1}{t_2}\mathcal{O}}(x)dx$. So the question is : given that $\mathcal{O}$ has volume 1, what volume does $\frac{t_1}{t_2}\mathcal{O}$ have?

Answer : $|\frac{t_1}{t_2}|$, where $|\ |$ is normalized as $(\#\frac{\mathcal{O}}{\pi\mathcal{O}})^{-\nu(\frac{t_1}{t_2})}$ with $\nu : \mathcal{O} \twoheadrightarrow \mathbb{Z}$. So in this case, $\delta_B(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}) = |\frac{t_1}{t_2}|^{-1} = |\frac{t_2}{t_1}|$.

More generally, for a group G, we obtain $\delta_B(t) = |\det(\mathrm{Ad}(t)\ |_{\mathfrak{n}})|^{-1}$.

13.3. **Back to Hecke algebras.** For now, G can be any locally profinite group. Recall :

$$\mathcal{H}(G) = \mathcal{C}_c^\infty(G) = \cup_K \mathcal{H}(G, K),$$

where the union is taken over the compact open subgroups K of G, and $\mathcal{H}(G, K) = \mathcal{H}_K$ is the subspace of K-bi-invariant functions. We observe that $\mathcal{H}_K$ is a subalgebra of $\mathcal{H}(G)$, and $\mathcal{H}_K = e_K * \mathcal{H}(G) * e_K$ where $e_K = \frac{\mathbb{1}_K}{\mathrm{vol}(K)}$. We can check that the e_K are idempotents : for $g \in G$, we have

$$(e_K * e_K)(g) = \int_G e_K(x)e_K(x^{-1}g)d\mu_G(x) = \frac{1}{\mathrm{vol}(K)} \int_K e_K(x^{-1}g)d\mu_G.$$

Since we consider now $x \in K$, this expression is zero if $x^{-1}g \notin K$, i.e., if $g \notin K$. In the case where $g \in K$, we have

$$(e_K * e_K)(g) = \frac{1}{\mathrm{vol}(K)} \int_K e_K(x^{-1}g)d\mu_G = (\frac{1}{\mathrm{vol}(K)})^2\mathrm{vol}(K) = \frac{1}{\mathrm{vol}(K)},$$

so we conclude that $e_K * e_K = \frac{\mathbb{1}_K}{\mathrm{vol}(K)} = e_K$.

13.4. **Construction of the equivalence $\mathcal{M}(G) \to \{$ smooth $\mathcal{H}(G)-$ modules$\}$.** Let $(\pi, V) \in \mathcal{M}(G)$. Let $f \in \mathcal{H}(G)$, and let $v \in V$. We define an action of $\mathcal{H}(G)$ on V by

$$\pi(f) \cdot v = \int_G f(g)\pi(g)(v)dg.$$

Exercise 13.8. The expression $\pi(f) \cdot v$ can be rewritten as a finite sum (Hint : use the fact that f is locally constant with compact support, and that π is a smooth representation.)

Exercise 13.9. This defines an action. Note that there is no unit in $\mathcal{H}(G)$, so there is no unit condition to check : we only want the associativity. So we need to check that $\pi(f_1 * f_2) \cdot v = \pi(f_1) \cdot (\pi(f_2) \cdot v)$, which can be done by developing both sides explicitely and using the left-invariance of the Haar measure on G.

What kinds of $\mathcal{H}(G)$-modules do we obtain?

Definition 13.10. An $\mathcal{H}(G)$-module V is smooth if for all $v \in V$, there exists a compact open subgroup K of G such that $e_K \cdot v = v$.

Proposition 13.11. *The map $\mathcal{M}(G) \to \{$ smooth $\mathcal{H}(G) -$ modules$\}$ is an equivalence of categories.*

Proof. First, we need to check that $(\pi, V) \in \mathcal{M}(G)$ gives a smooth $\mathcal{H}(G)$-module: we know that for each $v \in V$, there exists a compact open subgroup K of G with $\pi^K \neq 0$. Then we can check that $\pi(e_K) \cdot v = \int_G \frac{\mathbb{1}_K}{\mathrm{vol}(K)}(g)\pi(g)(v)dg = \int_K \frac{1}{\mathrm{vol}(K)}\pi(g)(v)dg = \pi(g)(v) = v$ since $g \in K$ from the previous equality and v is an element of π^K.

Then, we need to give a quasi-inverse map. Given a smooth $\mathcal{H}(G)$-module V, let $v \in V$ and $g \in G$. We have to define $\pi(g)$. Choose any K such that $e_K \cdot v = v$ (such a K exists by smoothness of V). Then we set $\pi(g) \cdot v = e_{gK} \cdot v$. We need to check that this is well-defined.

Let K' be another subgroup with $e_{K'} \cdot v = v$. We can reduce to the case where $K' \subset K$. Now if K' is a subgroup of K, we have

$$(e_{gK'} * e_K)(y) = \int_G \frac{\mathbb{1}_{gK'}(x)}{\mathrm{vol}(gK')} \frac{\mathbb{1}_K(x^{-1}y)}{\mathrm{vol}(K)} dx = \int_{gK'} \frac{\mathbb{1}_K(x^{-1}y)}{\mathrm{vol}(gK')\mathrm{vol}(K)} dx,$$

so we have here $x^{-1} \in K'g^{-1}$ and we get

$$(e_{gK'} * e_K)(y) = 0 \text{ if } y \notin gK,$$

$$(e_{gK'} * e_K)(y) = \frac{1}{\mathrm{vol}(K)} = \frac{1}{\mathrm{vol}(gK)} \text{ if } y \in gK,$$

so we have shown that $e_{gK'} * e_K = e_{gK}$.

In particular, $e_{gK'} \cdot v = (e_{gK'} * e_K) \cdot v = e_{gK} \cdot v$, since $e_K \cdot v = v$ by choice of K. So this is a well-defined action, and checking the equivalence is now easy. □

Our next goal is to show that we have the following bijection, for any K:

$$\{\text{Irreducible smooth } G\text{-rep. } (\pi, V) \text{ with } \pi^K \neq 0\} /_{\simeq} \to \{\text{Irreducible } \mathcal{H}_K\text{-modules}\} /_{\simeq}$$

induced by

$$V \mapsto V^K.$$

Lemma 13.12. *For all* $(\pi, V) \in \mathcal{M}(G)$, $\pi(e_K) : V \to V^K$ *is a projection. In particular,* $\mathcal{H}_K = e_K * \mathcal{H}(G) * e_K$ *acts on* V^K.

Proof. We already saw that e_K is an idempotent. The next exercises shows that the image of V by $\pi(e_K)$ lies indeed in V^K and concludes the proof. □

Exercise 13.13. Check that for all $k \in K$, we have $\pi(k) \cdot \pi(e_K)(v) = \pi(e_K)(v)$. (Hint: use a change of variable.)

Proposition 13.14. *The map sending* V *to* V^K *is a bijection on isomorphism classes of irreducible objects, as claimed.*

Proof. First, we check that if (π, V) is irreducible, then V^K is an irreducible $\mathcal{H}_K$-module. Since we consider our map as

$$\{\text{Irreducible smooth } G\text{-rep. } (\pi, V) \text{ with } \pi^K \neq 0\} /_{\simeq} \to \{\text{Irreducible } \mathcal{H}_K\text{-modules}\} /_{\simeq},$$

then $\pi^K \neq 0$. Suppose $0 \subsetneq M \subsetneq V^K$ is a proper $\mathcal{H}_K$-submodule. Then by definition of $M \subset V^K$ we have $M = \pi(e_K)M$, and $\mathcal{H}(G)M = V$ by the previous equivalence and the irreducibility of V. So we get

$$V^K = \pi(e_K) \cdot V = \pi(e_K)\mathcal{H}(G) \cdot M = \pi(e_K)\mathcal{H}(G)\pi(e_K) \cdot M = \mathcal{H}_K \cdot M = M,$$

which is a contradiction since we assumed $M \subsetneq V^K$. This implies that V^K is irreducible.

Now we construct a quasi-inverse. Given an irreducible $\mathcal{H}_K$-module M, then there exists an ideal I of $\mathcal{H}_K$ such that $M \simeq \mathcal{H}_K /_I$. We now consider $I \subset \mathcal{H}_K \subset \mathcal{H}(G)$ and form the $\mathcal{H}(G)$-submodule generated by I inside $\mathcal{H}(G)$, call it $\widetilde{I}$. We also denote by $\widetilde{\mathcal{H}_K}$ the $\mathcal{H}(G)$-module generated by $\mathcal{H}_K$. The quotient $\widetilde{\mathcal{H}_K} /_{\widetilde{I}}$ is a $\mathcal{H}(G)$-module but it might be too big. Using Zorn's lemma, we can take V to be any non-zero irreducible quotient, i.e., $\widetilde{\mathcal{H}_K} /_{\widetilde{I}} \twoheadrightarrow V$.

First, we check that V^K is non-zero : we apply $(\ )^K$ to the short exact sequence

$$0 \to U \to \widetilde{\mathcal{H}_K} /_{\widetilde{I}} \to V \to 0$$

and by exactness of $(\ )^K$ we have the short exact sequence

$$0 \to U^K \to (\widetilde{\mathcal{H}_K} /_{\widetilde{I}})^K \to V^K \to 0.$$

But we know that $(\widetilde{\mathcal{H}_K} /_{\widetilde{I}})^K = \mathcal{H}_K /_I = M$ by definition. So the exact sequence is

$$0 \to U^K \to M \to V^K \to 0.$$

If $V^K = 0$ we obtain $U^K = M$, and therefore U generates $\widetilde{\mathcal{H}_K} /_{\widetilde{I}}$. But this implies that $V = 0$, which is a contradiction. So we have $V^K \neq 0$ and by irreducibility of M we see that $V^K = M$ as desired. □

13.5. **A couple of loose ends.** For G a locally profinite group, if G/K is countable for some compact open K then :

(1) Any smooth $(\pi, V) \in \mathcal{M}(G)$ has countable dimension.

(2) Schur's lemme holds : if (π, V) is irreducible, then $\mathrm{End}_G(\pi) = \mathbb{C}$.

Proof. We will leave (1) as an exercise and prove (2). Since π is irreducible, $\mathrm{End}_G(\pi)$ is a divison algebra over $\mathbb{C}$, not a priori finite dimensional. Fix $v \in V \setminus \{0\}$. Any $\phi \in \mathrm{End}_G(\pi)$ is then determined by $\phi(v)$ since v generates (π, V). So by (1), $\mathrm{End}_G(\pi)$ has countable dimension. If $\phi \notin \mathbb{C}$, then we consider (note that $\mathbb{C}(\phi)$ is a field since $\mathrm{End}_G(\pi)$ is a division algebra)

$$\{\frac{1}{\phi - a} \mid a \in \mathbb{C}\} \subset \mathbb{C}(\phi).$$

All the elements $\frac{1}{\phi-a}$ are linearly independant over $\mathbb{C}$, and there are uncountably many of these. So $\mathrm{End}_G(\pi)$ would have uncountable dimension, which is a contradiction. This shows that $\mathrm{End}_G(\pi) = \mathbb{C}$. □

14. Statement of classical Satake isomorphism

Shiang Tang

Recall that if K is a compact open subgroup of G that is locally profinite, the set of isomorphism classes of smooth irreducible G-representations π such that $\pi^K \neq 0$ is equivalent to the set of isomorphism classes of irreducible $\mathcal{H}_K$-modules π^K. The case of basic interest to us is the following situation: F is a local field with ring of integers $\mathcal{O}$ and a uniformizer ϖ. $\mathbf{G}$ is a split connected reductive group scheme defined over $\mathcal{O}$, $G = \mathbf{G}(F)$, $K = \mathbf{G}(\mathcal{O})$. In this case, $\mathcal{H}_K$ can be explicitly described.

The following theorem gives us the first information about $\mathcal{H}_K$.

Theorem 14.1 (Cartan decomposition)**.** *For G, K as above,*

$$G = \bigsqcup_{\lambda \in \mathrm{X}_*(\mathrm{T})/\mathrm{W}} K\lambda(\varpi)K$$

(If we choose $B \supset T$, then we get $X_*(T)/W = X_*(T)_+$.)

Proof for GL_n. It uses the structure theory of modules over PID. Let $V = F^n \supset \Lambda = \mathcal{O}^n = \oplus_{i=1}^n e_i$. For $g \in GL_n(F)$, consider the translated lattice $g(\Lambda)$, these two lattices are commensurable so $\exists N$, such that $\varpi^N g(\Lambda) \subset \Lambda$. By theory of finite-generated $\mathcal{O}$-modules, $\exists$ a basis $f_1, \cdots, f_n$ of Λ and integers $a_1 \geq \cdots \geq a_n$ such that $\varpi^{a_1} f_1, \cdots, \varpi^{a_n} f_n$ form a basis of $\varpi^N g(\Lambda)$ and hence $\varpi^{a_1-N} f_1, \cdots, \varpi^{a_n-N} f_n$ form a basis of $g(\Lambda)$. Let B be the matrix of g under the basis $f_1, \cdots, f_n$ of Λ, so $g = SBS^{-1}$ for some $S \in GL_n(\mathcal{O})$. Since both of $gf_1, \cdots, gf_n$ and $\varpi^{a_1-N} f_1, \cdots, \varpi^{a_n-N} f_n$ are $\mathcal{O}$-bases of $g(\Lambda)$, there exists $C \in GL_n(\mathcal{O})$ such that $B = diag(\varpi^{a_1-N}, \cdots, \varpi^{a_n-N}) \cdot C$ and so $g = S \cdot diag(\varpi^{a_1-N}, \cdots, \varpi^{a_n-N}) \cdot CS^{-1} \in K diag(\varpi^{a_1-N}, \cdots, \varpi^{a_n-N})K$. □

Corollary 14.2. *$\mathcal{H}_K$ has a basis given by $\{\mathbf{1}_{K\lambda(\varpi)K}\}_{\lambda \in X_*(T)_+}$.*

Theorem 14.3 (Satake Isomorphism)**.** *There is an* ***algebra isomorphism***

$$\mathcal{H}_K \to \mathcal{H}_{T(\mathcal{O})}^W$$

where $\mathcal{H}_{T(\mathcal{O})} = \mathcal{H}(T(F), T(\mathcal{O}))$, given by

$$f \mapsto (\mathcal{S}f)(t) = \delta(t)^{1/2} \int_N f(tn)dn$$

where $\delta(t) = |\det(Ad(t)|_{\mathfrak{n}})|^{-1}$.

We make the following **important remarks**.

Remark 14.4. RHS$=H^W_{T(\mathcal{O})}$.

$$\mathbb{C}[X_*(T)] \cong C_c^\infty(T(F)/T(\mathcal{O}))$$
$$[\mu] \mapsto \mathbf{1}_{\mu(\varpi)T(\mathcal{O})}$$

is an isomorphism of algebras. So

$$\mathcal{H}_K \to \mathcal{H}^W_{T(\mathcal{O})} \cong \mathbb{C}[X_*(T)]^W = \mathbb{C}[X^*(T^\vee)]^W = R(G^\vee) \otimes \mathbb{C}$$

The isomorphism $\mathcal{H}_K \cong R(G^\vee) \otimes \mathbb{C}$ is what geometric Satake will categorify.

Remark 14.5. $\mathcal{H}_K$ is commutative!

Remark 14.6. Since $\mathcal{H}_K = H^W_{T(\mathcal{O})}$ is commutative, the following sets are in bijections:

$$\{\text{simple } \mathcal{H}_K\text{-modules}\} \leftrightarrow \{\text{1-dimensional representations of } H^W_{T(\mathcal{O})}\} \leftrightarrow$$

$$\{\mathbb{C}\text{-algebra homomorphisms: } H^W_{T(\mathcal{O})} = \mathbb{C}[X^*(T^\vee)]^W = \mathcal{O}_{T^\vee/W} \to \mathbb{C}\} \leftrightarrow (T^\vee/W)(\mathbb{C})$$

$$\leftrightarrow \{\text{semisimple conjugacy classes in } G^\vee\}$$

The last bijection holds since every semisimple element of $G^\vee$ is contained in some maximal torus, and all maximal tori are $G^\vee$-conjugate.

The point of Remark 1.6 is that it induces the unramified case of the local Langlands program, on which we give a short introduction: Let $\Pi(G)$ be the set of irreducible smooth admissible G-representations up to isomorphisms (the "automorphic side"). Let $\Phi(G)$ be the set of L-parameters for G up to equivalence (the "Galois side"). To define L-parameter, we start with the definition of Weil group:

Definition 14.7 (Weil group)**.** Given a non-archimedean local field F, we have the following exact sequence

$$1 \to I_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(\bar{k}_F/k_F) \to 1$$

where k_F is the residue field of F and $\mathrm{Gal}(\bar{k}_F/k_F) \cong \hat{\mathbb{Z}}$ is generated by the Frobenius map $\mathrm{Frob} : x \mapsto x^{|k_F|}$, I_F is the inertia group. Define W_F to be the preimage of $\mathbb{Z} =< \mathrm{Frob} >$, so it fits into an exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 0$$

This defines W_F as a group. As a topological group, we declare I_F to be an open subgroup with its natural profinite topology. Note that W_F is not a profinite group.

Definition 14.8. An L-parameter for G is a pair (r, N) where

$$r : W_F \to G^\vee(\mathbb{C})$$

is a continuous homomorphism and where N is a nilpotent element in $Lie(G)$ such that

$$r(g)Nr(g)^{-1} = q^m N$$

where the image of g in $\mathrm{Gal}(\bar{k}_F/k_F)$ is Frob^m.

Now we define $\Phi(G)$ to be

$$\{(r,N)|r(\text{Frob}) \text{ is semisimple}\}/\cong$$

We give a very crude form of Local Langlands Correspondence: There is a map

$$\mathrm{LL}_G : \Pi(G) \to \Phi(G)$$

all of whose fibers are finite non-empty subsets ("L-packets") of $\Pi(G)$ and satisfying various requirements without which the statement will not be meaningful. We omit the details here.

Example 14.9. For $G = GL_n$ such a correspondence is a theorem of Harris-Taylor, in which case LL_G is a bijection, and one way of characterizing the bijection is via L-functions and epsilon factors. The case of a general reductive group is still wide open.

Example 14.10 (Unramified LLC)**.** Let $\Phi_{unr}(G)$ be the set of equivalent classes of unramified homomorphisms $\gamma : W_F/I_F \to G^\vee(\mathbb{C})$ such that $r(\text{Frob})$ is semisimple. It is in bijection with the set of semisimple conjugacy classes in $G^\vee(\mathbb{C})$.

Let $\Pi_{unr}(G)$ be the set of irreducible smooth G-representations π such that there exists a connected reductive group scheme $\mathbf{G}$ defined over $\mathcal{O}$ with generic fiber G and $\pi^{\mathbf{G}(\mathcal{O})} \neq 0$. (Note that in general there may exist more than one such reductive integral model $\mathbf{G}$).

The unramified LLC is then the map

$$\Pi_{unr}(G) \to \Phi_{unr}(G)$$

induced by the Satake isomorphism. To be precise, let $\pi \in \Pi_{unr}(G)$, with $\mathbf{G}/\mathcal{O}$ the reductive group scheme such that $\pi^{\mathbf{G}(\mathcal{O})} \neq 0$. Then the Satake isomorphism $\mathcal{H}_{\mathbf{G}(\mathcal{O})} \xrightarrow{\sim} \mathcal{H}_{T(\mathcal{O})}^W$ associates to the $\mathcal{H}_{\mathbf{G}(\mathcal{O})}$-module $\pi^{\mathbf{G}(\mathcal{O})}$ a semisimple conjugacy class in $G^\vee(\mathbb{C})$, and thus an element of $\Phi_{unr}(G)$.

Remark 14.11 (To Example 1.10)*.* Even in the unramified setting, we can have L-packets that are not singletons. For instance, we can have smooth irreducible unramified representation $\tilde{\pi}$ on $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\tilde{\pi}|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ is not irreducible; in this case the constituents of the restriction form an L-packet. For example, the irreducible unramified representation

$$\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\varepsilon|\cdot|^{1/2} \otimes |\cdot|^{-1/2}),$$

where ε is the unramified character

$$\mathbb{Q}_p/\mathbb{Z}_p \to \{\pm 1\}$$
$$p \mapsto -1$$

, is reducible when restricted to $\mathrm{SL}_2(\mathbb{Q}_p)$.

Back to Satake isomorphism

$$\mathcal{S} : \mathcal{H}_K \to R(G^\vee) \otimes_{\mathbb{Z}} \mathbb{C}$$

LHS has a basis $c_\lambda = \mathbf{1}_{K\lambda(\varpi)K}$, $\lambda \in X_*(T)_+$, RHS has a basis $ch(V(\lambda))$, $\lambda \in X_*(T)_+$ where $V(\lambda)$ is the highest weight representation corresponding to λ.

Motivating Question: What in $\mathcal{H}_K$ maps to $ch(V(\lambda))$? A full understanding will come from geometric Satake.

Let us do some warm-up calculation before moving to the proof of Satake Isomorphism.

Lemma 14.12. *For all G, let $\lambda, \mu \in X_*(T)$. Write*

$$K\lambda(\varpi)K = \bigsqcup x_i K$$

We may assume by Iwasawa decomposition that $x_i \in B(F)$ and write $x_i = t(x_i)n(x_i) \in T \cdot N$. Let ρ be the half sum of positive roots and q be the cardinality of the residue field. Then

$$S(c_\lambda)(\mu(\varpi)) = q^{-<\mu,\rho>} \cdot |\{i \mid t(x_i) \equiv \mu(\varpi) \mod T(\mathcal{O})\}|$$

Proof.

$$S(c_\lambda)(\mu(\varpi)) = \delta(\mu(\varpi))^{1/2} \int_N \mathbf{1}_{K\lambda(\varpi)K}(\mu(\varpi)n)dn = q^{-<\mu,\rho>} \sum_i \int_N \mathbf{1}_{x_i K}(\mu(\varpi)n)dn$$

Note that

$$\mu(\varpi)n \in x_i K = t(x_i)n(x_i)K \iff n \in \mu(\varpi)^{-1}t(x_i)n(x_i)K = n(x_i)'\mu(\varpi)^{-1}t(x_i)K$$

i.e., $\exists\, k \in K$ such that $k = t(x_i)^{-1}\mu(\varpi)n(x_i)'^{-1}n$, which is equivalent to $t(x_i)^{-1}\mu(\varpi) \in T(\mathcal{O})$ and $n(x_i)'^{-1}n \in N(\mathcal{O})$. So

$$\begin{aligned} S(c_\lambda)(\mu(\varpi)) &= q^{-<\mu,\rho>} \sum_{\text{i such that } t(x_i) \equiv \mu(\varpi) \mod T(\mathcal{O})} \int_{n(x_i)'N(\mathcal{O})} dn \\ &= q^{-<\mu,\rho>} \cdot |\{i \mid t(x_i) \equiv \mu(\varpi) \mod T(\mathcal{O})\}| \end{aligned}$$

□

Now we turn to $G = PGL_2(F)$, then $\lambda \in X_*(T)_+$ are

$$ne_1^* : t \mapsto \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2(F)$$

Let us compute $S(c_0), S(c_{e_1^*}), S(c_{2e_1^*})$. It is clear that $S(c_0) = 1 (= 1 \cdot [0] \in \mathbb{C}[X_*(T)])$. For $S(c_{e_1^*})$, decompose

$$\begin{aligned} Ke_1^*(\varpi)K &= K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K \\ &= \bigsqcup_{x \in \mathcal{O}/\varpi} \begin{pmatrix} \varpi & x \\ 0 & 1 \end{pmatrix} K \cup \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} K \end{aligned}$$

i.e.,

$$S(c_{e_1^*}) = [e_1^*] \cdot q^{-<e_1^*,\rho>}|\mathcal{O}/\varpi| + [e_2^*] \cdot q^{-<e_2^*,\rho>} \cdot 1 = q^{1/2}(e_1^* + e_2^*) = q^{1/2}ch(V(e_1^*))$$

(Note W-invariance!)

Next, for $S(c_{2e_1^*})$, decompose

$$K \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix} K = \bigsqcup_{x \in \mathcal{O}/\varpi^2} \begin{pmatrix} \varpi & x \\ 0 & \varpi \end{pmatrix} K \cup \bigsqcup_{x \in (\mathcal{O}/\varpi)^\times} \begin{pmatrix} \varpi^2 & x \\ 0 & 1 \end{pmatrix} K \cup \begin{pmatrix} 1 & 0 \\ 0 & \varpi^2 \end{pmatrix} K$$

So

$$\begin{aligned} S(c_{2e_1^*}) &= [2e_1^*] \cdot q^{-1} \cdot q^2 + [0] \cdot (q-1) + [2e_2^*] \cdot q \cdot 1 \\ &= q[2e_1^*] + (q-1)[0] + q[2e_2^*] \quad \text{(Note } W\text{-invariance!)} \end{aligned}$$

$$= q \cdot ch(V(2e_1^*)) - ch(V(0))$$

where $V(2e_1^*))$ is the symmetric square representation of $PGL_2(F)$.

In general, we have

$$q^{n/2} ch(V(ne_1^*)) = \sum_{m \equiv n(2), m \leq n} S(c_{me_1^*})$$

for $G = PGL_2(F)$. This identity corresponds via geometric Satake to singularities of affine Schubert variety corresponding to ne_1^*.

15. Sketch of proof of the Satake isomorphism (Christian Klevdal)

In this section we will outline a proof of the Satake isomorphism. Throughout this section, F will be a local non archimedean field with ring of integers $\mathcal{O}$ and $\varpi \in \mathcal{O}$ a uniformizer. We will write $\underline{G}$ for a split connected reductive group over $\mathcal{O}$, $G = \underline{G}(F), K = \underline{G}(O)$ and we will fix a maximal torus and a Borel $T \subset B \subset \underline{G}$ with unipotent radical $N \subset B$. We write $\mathcal{H}_K = \mathcal{H}(G, K)$ is the K-spherical Hecke algebra, $W = N(T)/T$ the Weyl group and $\mathcal{H}_{T(\mathcal{O})}^W = \mathcal{H}(T(F), T(\mathcal{O}))^W$ the Weyl invariants in the $T(\mathcal{O})$-spherical Hecke algebra. Recall the isomorphism $\mathcal{H}_{T(\mathcal{O})}^W \cong \mathbb{C}[X_\bullet(T)]^W$. Finally, let $\delta_B(t) = |\det(\mathrm{Ad}(t)|_{\mathfrak{n}})|$.

Theorem 15.1 (Satake isomorphism). *There is an isomorphism of algebras*

$$\mathcal{S}\colon \mathcal{H}_K \xrightarrow{\sim} \mathcal{H}_{T(\mathcal{O})}^W \qquad f \mapsto \mathcal{S}(f)(t) = \delta_B(t)^{\frac{1}{2}} \int_N f(tn) dn$$

Recall also the Cartan decomposition

$$G = \bigsqcup_{\lambda \in X_\bullet(T)_+} K\lambda(\varpi)K$$

which implies that $\mathcal{H}_K$ has a basis of characteristic functions $c_\lambda = \mathbf{1}_{K\lambda(\varpi)K}$ with $\lambda \in X_\bullet(T)_+$. We also have that $\mathcal{H}_{T(\mathcal{O})}$ has a basis given by the characteristic functions $\mathbf{1}_{\mu(\varpi)}$ for $\mu \in X_\bullet(T)_+$. The basic calculation for the proof of the Satake isomorphism will be to determine $\mathcal{S}(c_\lambda)(\mu(\varpi))$ for $\mu, \lambda \in X_\bullet(T)_+$. The value is determined as follows: First, decompose the double coset $K\lambda(\varpi)K$ into a finite disjoint union of single cosets $\sqcup_{i=1}^n x_i K$. From the Iwasawa decomposition, we can assume each $x_i \in B(F)$. Thus we can write $x_i = t(x_i)n(x_i)$ for $t(x_i) \in T(F), n(x_i) \in N(F)$. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.

Lemma 15.2. *With notation as above,*

$$\mathcal{S}(c_\lambda)(\mu(\varpi)) = q^{-\langle \mu, \rho \rangle} |\{i\colon t(x_i) \equiv \mu(\pi) \mod T(\mathcal{O})\}|$$

Proof. First note that $\delta_B(\mu(\varpi))^{\frac{1}{2}} = q^{-\langle \mu, \rho \rangle}$. Thus

$$\mathcal{S}(c_\lambda)(\mu(\varpi)) = q^{-\langle \mu, \rho \rangle} \int_N \mathbf{1}_{K\lambda(\varpi)K}(\mu(\varpi)n) dn = q^{-\langle \mu, \rho \rangle} \sum_{i=1}^n \int_N \mathbf{1}_{x_i K}(\mu(\varpi)n)$$

We need to check when $\mu(\varpi)n \in x_i K$. But this happens if and only if $n \in \mu(\varpi)^{-1} t(x_i) n(x_i) K$. Since N normalizes T, we can write $\mu(\varpi)^{-1} t(x_i) n(x_i) K = n(x_i)' \mu(\varpi)^{-1} t(x_i) K$ for some $n(x_i)' \in N$. We now see that n is in this later set if and only if $t(x_i)^{-1} \mu(\varpi)(n(x_i)')^{-1} n^{-1} \in K$.

But this happens if and only if $n(x_i)'n \in N(\mathcal{O})$ and $t(x_i)^{-1}\mu(\varpi) \in T(\mathcal{O})$ (this last conidition is exactly that $t(x_i) = \mu(\varpi) \mod T(\mathcal{O})$). Once we note that $N(\mathcal{O})$ has volume 1 we have

$$\int_N \mathbf{1}_{x_iK} dn = \begin{cases} \int_{n(x_i)'N(\mathcal{O})} dn = 1 & \text{if } t(x_i) \equiv \mu(\varpi) \mod T(\mathcal{O}) \\ 0 & \text{otherwise} \end{cases}$$

Consequently, we have

$$\sum_i \int_N \mathbf{1}_{x_iK}(\mu(\varpi)n)dn = |\{i \colon t(x_i) \equiv \mu(\varpi) \mod T(\mathcal{O})\}|$$

□

As an easy consequence of the above lemma, $\mathcal{S}(c_\lambda)(\lambda(\varpi)) \neq 0$ since we can assume one of the x_i is $\lambda(\varpi)$.

Proof of Theorem 15.1. Our sketch procedes in 3 steps:

(1) Check that $\mathcal{S}$ is an algebra homomoprhism.

(2) Check that the image is Weyl invariant.

(3) Conclude $\mathcal{S}$ is an isomorphism using the facts:

   (a) $\mathcal{S}(c_\lambda)(\lambda(\varpi)) \neq 0$ (which we noted above).

   (b) For $\mu, \lambda \in X_\bullet(T)_+$

$$\mathcal{S}(c_\lambda)(\mu(\varpi)) \neq 0 \implies \mu \leq \lambda$$

   (we'll prove this later).

Proof of (3), given (1), (2) and facts (a),(b): Given $\mu \in X_\bullet(T)$, let $\mathrm{ch}(V(\mu))$ be the character of the representation of the Langlands dual group of G with highest weight μ. By highest weight theory, the set of such characters ranging over $\mu \in X_\bullet(T)_+$ gives a basis of $\mathcal{H}^W_{T(\mathcal{O})} = \mathbb{C}[X_\bullet(T)]^W$. Hence using W invariance

$$\mathcal{S}(c_\lambda) = \mathcal{S}(c_\lambda)(\lambda(\varpi))\mathrm{ch}(V(\lambda)) + \sum_{\mu<\lambda} a_{\lambda\mu}\mathrm{ch}(V(\mu))$$

We sum over $\mu < \lambda$ by fact (b) above. Now the fact that this is "upper triangular" implies that $\mathcal{S}$ is an isomorphism.

Explanation of fact (b): From the definition

$$\mathcal{S}(c_\lambda)(\mu(\varpi)) = \delta(\mu(\varpi))^{\frac{1}{2}} \int_N \mathbf{1}_{K\lambda(\varpi)K}(\mu(\omega)n)dn$$

we have $\mathcal{S}(c_\lambda)(\mu(\varpi)) \neq 0$ if and only if $N\mu(\varpi) \cap K\lambda(\varpi)K \neq \varnothing$ (since $N\mu(\varpi) = \mu(\varpi)N$). Thus (b) follows from the next proposition.

Proposition 15.3. *For $\mu, \lambda \in X_\bullet(T)_+$,*

$$N\mu(\varpi) \cap K\lambda(\varpi)K \neq \varnothing \implies \mu \leq \lambda$$

Proof. We will only sketch a proof in this in the case that $\mathcal{O}$ is equicharacteristic (e.g. $\mathcal{O} = k[\![t]\!], F = k((t))$ which is the case we will be interested in for geometric Satake). The reference in general is Bruhat-Tits. What we gain in equal characetistic is the inclusion $\underline{G}(k) \subset \underline{G}(k[\![t]\!]) \subset \underline{G}(F)$.

Define a left action of $\mathbb{G}_m$ on G/K where $s \in \mathbb{G}_m$ acts by left multiplication by $2\rho^\vee(s)$ (where $2\rho^\vee$ is the sum of positive coroots). For $x \in N\mu(\varpi)$

$$\lim_{s\to 0} 2\rho^\vee(s)x = \mu(\varpi) \in G/K$$

If $k = \mathbb{C}$ taking the limit makes sense, but for general k we won't make sense of the limit until later when we realize G/K as an object in algebraic geometry. To see the above formula, write $x = n\mu(\varpi)$ with $n \in N$. Then

$$2\rho^\vee(s)n\mu(\varpi) = 2\rho^\vee(s)n(2\rho^\vee(s))^{-1}2\rho^\vee(s)\mu(\varpi)$$

Since we are working in G/K we can replace the $2\rho^\vee(s)\mu(\varpi)$ with just $\mu(\varpi)$ on the right hand side since $2\rho^\vee(s)$ is an element of K. Hence we get the s action sends $n\mu(\varpi)$ to $\mathrm{Ad}2\rho^\vee(s)(n)\mu(\varpi)$. Now one sees that the limit as $s \to 0$ of $\mathrm{Ad}2\rho^\vee(s)(n) = 1$ and we conclude the formula. For example, in the Sl_3 case, we get n is a unipotent matrix, and $2\rho^\vee(s)$ is the matrix with $s^2, 1, s^{-2}$ on the diagonal. We compute

$$\mathrm{Ad}\begin{pmatrix} s^2 & & \\ & 1 & \\ & & s^{-2}\end{pmatrix} \cdot \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1\end{pmatrix} = \begin{pmatrix} 1 & s^2x & s^4y \\ & 1 & s^2z \\ & & 1\end{pmatrix}$$

and this clearly goes to the identity as $s \to 0$.

As a consequence, we have $S_\mu = N\mu(\varpi)K/K = \{x \in G/K\colon \lim_{s\to 0} 2\rho^\vee(s)x = \mu(\varpi)\}$. So if we have $x \in S_\mu \cap K\lambda(\varpi)K$ then $2\rho^\vee(s)x$ preserves $K\lambda(\varpi)K$ and hence $\mu(\varpi) = \lim_{s\to 0} 2\rho^\vee(s)x \in \overline{K\lambda(\varpi)K}/K$ (we will make sense of the closure later). We now use the following fact that

$$\overline{K\lambda(\varpi)K}/K = \sqcup_{\nu\leq\lambda} K\nu(\varpi)K/K$$

Hence we conclude that $\mu \leq \lambda$. (This argument will be made rigorous when we discuss the affine Grassmannian and Schubert varieties.) $\square$

<u>Proof of (1):</u> Consider the factorization of $\mathcal{S}$ as

$$\mathcal{H}_K \xrightarrow{\mathrm{res}_B} \mathcal{H}(B(F), B(\mathcal{O})) \xrightarrow{\beta} \mathcal{H}_{T(\mathcal{O})} \xrightarrow{\cdot\delta_B^{\frac{1}{2}}} \mathcal{H}_{T(\mathcal{O})}$$

Where the map β is $\beta(f)(t) = \int_N f(tn)dn$. We will prove that each individual map is an algebra homomorphism. Given functions $f_1, f_2 \in \mathcal{H}_K$ we have that $\mathrm{res}_B(f_1 * f_2)$ is the map

$$p \mapsto \int_G f_1(x)f_2(x^{-1}p)d_G g$$

where d_G is a normalized left Haar measure on G. If we likewise denote d_K, d_B as normalized left Haar measures on K, B then the Iwasawa decomposition gives rise to the following integration formula (see Cartier, Representations of p-adic Groups)

$$\begin{aligned}\int_G f_1(x)f_2(x^{-1}p)d_G g &= \int_K\int_B f_1(bk)f_2(k^{-1}b^{-1}p)d_K k d_B b && \text{(Cartier)}\\ &= \int_B f_1(b)f_2(b^{-1}p)d_B b && \text{since } f_1, f_2 \text{ are K invariant}\\ &= \mathrm{res}_B(f_1) * \mathrm{res}_B(f_2)\end{aligned}$$

Hence the first map is an algebra homomorphism. For the second map β, we have

$$\begin{aligned}\beta(f_1 * f_2)(s) &= \int_N f_1 * f_2(sn)dn = \int_N\int_B f_1(b)f_2(b^{-1}sn)db\ dn\\ &= \int_N\int_T\int_N f_1(tn_1)f_2(n_1^{-1}t^{-1}sn)dn_1\ dt\ dn\\ &= \int_N\int_T\int_N f_1(tn_1)f_2(t^{-1}sn)dn\ dt\ dn && dn \text{ is left invariant}\\ &= \int_T\left(\int_N f_1(tn_1)dn_1\right)\left(\int_N f_2(t^{-1}sn)dn_2\right)dt\\ &= \int_T \beta(f_1)(t)\beta(f_2)(t^{-1}s)dt = \beta(f_1) * \beta(f_2)(s)\end{aligned}$$

This shows that the second map is an algebra homomorphism. It is clear that the third map is an algebra homomorphism. Hence we have proven (1).

Proof of (2): We need to show that $\mathrm{im}(\mathcal{S})$ is Weyl invariant. Recall $W = N(T)/T = N(T)\cap K/T\cap K$ (this holds because G is split) and $T\cap K = T(\mathcal{O})$. So we need to show

$$\mathcal{S}(f)(t) = \mathcal{S}(f)(xtx^{-1})$$

for all $x \in N(T)\cap K$ and all $t \in T(F)$. By continuity, it suffices to check this on the dense set of regular t, i.e. $t \in T$ such that $\alpha(t) \neq 1$ for all $\alpha \in \Phi$. (For Gl_n these are the diagonal matrices with distinct entries). We will use without proof the following lemma, (see Cartier, Representations of p-adic Groups).

Lemma 15.4. *For t regular, if one writes $D(t) = |\det((\mathrm{Ad}(t)-1)|_{\mathfrak{n}})|\delta(t)^{\frac{-1}{2}}$, then*

$$\mathcal{S}(f)(t) = D(t)\int_{G/T} f(gtg^{-1})d_{G/T}g$$

Perhaps a little bit of explanation is necessary. The groups G, T are unimodular, and we can define a left G-invariant quotient measure $d_{G/T}$ allowing us to integrate T-invariant function over G/T. This type of integral over conjugacy classes is usually called an orbital integral. Also note that $f(gtg^{-1})$ is (right) T-invariant as a function of g. We will now prove (3) using the lemma. First we show Weyl invariance of $D(t)$ (we've written $\mathfrak{n}'$ for the opposite unipotent radical of $\mathfrak{n}$):

$$\begin{aligned} D(t)^2 &= |\det((\mathrm{Ad}(t)-1)|_{\mathfrak{n}})|^2 |\det\left((\mathrm{Ad}(t^{-1})|_{\mathfrak{n}}\right)| \\ &= |\det((\mathrm{Ad}(t)-1)|_{\mathfrak{n}})||\det\left((\mathrm{Ad}(t^{-1})-1)|_{\mathfrak{n}}\right)| \\ &= |\det((\mathrm{Ad}(t)-1)|_{\mathfrak{n}})||\det((\mathrm{Ad}(t)-1)|_{\mathfrak{n}'})| \\ &= |\det\left((\mathrm{Ad}(t)-1)|_{\mathfrak{g}/\mathfrak{t}}\right)| \end{aligned}$$

This expression is clearly Weyl-invariant. Finally,

$$\begin{aligned} \int_{G/T} f(gxtx^{-1}g^{-1}) d_{G/T}g &= \int_{G/T} (f(x^{-1}gx)t(x^{-1}gx)^{-1}) d_{G/T}g \\ &= \int_{G/T} f(gtg^{-1}) d_{G/T}g \end{aligned}$$

where the last equality is by the change of variables $g \mapsto xgx^{-1}$ (here we use that $x \in K$, which is compact, to show the quotient measure is preserved under x-conjugation).

□

16. Function sheaf dictionary

With notation as in the previous section, the classical Satake isomorphism is giving an isomorphism of $\mathcal{S}\colon \mathcal{H}_K \xrightarrow{\sim} \mathcal{H}(T, T(\mathcal{O}))^W$. We can phrase this isomorphism slightly differently which will show us how to categorify the classical Satake isomorphism to get geometric Satake. We begin by noting that $\mathcal{H}(T, T(\mathcal{O}))^W \cong \mathbb{C}[X_\bullet(T)]^W$, which by heighest weight theory is precisely the complexification of the Grothendieck group of $\mathrm{Rep}(G^\vee)$ where $G^\vee$ is the Langlands dual of G. The K-spherical Hecke algebra is $\mathcal{H}_K = C^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})) = C^\infty(G(F)/G(\mathcal{O}))^{G(\mathcal{O})}$. These are the $G(\mathcal{O})$-invariant functions on $G(F)/G(\mathcal{O})$. So the Satake isomorphism is an isomorphism

$$\mathcal{S}\colon C^\infty(G(F)/G(\mathcal{O}))^{G(\mathcal{O})} \xrightarrow{\sim} K_0(\mathrm{Rep}(G^\vee)) \otimes \mathbb{C}$$

If we wanted to categorify this isomorphism, that is to turn this into an equivalence of categories, the obvious thing to put on the right hand side is $\mathrm{Rep}(G^\vee)$. It is much less clear what we should put on the left hand side. From now on we specialize to the case where $F = k((t))$ and $\mathcal{O} = k[\![t]\!]$. In the next section, we will introduce a geometric object over k, the affine Grassmannian Gr_G, which has the property that $\mathrm{Gr}_G(k) = G(F)/G(\mathcal{O}) = G(k((t)))/G(k[\![t]\!])$. Under this identification the left hand side becomes a collection of $K = G(\mathcal{O})$ invariant functions on the affine Grassmannian. But what kind of "functions"? The answer to this question will be provided using the function sheaf dictionary, which we explain now.

Take $k = \mathbb{F}_q$ and X a finite type separated scheme over $\mathbb{F}_q$. A constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on X is a $\mathbb{Q}_\ell$-sheaf (we are not being very precise here) $\mathcal{F}$ such that there exists a partition X into a finite disjoint union of locally closed subschemes X_α and $\mathcal{F}|_{X_\alpha}$ is locally constant. Recall that locally constant $\mathbb{Q}_\ell$-sheaves on a connected scheme Y are equivalent to continuous $\mathbb{Q}_\ell$ representations of $\pi_1^{\text{ét}}(Y)$ (in what follows we omit reference to base-points). In particular, a locally constant sheaf $\mathcal{G}$ on $\mathrm{Spec}(k)$ is the same thing as a continouous ℓ-adic representation of $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ which by abuse of notation we will also denote by $\mathcal{G}$. In particular, given

$\mathcal{F}$ constructible and $x \in X(k)$ we can view $x^*\mathcal{F}$ as a Γ_k representation. We can now define the function associated to a constructible sheaf $\mathcal{F}$ on X.

Definition 16.1. If $\mathcal{F}$ is a constructible $\mathbb{Q}_\ell$-sheaf the the function associated to $\mathcal{F}$ is

$$\begin{aligned} \operatorname{tr}_{\mathcal{F}} \colon X(k) &\to \mathbb{Q}_\ell \\ (x \colon \operatorname{Spec}(k) \to X) &\mapsto \operatorname{tr}(\operatorname{Fr}_q | x^*\mathcal{F}) \end{aligned}$$

where $\operatorname{Fr}_q \in \Gamma_k$ is the geometric Frobenius, and we think of $x^*\mathcal{F}$ as an ℓ-adic Galois representation. More generally, given $K \in D^b_c(X)$ the bounded derived category of ℓ-adic sheaves, the function associated to K is

$$\begin{aligned} \operatorname{tr}_{\mathcal{F}} \colon X(k) &\to \mathbb{Q}_\ell \\ x &\mapsto \sum_i (-1)^i \operatorname{tr}_{H^i(K)}(x) \end{aligned}$$

This turns out to be a reasonable definition because of the following

Proposition 16.2. *The following compatibilities hold.*

(1) *If F, G are in $D^b_c(X)$, the bounded derived categry of constructible sheaves on X, then*
$$\operatorname{tr}_{F \oplus G} = \operatorname{tr}_F + \operatorname{tr}_G$$
and
$$\operatorname{tr}_{F \otimes G} = \operatorname{tr}_F \cdot \operatorname{tr}_G .$$

(2) *If $f \colon X \to Y$ is a morphism and $F \in D^b_c(Y)$ then*
$$f^* \operatorname{tr}_{\mathcal{F}} = \operatorname{tr}_{f^*\mathcal{F}}$$

(3) *With f as above we can form for any $K \in D^b_c(X)$ the sheaf $Rf_!K \in D^b_c(Y)$ and we have for $y \in Y(k)$*
$$\operatorname{tr}_{Rf_!K}(y) = \sum_{x \in f^{-1}(y) \cap X(k)} \operatorname{tr}_K(x) \qquad (\star)$$

This proposition shows that operations on sheaves translate to the analogous operations on fuctions. The first two are fairly easy, while the last formula is difficult; it can be thought of as saying that pushforward is integration along the fibers. We will now give examples of the function sheaf dictionary and investigate some consequences of the proposition.

Example 16.3 (Sheaves associated to characters)**.** Suppose A is a commutative algebraic group scheme over k. We can define a map $L \colon A \to A$ called the Lang isogeny given by $x \mapsto x - Fx$, where F is the k-morphism given by $x \mapsto x^q$ on structure sheaves. The Lang isogeny is a finite étale cover with kernel equal to $A(k)$. In the case of $\mathbb{G}_a$ the Lang isogeny is the Artin-Scheier map $x \mapsto x - x^q$.

The deck transformations of L over A are given by multiplication by elements of $A(k)$, hence we get a surjection $\pi_1^{\text{ét}}(A) \to \operatorname{Aut}(L) \cong A(k)$. Now given any character $\chi \colon A(k) \to \overline{\mathbb{Q}_\ell}^\times$ is an ℓ-adic representation, hence corresponds to a locally constant ℓ adic sheaf $\mathcal{L}_\chi$. We claim that $\operatorname{tr}_{\mathcal{L}_\chi} = \chi$. Indeed, let $x \in A(k)$. Then $x^*\mathcal{L}_\chi$ is the unique continuous representation $\Gamma_k \to \overline{\mathbb{Q}}_\ell$

sending $\mathrm{Fr}_q \mapsto \chi(x)$. Indeed, given a point $x \in A(k)$ fix a geometric point $\overline{x} \to x$ (which amounts to fixing an algebraic closure $\overline{k}$ of k). The corresponding ℓ-adic representation is

$$\Gamma_k = \pi_1(x) \to \pi_1(A) \to A(k) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$$

(we have ommitted the basepoint $\overline{x}$ and the superscripts ét). So we are done as soon as we see that $\mathrm{Fr}_q \mapsto x \in A(k)$ under the composition

$$\pi_1(x) \to \pi_1(A) \to \mathrm{Aut}(L) \cong A(k)$$

Given $y \in L^{-1}(x)$ then $y - F(y) = x$ or equivalently $F(y) = y - x$. But the action of F on $L^{-1}(x)$ is the same as the action of $\mathrm{Fr}_q^{-1} \in \pi_1(x)$. Hence we see that Fr_q acts as translation by x on $L^{-1}(x)$ and hence Fr_q maps to $x \in A(k)$ as required.

Example 16.4 (Lefschetz fixed point formula)**.** A key result in proving the Weil conjectures is the Lefschetz fixed point formula

$$\sum_i (-1)^i \operatorname{tr}(\mathrm{Fr}_q | H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell)) = |X(k)|$$

This is the basic case of the push-forward formula in the previous Proposition: take f to be the structure morphism $f\colon X \to Y = \mathrm{Spec}(k)$ and $K = \mathbb{Q}_\ell \in D_c^b(X)$. For the one point $y \in Y(k)$ we have $f(x) = y$ for all $x \in X(k)$, hence the right hand side of 3 is just

$$\sum_{X(k)} \operatorname{tr}_{\mathbb{Q}_\ell}(x) = \sum_{X(k)} 1 = |X(k)|$$

But we can also compute $\operatorname{tr}_{Rf_!\mathbb{Q}_\ell}(y)$ from the definition. Since $H^i(Rf_!\mathbb{Q}_\ell) = H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ we have the left hand side of 3 is

$$\operatorname{tr}_{Rf_!\mathbb{Q}_\ell} = \sum_i (-1)^i \operatorname{tr}(\mathrm{Fr}_q | H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell))$$

Example 16.5 (Gauss sums)**.** Gauss considered sums of the form

$$G(\chi, \psi) = \sum_{x \in k^\times} \chi(x)\psi(x)$$

where $\chi\colon k^\times \to \mathbb{C}^\times$ and $\psi\colon k \to \mathbb{C}^\times$ are characters. Gauss proved that $|G(\chi,\psi)| = \sqrt{q}|$. This is a very elementary computation, but we will prove it geometrically. The idea is to use the sheaf $\mathcal{F} = \mathcal{L}_\chi \otimes j^*\mathcal{L}_\psi$ where $\mathcal{L}_\chi$ is the sheaf associated to χ on $\mathbb{G}_m$ (example 16.3), $\mathcal{L}_\psi$ is the sheaf associated to ψ on $\mathbb{G}_a$ and $j\colon \mathbb{G}_m \to \mathbb{G}_a$ is the open immersion. Then it follows that $\operatorname{tr}_{\mathcal{F}}(x) = \chi(x)\psi(x)$, hence

$$G(\chi,\psi) = \sum_{x \in k^\times} \operatorname{tr}_{\mathcal{F}}(x) = \sum_i (-1)^i \operatorname{tr}(\mathrm{Fr}_q | H_c^i(\mathbb{G}_m, \mathcal{F}))$$

where the second equality is by the theorem and a similar argument as was used in the previous example. We can compute that $H_c^0(\mathbb{G}_m, \mathcal{F}) = H_c^2(\mathbb{G}_m, \mathcal{F}) = 0$ and $H_c^1(\mathbb{G}_m, \mathcal{F})$ is 1 dimensional. A little argument using Deligne's Weil II shows the eigenvalue of Frobenius on first cohomology is $\sqrt{q}$ and consequently $|G(\chi,\psi)| = \sqrt{q}$.

Example 16.6. What we did for Gauss sums was very silly, since it is extremely elementary to prove that $|G(\chi,\psi)| = \sqrt{q}$. However, the idea we used can be used to bound absolute values of Kloosterman sums. Given a character $\psi\colon k \to \mathbb{C}^\times$ the n-th Kloosterman sum is

$$\mathrm{Kl}_n(a) = \sum_{x_1 \cdot x_2 \cdots x_n = a} \psi(x_1 + \cdots + x_n)$$

Using an idea similar to the previous example, Deligne showed that $|\mathrm{Kl}_n(a)| \leq nq^{(n-1)/2}$. The idea is to consider $X = \{(x_1, \ldots, x_n)\colon x_1 \cdots x_n = a\} \subset \mathbb{A}^n$ and take $f\colon X \to \mathbb{A}^1$ given by $(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n$. Then

$$\mathrm{Kl_n(a)} = \sum_i \mathrm{tr}(\mathrm{Fr}_q | H^i_c(X_{\overline{k}}, f^*\mathcal{L}_\psi))$$

In order to bound this sum, Deligne shows the cohomology on the right hand side is 0 for $i \neq n-1$ and then uses information (Weil II) about the eigenvalues of frobenius on étale cohomology.

17. Introduction to Affine Grassmanians (Notes by Marin Petkovic)

Let k be a field and let G be a split connected reductive group over k. Let $F = k((t))$ and $\mathcal{O} = k[[t]]$. The aim of this section is to define the affine Grassmanian of G as a geometric object. That is, we want to define Gr_G so that $\mathrm{Gr}_G(k) = G(F)/G(\mathcal{O})$. Before defining the Grassmanian for general groups, we are going to consider the special case of GL_n.

17.1. The affine Grassmannian of GL_n.

Recall that by the Cartan decomposition

$$\mathrm{GL}_n(F) = \coprod_{\lambda \in X_\bullet(T)_+} \mathrm{GL}_n(\mathcal{O})\lambda(t)\mathrm{GL}_n(\mathcal{O}).$$

We proved this using the lattice $\Lambda_0 = \bigoplus_{i=1}^n k[[t]]e_i \subset \bigoplus_{i=1}^n k((t))e_i$. Consider the group action

$$\begin{aligned} \mathrm{GL}_n(F) &\longrightarrow \{\mathcal{O}\text{-lattices in } V\} \\ g &\longmapsto g \cdot \Lambda_0. \end{aligned}$$

Then $\mathrm{Stab}(\Lambda_0) = \mathrm{GL}_n(\mathcal{O})$ and we get a bijection

$$\mathrm{GL}_n(F) \big/ \mathrm{GL}_n(\mathcal{O}) \xrightarrow{\sim} \{\mathcal{O}\text{-lattices in } V\}.$$

This motivates the following definition.

Definition 17.1. The affine Grassmannian of the group GL_n is the functor

$$\mathrm{Gr}_{\mathrm{GL}_n} : k\text{-alg} \longrightarrow \mathrm{Set}$$

given by

$$R \longmapsto \left\{ \begin{array}{c} \text{f. g. projective } R[[t]]\text{-submodules } \Lambda \subset k((t))^n \\ \text{such that } \Lambda \otimes_{R[[t]]} R((t)) = R((t))^n \end{array} \right\}$$

Definition 17.2. A k-space is a functor $\mathcal{F} : k\text{-alg} \to \mathrm{Set}$ that is a sheaf in fpqc topology, i.e.

(1) $\mathcal{F}$ is a sheaf in the Zariski topology, and

(2) for all homomorphisms $R \to R'$ faithfully flat, the sequence
$$\mathcal{F}(R) \longrightarrow \mathcal{F}(R') \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathcal{F}(R' \otimes_R R')$$
is exact.

Remark 17.3. Every scheme over k is a k-space. (fpqc descent)

Definition 17.4. An ind-scheme is a directed system $(X_i)_{i \in I}$ of schemes such that all transition maps $X_i \to X_j$ are closed immersions. We say that an ind-scheme is ind-projective, if X_i's can be chosen to be projective. Similarly we define other properties of ind-schemes.

Example 17.5. An example of ind-scheme is
$$\mathbb{P}_\infty = \varinjlim \mathbb{P}_n.$$

It will turn out that $\mathrm{Gr}_{\mathrm{GL}_n}$ is not representable by a scheme, but it is an ind-scheme.

Theorem 17.6. *The affine Grassmanian* $\mathrm{Gr}_{\mathrm{GL}_n}$ *is an ind-projective ind-scheme.*

Proof. We can write $\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_{N \geq 1} \mathrm{Gr}^{(N)}$ where $\mathrm{Gr}^{(N)}$ is a subfunctor defined as
$$\mathrm{Gr}^{(N)} : R \longrightarrow \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}(R) \mid t^N R[[t]]^n \subset \Lambda \subset t^{-N} R[[t]]^n\}$$
Define functors $Q^{(N)}$ with
$$Q^{(N)}(R) = \left\{ \begin{array}{c} R[[t]]\text{-quotients } \bar{\Lambda} \text{ of } t^{-N}R[[t]]^n \big/ t^N R[[t]]^n \\ \text{such that } \bar{\Lambda} \text{ is a projective } R\text{-module} \end{array} \right\}$$
and a morphism of functors $\mathrm{Gr}^{(N)} \longrightarrow Q^{(N)}$ defined with
$$\Lambda \longmapsto \bar{\Lambda} = t^{-N}R[[t]]^n / \Lambda.$$
We claim that this is an isomorphism of functors. Granted this, we are done, because
$$Q^{(N)} \hookrightarrow \mathrm{Grass}\left(t^{-N}k[[t]]^n \big/ t^N k[[t]]^n\right)$$
identifies $Q^{(N)}$ as a closed subscheme of t-stable quotients of $t^{-N}k[[t]]^n \big/ t^N k[[t]]^n$.

To prove the claim, we first check that $t^{-N}R[[t]]^n/\Lambda$ is a projective R-module. For that, consider the short exact sequence
$$0 \longrightarrow t^{-N}R[[t]]^n / \Lambda \longrightarrow R((t))^n / \Lambda \longrightarrow R((t))^n \big/ t^{-N}R[[t]]^n \longrightarrow 0.$$
Since Λ is projective, so is $\Lambda/t\Lambda$.
$$R((t))^n / \Lambda \simeq \bigoplus t^{-k}\Lambda / t^{-k+1}\Lambda$$
is projective since each of the summands on the right is isomorphic to $\Lambda/t\Lambda$ as R-modules. The same reasoning shows that $R((t))^n \big/ t^{-N}R[[t]]^n$ is projective. From the short exact sequence above it follows that $t^{-N}R[[t]]^n/\Lambda$ is projective.

To show bijectivity, let $Q \in Q^{(N)}(R)$. As we have seen, $Q^{(N)}$ is of finite type over k, so for all k-algebras R we have

$$Q^{(N)}(R) = \operatorname*{colim}_{\text{f.g. } R_i \subset R} Q^{(N)}(R_i).$$

We may therefore assume R is finitely-generated over k.

We have

$$t^{-N}R[[t]]^n \twoheadrightarrow t^{-N}R[[t]]^n \Big/ t^N R[[t]]^n \twoheadrightarrow Q,$$

so let Λ_Q be the kernel of the composition above. We need to show that $\Lambda_Q \in \mathrm{Gr}^{(N)}(R)$. Since $t^N R[[t]]^n \subset \Lambda_Q$, we know that $\Lambda \otimes R((t)) = R((t))^n$, so it remains to show that Λ_Q is finitely generated projective $R[[t]]$-module. Since R is finitely generated over k, it is Noetherian. In particular $R[t] \longrightarrow R[[t]]$ is flat.

Since

$$t^{-N}R[t]^n \twoheadrightarrow t^{-N}R[t]^n \Big/ t^N R[t]^n \xrightarrow{\sim} t^{-N}R[[t]]^n \Big/ t^N R[[t]]^n \twoheadrightarrow Q$$

we can define $\Lambda_{Q,\text{fin}} = \ker(t^{-N}R[t]^n \twoheadrightarrow Q)$. Since $R[t] \longrightarrow R[[t]]$ is flat, $\Lambda_{Q,\text{fin}} \otimes_{R[t]} R[[t]] = \Lambda_Q$. Hence it suffices to show that $\Lambda_{Q,\text{fin}}$ is finitely generated projective $R[t]$-module. It is clearly finitely generated, since it is contained in $t^{-N}R[t]^n$. To show $\Lambda_{Q,\text{fin}}$ is projective, we will show it is flat over $R[t]$.

For this, notice that $\forall \mathfrak{p} \in \mathrm{Spec}(R)$, $\Lambda_{Q,\text{fin}} \otimes_{R[t]} \kappa(\mathfrak{p})[t]$ is flat (and therefore free) over $\kappa(\mathfrak{p})[t]$. Indeed, since Q is R-flat, we have

$$\Lambda_{Q,\text{fin}} \otimes_{R[t]} \kappa(\mathfrak{p})[t] = \ker\left(t^{-N}\kappa(\mathfrak{p})[t]^n \longrightarrow Q \otimes_R \kappa(\mathfrak{p})\right).$$

Since $t^{-N}\kappa(\mathfrak{p})[t]^n$ is torsion free, so is $\Lambda_{Q,\text{fin}} \otimes_{R[t]} \kappa(\mathfrak{p})[t]$, and hence it is flat. Now apply the following lemma on $R_\mathfrak{p} \longrightarrow R[t]_\mathfrak{q}$ and $M = \Lambda_{Q,\text{fin}} \otimes R[t]_\mathfrak{q}$, for all primes $\mathfrak{q}$ of $R[t]$.

□

Lemma 17.7. *Let $R \longrightarrow S$ be a local homomorphism of Noetherian local rings, $\mathfrak{m} \subset R$ maximal ideal and M a finite S-module. If M is flat over R and $M/\mathfrak{m}M$ is free over $S/\mathfrak{m}S$, then M is free over S and S is flat over R.*

Proof. Stacks Tag 00MH □

17.2. **Affine Grassmanian for general groups.** Let $\mathcal{D}_R = \mathrm{Spec}(R[[t]])$ (family of disks parametrized by R) and $\mathcal{D}_R^* = \mathrm{Spec}(R((t)))$ (family of punctured disks parametrized by R).

We can identify

$$\mathrm{Gr}_{\mathrm{GL}_n}(R) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ is a vector bundle of rank } n \text{ on } \mathcal{D}_R \\ \text{and } \beta : \mathcal{E}\big|_{\mathcal{D}_R^*} \xrightarrow{\sim} \text{ the trivial bundle on } \mathcal{D}_R^* \end{array} \right\} \Big/ \simeq.$$

Proposition 17.8. *Let G be a linear group over k and S quasi-compact scheme. The following categories are equivalent:*

(1) *fpqc sheafs $\mathcal{P}$ on Aff/S which are right G-torsors, i.e. $\mathcal{P}$ has a right G action such that*

$$\begin{aligned}\mathcal{P} \times G &\longrightarrow \mathcal{P} \times_S \mathcal{P}\\ (s,g) &\longmapsto (s,sg)\end{aligned}$$

is an isomorphism, and for which there exists an fpqc cover $S' \longrightarrow S$ such that $\mathcal{P}(S') \neq \emptyset$.

(2) *scheme maps $\pi : \mathcal{E} \longrightarrow S$ with a right G-action on $\mathcal{E}$ equivariant for π such that there exists a faithfully flat cover $S' \longrightarrow S$ such that $\mathcal{E} \times_S S' \simeq G \times S'$ as G-torsors.*

(3) *faithfully flat maps $\mathcal{E} \longrightarrow S$ with a right G-action such that $\mathcal{E} \times_S \mathcal{E} \simeq G \times \mathcal{E}$ as G-torsors.*

(4) *If G is smooth, as in (2) but require $S' \longrightarrow S$ étale.*

Remark 17.9. All the non-obvious implications follow from faithfully flat descent theory.

Definition 17.10. A G-bundle is an object in any of the four equivalent categories above (we will use (4)). *The* trivial G-bundle over S is $\mathcal{E}^0 = G \times_k S$ with G acting on itself by right multiplication.

Definition 17.11. The affine Grassmannian of G is the functor

$$\mathrm{Gr}_G : k\text{-alg} \longrightarrow \mathrm{Set}$$

given by

$$R \longmapsto \left\{ (\mathcal{E},\beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ is a } G\text{-bundle on } \mathcal{D}_R \text{ and} \\ \beta : \mathcal{E}\big|_{\mathcal{D}_R^*} \xrightarrow{\sim} \mathcal{E}^0 \text{ is an isomorphism of } G\text{-bundles} \end{array} \right\} /\simeq .$$

Relation to previous definition for GL_n

For any S there is an equivalence:

$$\begin{aligned}\{\mathrm{GL}_n\text{-bundles over } S\} &\longrightarrow \{\text{rank } n \text{ vector bundles over } S\}\\ (\mathcal{E} \to S) &\longmapsto \mathcal{E} \overset{\mathrm{GL}_n}{\times} \mathbb{A}^n\end{aligned}$$

where $\mathcal{E}\ {}^{\mathrm{GL}_n}\!\times \mathbb{A}^n = {\mathrm{GL}_n}\backslash{\mathcal{E} \times \mathbb{A}^n}$ with the action given by $g.(s,v) = (sg^{-1}, gv)$. Its quasi-inverse is given by

$$\mathcal{V} \longmapsto \left(T \mapsto \underset{\mathrm{GL}_n \circlearrowleft}{\mathrm{Isom}_T(\mathcal{O}_T^n, \mathcal{V})} \right),$$

for all $\mathcal{V}$ rank n vector bundle over S.

Theorem 17.12. *For any affine G/k the presheaf Gr_G is representable by an ind-scheme of ind-finite type. If G is reductive, then Gr_G is moreover ind-projective.*

Strategy of proof. Choose an embedding $G \longrightarrow \mathrm{GL}_n$ such that GL_n/G is quasi-affine. In this case, one can show that the map

$$\begin{aligned} \mathrm{Gr}_G &\longrightarrow \mathrm{Gr}_{\mathrm{GL}_n} \\ \mathcal{E} &\longmapsto \mathcal{E} \overset{G}{\times} \mathrm{GL}_n \end{aligned}$$

is a locally closed embedding.

When G is reductive, GL_n/G is affine and the map above is a closed embedding. □

For general homomorphisms $H \longrightarrow G$, the map $\mathrm{Gr}_H \longrightarrow \mathrm{Gr}_G$ may be very strange.

Example 17.13. Let $B < G$ be a Borel subgroup. If G is reductive, the Iwasawa decomposition says $G(k((t))) = B(k((t))) \cdot G(k[[t]])$, so $\mathrm{Gr}_B \longrightarrow \mathrm{Gr}_G$ is a bijection on k-points. This map is far from being an isomorphism.

17.3. **Another description of Gr_G.** We would like a description of Gr_G that is closer to our intuitive understanding of it as the "quotient" $G(k((t)))/G(k[[t]])$. Define the loop group of G to be

$$\begin{aligned} \mathrm{L}G : k\text{-alg} &\longrightarrow \mathrm{Grp} \\ R &\longmapsto G(R((t))) \end{aligned}$$

and the positive loop group

$$\begin{aligned} \mathrm{L}^+G : k\text{-alg} &\longrightarrow \mathrm{Grp} \\ R &\longmapsto G(R[[t]]) \end{aligned}$$

Lemma 17.14. *L^+G is a scheme over k and $\mathrm{L}G$ is an ind-scheme over k.*

Proof for GL_n. Notice that

$$\mathrm{L}^+G(R) = \mathrm{GL}_n(R[[t]]) = \left\{ \left(\sum_{m\geq 0} b_{i,j,m} t^m \right)_{i,j} \;\middle|\; b_{i,j,m} \in R,\ \det \in R^\times \right\}$$

can be embedded into the affine scheme $R[\{x_{i,j,m}\}, \det]$, and

$$\mathrm{L}G(R) = \bigcup_{r\geq 0} \{\text{matrices with all entries having poles of order at most } r\}$$

and all the objects on the right are schemes. □

Now consider the presheaf

$$\mathrm{L}G \big/ \mathrm{L}^+G : R \longmapsto \mathrm{L}G(R) \big/ \mathrm{L}^+G(R).$$

Let $\left[\mathrm{L}G / \mathrm{L}^+G\right]$ be the sheafication for the fpqc (or fppf, or étale) topology. As we will see, when G is smooth the choice of topology does not matter.

Proposition 17.15. *For any smooth linear algebraic group G/k, Gr_G is naturally isomorphic to $\left[\mathrm{L}G / \mathrm{L}^+G\right]$.*

Before the proof, we apply the Proposition to an example:

Example 17.16. Consider $\mathrm{L}\mathbb{G}_m/\mathrm{L}^+\mathbb{G}_m$. Let $f = \sum a_n t^n \in \mathrm{L}\mathbb{G}_m(R) = R((t))^\times$. Then there exists n_0 such that $a_{n_0} \in R^\times$ and a_n is nilpotent for all $n < n_0$ (eg. if $R = k[\epsilon]/\epsilon^2$, then $\epsilon t^{-1} + 1 \in R((t))^\times$ since $(\epsilon t^{-1}+1)(-\epsilon t^{-1}+1) = 1$). Write $f(t) = a_{n_0} t^{n_0} g(t)$ where the negative powers of t in $g(t)$ have nilpotent coefficients. Then $g(t)$ can be factored uniquely as a product of a polynomial with degrees ≤ 0 and nilpotent coefficients with a polynomial in $1 + tR[[t]]$. Hence we obtain a decomposition

$$\mathrm{L}\mathbb{G}_m = \mathbb{G}_m \times \mathbb{Z} \times (R \mapsto 1 + tR[[t]]) \times (\text{some highly non-reduced ind-scheme}).$$

Under this composition $\mathrm{L}^+\mathbb{G}_m = \mathbb{G}_m \times \mathbb{A}^\infty$, so

$$\mathrm{L}\mathbb{G}_m \big/ \mathrm{L}^+\mathbb{G}_m \simeq \mathbb{Z} \times (\text{some highly non-reduced ind-scheme}).$$

In particular, $\mathrm{Gr}_{\mathbb{G}_m}$ is not reduced and $\pi_0 \mathrm{Gr}_{\mathbb{G}_m} = \mathbb{Z}$.

Proof of proposition. There is an isomorphism

$$\begin{aligned} \mathrm{L}G(R) &\xrightarrow{\sim} \left\{ (\mathcal{E}, \beta, \epsilon) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ is a } G\text{-bundle on } \mathcal{D}_R \\ \beta : \mathcal{E}\big|_{\mathcal{D}_R^*} \xrightarrow{\sim} \mathcal{E}^0 \\ \epsilon : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0 \end{array} \right\} /\simeq \\ g &\longmapsto (\mathcal{E}^0, g, \mathrm{id}) \end{aligned}$$

where we interpret g as an automorphism of $\mathcal{E}^0$ over $\mathcal{D}_R^*$. Let $\mathrm{L}G \longrightarrow \mathrm{Gr}_G$ be the obvious morphism that forgets ϵ. Since $\mathrm{L}^+G(R) = \mathrm{Aut}(\mathcal{E}^0/\mathcal{D}_R)$, this map factors and we obtain $\mathrm{L}G \big/ \mathrm{L}^+G \longrightarrow \mathrm{Gr}_G$. Since Gr_G is a sheaf (for fpqc topology and hence for étale topology) this induces a map of sheaves

$$\left[\mathrm{L}G \big/ \mathrm{L}^+G\right] \longrightarrow \mathrm{Gr}_G.$$

It is easy to see that this is injective, so it remains to prove surjectivity. This follows from the following lemma. ☐

Lemma 17.17. *Any G-bundle $\mathcal{E}$ on $\mathcal{D}_R$ can be trivialized étale locally on R, i.e. there exists étale cover $R \longrightarrow R'$ such that $\mathcal{E} \otimes_{R[[t]]} R'[[t]]$ is isomorphic to the trivial bundle on $\mathcal{D}_{R'}$.*

Proof. We claim that $\mathcal{E}$ is trivial over $\mathcal{D}_R$ iff $\mathcal{E} \times_{R[[t]]} R$ is trivial over R. To see this, recall that $\mathcal{E}\big|_{S'}$ is trivial iff $\mathcal{E}(S') \neq \emptyset$. Since $\mathcal{E}$ is smooth, the maps $\mathcal{E}(R[t]/t^{n+1}) \longrightarrow \mathcal{E}(R[t]/t^n)$ are all surjections. Since $\mathcal{E}$ is affine, $\mathcal{E}(R[[t]]) = \varprojlim \mathcal{E}(R[t]/t^n)$. Thus, if $(\mathcal{E} \times_{R[[t]]} R)(R) \neq 0$, then $\mathcal{E}(R[[t]]) \neq 0$, which is equivalent to $\mathcal{E}$ being trivial over $\mathcal{D}_R$.

Having established the claim, the lemma follows since $\mathcal{E} \otimes_{R[[t]]} R$ is a G-bundle over R, so it can be trivialized étale locally on R. ☐

Let G be connected reductive group and $T \subset B \subset G$ a maximal torus and Borel subgroup. Let $\lambda \in X_\bullet(T)_+$. Then $\lambda(t) \in \mathrm{Gr}_G(k)$. The positive loop group L^+G acts on $\mathrm{Gr}_G = \left[\mathrm{L}G \big/ \mathrm{L}^+G\right]$ by left multiplication.

Definition 17.18. Let $\mathrm{Gr}_{G,\lambda}$ be the L^+G-orbit of $\lambda(t)$, regarded as a locally closed reduced subscheme of Gr_G. Define $\mathrm{Gr}_{G,\leq\lambda} = \overline{\mathrm{Gr}_{G,\lambda}} \hookrightarrow \mathrm{Gr}_G$.

Now we can answer the motivating question from classical Satake:

There is a complex of sheaves IC_λ called the intersection complex on $\mathrm{Gr}_{G,\leq\lambda}$ such that under the sheaf-function dictionary:

$$\begin{aligned} \mathcal{H}_{G(k[[t]])} &\xrightarrow{\sim} R(G^\vee) \\ \mathrm{tr}(IC_\lambda) &\longmapsto ch(V(\lambda)). \end{aligned}$$

The intersection complex is the basic example of a perverse sheaf; we turn to this subject in the next part of the course.

18. Review of homological algebra (Allechar Serrano López)

1. Let $\mathcal{C}$ be an additive category. Set $C(\mathcal{C})$ = category of complexes

$$\cdots \to X^n \to X^{n+1} \to \cdots$$

in $\mathcal{C}$ and $K(\mathcal{C})$ = category of complexes modulo chain homotopy equivalence, i.e.

$$\begin{aligned} \mathrm{ob}K(\mathcal{C}) &= \mathrm{ob}C(\mathcal{C}) \\ \mathrm{Hom}_{K(\mathcal{C})}(X,Y) &= \mathrm{Hom}_{C(\mathcal{C})}(X,Y)/\sim \end{aligned}$$

where $\sim$ is given by $f \sim g$ if there exists a chain homotopy between them.

Recall

$$\begin{tikzcd} \cdots \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r] \arrow[dl, "s"] & X^{n+1} \arrow[r] \arrow[dl, "s"] & \cdots \\ \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

a chain homotopy between maps $f, g : X \to Y$ is a collection $s_n : X^n \to Y^{n-1}$ such that $f - g = ds + sd$.

We will write $C^?(\mathcal{C})$ and $K^?(\mathcal{C})$ to denote the analogous categories of complexes bounded from below $(? = +)$, from above $(? = -)$, or from both directions $(? = b)$.

Example 18.1. If $\mathcal{C}$ is an abelian category. Any complex $X \in C(\mathcal{C})$ has (co)homology objects in $\mathcal{C}$, namely

$$H^n(X) = \frac{\ker(d_X^n)}{\mathrm{image}(d_X^{n-1})} \in \mathcal{C}$$

Maps in $C(\mathcal{C})$ induce maps on homology, and homotopic maps induce the same map on H^n.

2. Additional structure on $C(\mathcal{C})$, $K(\mathcal{C})$:

   shift autoequivalences $X \to X[1]$ where $(X[1])^n = X^{n+1}$. More generally, for all $k \in \mathbb{Z}$, we have $X[k] \in C(\mathcal{C})$ (or $K(\mathcal{C})$) with $(X[k])^n = X^{n+k}$ and $d^n_{X[k]} = (-1)^k d^{n+k}_X$. These shifts are indeed functors: if $f : X \to Y$, then $f[k] : X[k] \to Y[k]$ by $f[k]^n = f^{n+k}$.

   New objects from old: (In parallel with algebraic topology)

   Let $f : X \to Y$ be a map in $C(\mathcal{C})$. Define a cone of f by $C(f) \in C(\mathcal{C})$ such that $C(f)^n = X^{n+1} \oplus Y^n$ with

$$d^n_{C(f)} : X^{n+1} \oplus Y^n \longrightarrow X^{n+2} \oplus Y^{n+1}$$

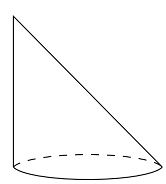
   where the map on X^{n+1} is given by $(d^n_{X[1]}, f)$ and on Y^n it is given by $(0, d^n_Y)$.

   Exercise 18.2. Check that $d^2 = 0$.

   We have that

$$\begin{pmatrix} d^{n+1}_{X[1]} & f \\ 0 & d^{n+1}_Y \end{pmatrix}\begin{pmatrix} d^n_{X[1]} & f \\ 0 & d^n_Y \end{pmatrix} = \begin{pmatrix} d^{n+1}_{X[1]} \circ d^n_{X[1]} & d^{n+1}_{X[1]} \circ f + f \circ d^n_Y \\ 0 & d^{n+1}_Y \circ d^n_Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

18.1. **Topological analogue.** If X is a topological space, recall that the cone of X,



$$C(X) = \frac{X \times I}{X \times \{1\}}$$

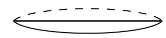
and if $f : X \to Y$ is a map of topological spaces, then $C(f)$ is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[d] \\ C(X) \arrow[r] & C(f) \end{tikzcd}$$

i.e. $C(f) = \dfrac{C(X) \cup Y}{(x, 0) \sim f(x)}$

Example 18.3.

(**Top**) $Y = \{*\}$ and $f : X \to \{*\}$. Then



$$C(f) = \text{suspension of } X$$

(**Alg**) Y = the zero complex, $f : X \to 0$. Then $C(f)^n = X^{n+1}$ with boundary map $d^n_{X[1]}$, i.e., $C(F) = X[1]$.

Example 18.4.

(**Top**) $C(\mathrm{id}_X) = C(X)$

(**Alg**)

Exercise 18.5. $\mathcal{C}$ abelian, $X \in C(\mathcal{C})$. Then $C(\mathrm{id}_X)$ has trivial homology.

The identity morphism on the mapping cone is homotopic to zero via the map

$$s_n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{X^n} & 0 \end{pmatrix}$$

So, the identity $id_{C(id_X)}$ is equal to the zero map and the mapping cone is isomorphic to the zero complex. Hence, it has trivial homology and it is irreducible.

Example 18.6. Cones as substitutes for cokernels in non-abelian settings

(**Top**) If $f : X \hookrightarrow Y$ is a nice inclusion (cofibration), then $C(f) \xrightarrow{\text{collapse } C(X)} Y/X$ is a homotopy equivalence.

(**Alg**) Let $\mathcal{C}$ = the abelian category of $R-$ modules for some ring R. Let $M, N \in \mathcal{C}$ and $f : M \to N$. Regard this as a map between complexes concentrated in degree zero. Then $C(f)^n = (M)^{n+1} \oplus (N)^n$, i.e.,

$$\cdots \to 0 \to C(f)^{-1} = M \xrightarrow{f} C(f)^0 = N \to 0 \to \cdots$$

i.e., $C(f)$ is a complex such that

$$H^{-1}(C(f)) = \ker(f)$$
$$H^0(C(f)) = \mathrm{coker}(f)$$

In particular, if f is an inclusion $H^n(C(f)) = 0$ for all $n \neq 0$ and $H^0(C(f)) = N/M$.

3. What happens in $K(\mathcal{C})$ that does not happen in $C(\mathcal{C})$?

From the cone construction, for any $f : X \to Y$ we get maps of complexes

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} X[1]$$

Exercise 18.7. In $K(\mathcal{C})$, we have isomorphisms $X[1] \overset{\psi}{\xrightarrow{\sim}} C(i(f))$ such that

$$\begin{tikzcd}
Y \arrow[r, "i(f)"] \arrow[d, equal] & C(f) \arrow[r, "\pi(f)"] \arrow[d, equal] & X[1] \arrow[r, "-f[1]"] \arrow[d, "\psi"', "\sim"] & Y[1] \arrow[d, equal] \\
Y \arrow[r, "i(f)"] & C(f) \arrow[r, "i(i(f))"] & C(i(f)) \arrow[r, "\pi(i(f))"] & Y[1]
\end{tikzcd}$$

commutes in $K(\mathcal{C})$.

We want to show that the rotated triangle

$$Y \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} X[1] \xrightarrow{-f[1]} Y[1]$$

is isomorphic in $K(C)$ to the standard triangle for $i(f)$

$$Y \xrightarrow{i(f)} C(f) \xrightarrow{i(i(f))} C(i(f)) \xrightarrow{\pi(i(f))} Y[1]$$

Define $\phi = (\phi_n) : X[1] \to C(i(f))$ by setting $\phi = (-f, \mathrm{id}_X, 0)$ and define $\psi : C(i(f)) \to X[1]$ by setting $\psi : (0, \mathrm{id}_X, 0)$. These gives morphisms of triangles since we have $\pi(i(f)) \circ \phi = -f[1]$ and $\phi \circ \pi(f) \sim i(i(f))$. Similarly, ψ is a morphism of triangles since $\pi(f) = \psi \circ (i(i(f)))$ and $-f[1] \circ \psi \sim \pi(i(f))$. Now, this morphisms are isomorphisms since $\psi \circ \phi = \mathrm{id}_{X[1]}$ and $\phi \circ \psi \sim \mathrm{id}_{C(i(f))}$.

Definition 18.8. A triangle, $\triangle$, in $K(\mathcal{C})$ is any sequence of maps

$$X \to Y \to Z \to X[1]$$

A distinguished triangle is a triangle that is isomorphic to one of the form

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} X[1]$$

for some map f.

Theorem 18.9. *(Verdier) For an additive category $\mathcal{C}$, $K(\mathcal{C})$ equipped with its shift autoequivalence $[1]$ and its collection of distinguished triangles is a triangulated category.*

Here's what this means:

Definition 18.10. A triangulated category is an additive abelian category $\mathcal{D}$ with autoequivalence $[1] : \mathcal{D} \to \mathcal{D}$ and a designated collection of "distinguished triangles" inside the collection of all triangles, satisfying

(TR1)
- For all $X \in \mathcal{D}$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle
- any triangle isomorphic to a distinguished triangle is itself distinguished
- any map $f : X \to Y$ can be completed to a distinguished triangle, i.e., there exist Z and maps such that $X \xrightarrow{f} Y \to Z \to X[1]$ is a distinguished triangle; such a Z is called a cone of f.

Example 18.11. For $K(\mathcal{C}) = \mathcal{D}$, we have to check that $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is distinguished, which is equivalent to the exercise of showing that the cone of the identity map is contractible.

(TR2) A triangle is distinguished if and only if its rotation is distinguished, i.e.,

$$\triangle : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
$$\mathrm{rot}(\triangle) : Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

Remark 18.12. For $K(\mathcal{C}) = \mathcal{D}$, this axiom is equivalent to Exercise 18.7.

(TR3) Given two distinguished triangles and maps α and β between their terms making this diagram commute

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

then there exists $\gamma : Z \to Z'$ such that all squares commute.

Example 18.13. For $K(\mathcal{C})$ this is easy to check since we can reduce to triangles of the form $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} X[1]$.

(TR4) Octahedron axiom. We won't state this in detail.

Motto: triangulated analogue of third isomorphism theorem: $(Y/X)/(Z/X) \cong Y/Z$.

We will often use the following:

Exercise 18.14. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then for any $U \in \mathcal{D}$, the induced sequence

$$\cdots \to \mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y) \to \mathrm{Hom}(U, Z) \to \mathrm{Hom}(U, X[1]) \to \cdots$$

is exact.

Proof. Indeed, if $l : U \to Y$ is a morphism such that $g \circ l = 0$, then we have a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "l"'] & 0 \arrow[r] \arrow[d] & U[1] \arrow[r, "-1"] & U[1] \arrow[d, "-l[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \arrow[r, "-f[1]"] & Y[1]
\end{tikzcd}$$

where the top and bottom rows are distinguished triangles. Hence there is a morphism $\phi : U \to X$ such that $-\phi[1]$ makes the diagram commute. In particular, $l = f \circ \phi$. The same holds for $\mathrm{Hom}(\cdot, U)$: we say that $\mathrm{Hom}(U, \cdot)$ and $\mathrm{Hom}(\cdot, U)$ are cohomological functors. $\square$

4. Now let $\mathcal{C}$ be a abelian category. Then $K(\mathcal{C})$ and $C(\mathcal{C})$ have more structure: they have homology functors

$$\begin{tikzcd} H^n : C(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{C} \\ K(\mathcal{C}) \arrow[ur, "\text{factors through } K(\mathcal{C})"'] & \end{tikzcd}$$

and more generally, we have <u>truncation</u> functors $\tau^{\leqslant n}$, $\tau^{\geqslant n}$ given by

$$\tau^{\leqslant n}(X) = (\cdots \to X^{n-2} \to X^{n-1} \to \ker(d_X^n) \to 0 \to 0 \to \cdots)$$
$$\tau^{\geqslant n}(X) = (\cdots \to 0 \to 0 \to \mathrm{coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots)$$

so $\tau^{\leqslant n}$ has the same cohomology as X in degrees $\leqslant n$ and $\tau^{\geqslant n}$ has the same cohomology as X in degrees $\geqslant n$.

The familiar statement that "a short exact sequence of complexes yields a long exact sequence in cohomology" generalizes to

Lemma 18.15. *$H^0 : K(\mathcal{C}) \to \mathcal{C}$ is a cohomological functor, i.e., for any distinguished triangle $X \to Y \to Z \to X[1]$ we get an exact sequence*

$$H^0(X) \to H^0(Y) \to H^0(Z)$$

Proof. Exercise. $\square$

Corollary 18.16. *For any distinguished triangle, we get a long exact sequence*

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$$

Proof. Use previous lemma and the fact that a rotation of a distinguished triangle is also a distinguished triangle. $\square$

Example 18.17. An example of a short exact sequence in $C(\mathcal{C})$ whose image in $K(\mathcal{C})$ is not a distinguished triangle. Consider $\mathcal{C} = \underline{\mathrm{Ab}}$. Take the short exact sequence of complexes

$$0 \to \mathbb{Z}/2 \xrightarrow{[2]} \mathbb{Z}/4 \xrightarrow{\text{proj}} \mathbb{Z}/2 \to 0$$

Proposition 18.18. *In $K(\underline{\mathrm{Ab}})$, $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ cannot be completed to a distinguished triangle.*

Proof. Suppose it could be completed. Then we would have the distinguished triangles

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, "{[2]}"] \arrow[d, equal] & \mathbb{Z}/4 \arrow[r] \arrow[d, dashed, "\beta"] & \mathbb{Z}/2 \arrow[r, "h=0"] \arrow[d, "0"] & \mathbb{Z}/2[1] \arrow[d, equal] \\
\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2[1]
\end{tikzcd}$$

Then in $K(\underline{\text{Ab}})$ we can fill in β making all squares commute (TR3). In $K(\underline{\text{Ab}})$, $\beta \circ [2] = \text{id} : \mathbb{Z}/2 \to \mathbb{Z}/2$.

So in $C(\underline{\text{Ab}})$, there exists a chain homotopy s from $\text{id}_{\mathbb{Z}/2}$ to $\beta \circ [2]$. Since both complexes are concentrated in degree 0, s must be zero, so $\beta \circ [2] = \text{id}_{\mathbb{Z}/2}$ in $C(\underline{\text{Ab}})$, hence $\underline{\text{Ab}}$. This implies that $\mathbb{Z}/4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, a contradiction. $\square$

Remark 18.19. The short exact sequence of complexes still gives a long exact sequence in cohomology even though it does not give a distinguished triangle in $K(\underline{\text{Ab}})$. The derived category will fix this lack of distinguished triangles.

Let $\mathcal{C}$ be an abelian category, and let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $C(\mathcal{C})$. We want to see what would be needed to complete this to a distinguished triangle. Look at

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & C(f) \arrow[r] \arrow[d, "{\phi=(0,g)}"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, dashed, "?"] & X[1]
\end{tikzcd}$$

If ϕ were invertible, then we could fill in the bottom row to complete the short exact sequence to a distinguished triangle. As we've seen, this may not be possible in $K(\mathcal{C})$, so instead we look for a related category in which ϕ becomes invertible.

Lemma 18.20. *For a short exact sequence* $0 \to X \xrightarrow{f} Y \to Z \to 0$*,* $\phi : C(f) \to Z$ *is an isomorphism on cohomology.*

Definition 18.21. A map of complexes inducing isomorphisms on cohomology is called a quasi-isomorphism.

Proof. We have a short exact sequence of complexes $0 \to C(\text{id}_X) \to C(f) \xrightarrow{\phi} Z \to 0$ given in degree n by

$$0 \to X^{n+1} \oplus X^n \xrightarrow{\text{id}_X \oplus f} X^{n+1} \oplus Y^n \xrightarrow{(0,g)} Z^n \to 0$$

We get a long exact sequence on H^n, and then conclude the proof using that $H^n(C(\text{id}_X)) = 0$ for all n. $\square$

We will now pass to a new category, $D(\mathcal{C})$, where quasi-isomorphisms become isomorphisms.

Definition 18.22. Let $\mathcal{C}$ be an abelian category. There exist (see below for a set-theoretic caveat) a category $D(\mathcal{C})$ and a functor $Q : K(\mathcal{C}) \to D(\mathcal{C})$ satisfying

1) Q(quasi-isomorphism) = isomorphism

2) For all functors F such that $F(\text{quasi-isomorphism}) = \text{isomorphism in } \mathcal{D}$

$$\begin{tikzcd} K(\mathcal{C}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(\mathcal{C}) \arrow[dl, dashed, "\exists!"] \\ & \mathcal{D} & \end{tikzcd}$$

we can uniquely complete the diagram such that it commutes.

In other words, $D(\mathcal{C})$ is the localization of $K(\mathcal{C})$ with respect to quasi-isomorphisms (more on this later).

Moreover,

1. $D(\mathcal{C})$ is a triangulated category
   - additive
   - $[1]$ induced from $K(\mathcal{C})$
   - distinguished triangles are triangles in $D(\mathcal{C})$ isomorphic in $D(\mathcal{C})$ to the image of distinguished triangles in $K(\mathcal{C})$
2. Cohomology functors and truncations descend to $D(\mathcal{C})$, i.e., we get functors

$$\begin{tikzcd} H^n : D(\mathcal{C}) \arrow[rr] & & \mathcal{C} \\ & K(\mathcal{C}) \arrow[ul, "Q"] \arrow[ur, "H^n"'] & \end{tikzcd}$$

   and $\tau^{\leqslant n} : D(\mathcal{C}) \to D^-(\mathcal{C})$, $\tau^{\geqslant n} : D(\mathcal{C}) \to D^+(\mathcal{C})$ again by factoring the earlier $\tau^{\leqslant n}, \tau^{\geqslant n}$.

18.2. **The construction of $D(\mathcal{C})$.** We have $\mathrm{ob}D(\mathcal{C}) = \mathrm{ob}K(\mathcal{C}) = \mathrm{ob}C(\mathcal{C})$. For $X, Y \in D(\mathcal{C})$, $\mathrm{Hom}_{D(\mathcal{C})}(X, Y) = \{(s, f)\}/\sim$, where s, f are maps in $K(\mathcal{C})$

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

such that s is a quasi-isomorphism (f is any map in $K(\mathcal{C})$). The equivalence relationship is given by $(s, f) \sim (t, g)$ if there exists a diagram

$$\begin{tikzcd} & & X''' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X'' \arrow[dlll, "t" near end] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd}$$

where u is a quasi-isomorphism such that the triangles commute.

There is a set-theoretic subtlety here: for general $\mathcal{C}$, $\mathrm{Hom}_{D(\mathcal{C})}(X, Y)$ may not be a set, but in the cases of interest to us it will be.

To define composition, we use the following property of the class of quasi-isomorphisms:

Definition 18.23. In any category $\mathcal{B}$, a localizing class of morphisms is a class S of maps satisfying

1) S is closed under composition and $\mathrm{id}_X \in S$ for all X
2) For all $Z \xrightarrow{s \in S} Y \xleftarrow{f} X$, there exists a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $s \in S$, and likewise

$$\begin{tikzcd} W & Z \arrow[l, dashed, "g"'] \\ X \arrow[u, dashed, "t"] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd}$$

3) For any two maps $f, g : X \to Y$, there exists $s \in S$ such that $sf = sg \Leftrightarrow$ there exists $t \in S$ such that $ft = gt$.

Now, we use the fact that $S = \{\text{quasi-isomorphisms}\} \subset K(\mathcal{C})$ is a localizing class (this requires proof, but we omit it) to define composition in $D(\mathcal{C})$. Given maps F, G represented by roofs

$$F = \left\{ \begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right. \quad \text{and } G = \left\{ \begin{tikzcd} & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd} \right.$$

there exists a commutative diagram (since quasi-isomorphisms are a localizing class)

$$\begin{tikzcd} & & X''' \arrow[dl, "u \in S"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z. \end{tikzcd}$$

Then $G \circ F \in \mathrm{Hom}_{D(\mathcal{C})}(X, Z)$ is represented by the roof

$$\begin{tikzcd} & X''' \arrow[dl, "sou"'] \arrow[dr, "goh"] & \\ X & & Z \end{tikzcd}$$

Remark 18.24. For any category $\mathcal{B}$ and localizing class S of morphisms, we can (modulo set-theoretic difficulties! we need some statement like "all morphisms are equivalent to those defined using only a *set* S") define $\mathcal{B}[S^{-1}]$ by the same formulas. This comes with a canonical (localization) functor $Q\colon \mathcal{B} \to \mathcal{B}[S^{-1}]$ satisfying the universal property that the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{D} \\ & \mathcal{B}[S^{-1}] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

can be uniquely completed for any functor F such that $F(S) \subset$ isomorphisms in $\mathcal{D}$.

18.3. **Truncation structure on $D(\mathcal{C})$ and t-structures.**

Definition 18.25. Let $\mathcal{C}$ be any triangulated category. A t-structure on $\mathcal{C}$ is a pair $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ of full subcategories of $\mathcal{C}$ satisfying the following properties:

Set $\mathcal{C}^{\leqslant 0}[n] := \mathcal{C}^{\leqslant -n}$ and $\mathcal{C}^{\geqslant 0}[n] := \mathcal{C}^{\geqslant -n}$

i) $\mathcal{C}^{\leqslant -1} \subseteq \mathcal{C}^{\leqslant 0}$ and $\mathcal{C}^{\geqslant 1} \subseteq \mathcal{C}^{\geqslant 0}$

ii) $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 1}) = 0$

iii) For all $X \in \mathcal{C}$, there exists a distinguished triangle $X_{\leqslant 0} \to X \to X_{\geqslant 1} \to X_{\leqslant 0}[1]$ with $X_{\leqslant 0} \in \mathcal{C}^{\leqslant 0}$ and $X_{\geqslant 1} \in \mathcal{C}^{\geqslant 1}$.

Proposition 18.26. *For an abelian category $\mathcal{A}$ and $\mathcal{C} = D(\mathcal{A})$, set*

$$\mathcal{C}^{\leqslant 0} = \{\textit{full subcategory of } D(\mathcal{A}) \textit{ of complexes } X \textit{ such that } H^n X = 0 \forall n > 0\}$$
$$\mathcal{C}^{\geqslant 0} = \{\textit{full subcategory of } D(\mathcal{A}) \textit{ of complexes } X \textit{ such that } H^n X = 0 \forall n < 0\}$$

Then $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ is a t-structure on $D(\mathcal{A})$.

Proof.

i) obvious

iii) In $C(\mathcal{A})$ we can form $\operatorname{coker}(\tau^{\leqslant 0} X \to X)$. This cokernel is quasi-isomorphic to $\tau^{\geqslant 1} X$. Use the fact that short exact sequences of complexes yield a distinguished triangle in $D(\mathcal{A})$.

ii) interesting: see subsequent discussion for the proof.

∎

We will expand on this point (ii):

Definition 18.27. For all $X, Y \in D(\mathcal{A})$, define $\operatorname{Ext}^i_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{D(\mathcal{A})}(X, Y[i])$.

Lemma 18.28. *Given $X, Y \in D(\mathcal{A})$ and $a, b \in \mathbb{Z}$ such that $H^i(X) = 0$ for $i > a$ (i.e. $X \in D(\mathcal{A})^{\leqslant a}$) and $H^j(Y) = 0$ for $j < b$ (i.e. $Y \in D(\mathcal{A})^{\geqslant b}$). Then $\operatorname{Ext}^n_{\mathcal{A}}(X, Y) = 0$ for $n < b - a$ and $\operatorname{Ext}^{b-a}_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(H^a(X), H^b(Y))$.*

Proof. By assumption, $Y \to \tau^{\geqslant b} Y$ is a quasi-isomorphism, so we can replace Y by a complex that is zero in degrees $< b$. Also $\tau^{\leqslant a} X \to X$ is a quasi-isomorphism, so we can replace X by a complex that is zero in degrees $> a$. Now any map $X \to Y[n]$ in $D(\mathcal{A})$ can be represented by

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y[n] \end{tikzcd}$$

where s is a quasi-isomorphism. We can assume $L^i = 0$ for $i > a$. So $f^i : L^i \to Y^{n+i}$ and claim this is zero for $n < b - a$. If $i > a$, then $L^i = 0$ and we obtain the result. If $i \leqslant a$, then $n + i \leqslant n + a < b$ so $Y^{n+i} = 0$ and we obtain the result.

Exercise 18.29. For $n = b - a$ the map f induces and is equivalent to a map $H^a(X) \to H^b(Y)$.

$\square$

Note that for $n > b - a$, in general we do have interesting $\operatorname{Ext}^n_{\mathcal{A}}$'s.

Remark 18.30. Definition 18.27 gives us a definition of Ext groups even for abelian categories not having enough injectives. If $\mathcal{A}$ has enough injectives, then we also have the classical definition of $\operatorname{Ext}^n(X, Y)$ for $X, Y \in \mathcal{A}$ as n^{th} derived functor of $\operatorname{Hom}(X, \cdot)$ and in this setting

$$\underbrace{\operatorname{Ext}^n_{\mathcal{A}} \overbrace{(X, Y)}^{\substack{\text{objects of} \\ \mathcal{A} \text{ placed} \\ \text{in degree } 0}}}_{D(\mathcal{A}) \text{ definition}} \simeq \underbrace{\operatorname{Ext}^n(X, Y)}_{\text{classical definition}}$$

We can check this via the universal δ-functor formalism: for $n = 0$, both sides are $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ and so it suffices to show both families of higher Ext's are universal δ-functors. We know this for the "classical" Ext-groups, so we just have to check that $\operatorname{Ext}^n_{\mathcal{A}}(X, \cdot)$ is effaceable for all $n > 0$. For this (since $\mathcal{A}$ has enough injectives) it is enough to show $\operatorname{Ext}^n_{\mathcal{A}}(X, I) = 0$ for all $n > 0$ where I is an injective, i.e., we must show

$$\operatorname{Hom}_{D(\mathcal{A})}(X, I[n]) = 0$$

for all $n > 0$. We'll show much more in the following lemma and theorem.

Lemma 18.31. *For $X \in K^+(\mathcal{A})$, I a bounded below complex of injectives $\in K^+$(injective objects), $\mathrm{Hom}_{K(\mathcal{A})}(X, I) \to \mathrm{Hom}_{D(\mathcal{A})}(X, I)$ is an isomorphism.*

In particular, for $X \in \mathcal{A}$, $I \in \mathcal{A}$, $n > 0$.

$$\underbrace{\mathrm{Hom}_{K(\mathcal{A})}(X, I[n])}_{\substack{= 0 \text{ (for degree reasons)} \\ I[n] \text{ concentrated in degree } -n, \\ \text{so no maps in } C(\mathcal{A})}} = \mathrm{Hom}_{D(\mathcal{A})}(X, I[n])$$

We will see this in the course of proving the following theorem:

Theorem 18.32. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{I}$ = full subcategory consisting of injective objects. Then the natural functor*

$$K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is an equivalence.

Proof. Essential surjectivity follows from $\mathcal{A}$ having enough injectives. Hence, any object of $D^+(\mathcal{A})$ is quasi-isomorphic to a complex of injectives.

For full faithfulness, we described maps $X \to Y$ in $D^+\mathcal{A}$ as equivalence classes of

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where s is a quasi-isomorphism and f is any map in $K(\mathcal{A})$. Since quasi-isomorphisms form a localizing class, we can instead take equivalence classes of

$$\begin{tikzcd} & X' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

Now suppose we have $X, Y \in K^+\mathcal{I}$.

We will check that any pair $(f, s), (g, t)$ of maps between X and Y that become equivalent in $D^+\mathcal{A}$ are already equivalent in $K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}]$; the argument that we are about to give easily adapts to prove Lemma 18.31 as well, and one then deduces $K^+\mathcal{I} \xrightarrow{\sim} K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}]$ to complete the proof of the theorem (these final details are left as an exercise).

The key point will be the following assertion, whose proof is postponed until the end of the argument:

(*) For all $I \in K^+(\mathcal{I}$, $X \in K^+(\mathcal{A})$, if $s \colon I \to X$ is a quasi-isomorphism, then there exists $t \colon X \to I$ such that $t \circ s = \mathrm{id}_I \in K^+(\mathcal{I})$.

Suppose there exists $X''' \in D^+\mathcal{A}$ and maps $h, u \in K^+\mathcal{A}$ with u a quasi-isomorphism such that the following diagram commutes

$$\begin{tikzcd}
 & & X''' & & \\
 & X' \arrow[ur, "h"] & & X'' \arrow[ul, "u"'] & \\
X \arrow[ur, "f"] \arrow[urrr, "g"'] & & & & Y \arrow[ul, "t"'] \arrow[ulll, "s"]
\end{tikzcd}$$

Apply (*) to $\begin{tikzcd} Y \arrow[r, "u \circ t"] \arrow[rr, bend right, "\sim \mathrm{id}_Y"'] & X''' \arrow[r, dashed, "i"] & Y \end{tikzcd}$

Replace the previous roof diagram with its composite with $X''' \xrightarrow{i} Y$ to conclude (f, s) and (g, t) are equivalent in $K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}]$. This shows that

$$K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}] \to D^+\mathcal{A}$$

is faithful.

Now, we need to show it is full. Given $X, Y \in K^+\mathcal{I}$ and a map between them in $D^+\mathcal{A}$

$$\begin{tikzcd}
 & X' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"']
\end{tikzcd}$$

for $X' \in D^+\mathcal{A}$.

Apply (*) again: $\begin{tikzcd} Y \arrow[r, "s"] \arrow[rr, bend right, "\simeq \mathrm{id}_Y"'] & X' \arrow[r, "\exists t"] & Y \end{tikzcd}$

Then $(t \circ f, t \circ s = \mathrm{id}_Y)$ is a map $X \to Y$ in $K^+\mathcal{I}$ and it is equivalent in $D^+\mathcal{A}$ to (f, s) via

$$\begin{tikzcd}
 & & Y & & \\
 & Y \arrow[ur, equal] & & X'' \arrow[ul, "t"'] & \\
X \arrow[ur, "t \circ f"] \arrow[urrr, "f"'] & & & & Y \arrow[ul, "s"'] \arrow[ulll, equal]
\end{tikzcd}$$

Thus $K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}] \to D^+\mathcal{A}$ is fully faithful. As noted above, we are done since (by Lemma 18.31)

$$K^+\mathcal{I} \to K^+\mathcal{I}[(\text{quasi-isomorphisms})^{-1}]$$

is an isomorphisms of categories. □

Exercise 18.33. Use a similar argument to the above to prove Lemma 18.31: If $I \in K^+(\mathcal{I})$, then for any $X \in D^+\mathcal{A}$

$$\mathrm{Hom}_{K(\mathcal{A})}(X, I) \to \mathrm{Hom}_{D(\mathcal{A})}(X, I)$$

is an isomorphism.

Consider $g^{-1}u : X \xrightarrow{u} Z \xleftarrow{g} Y$ is equivalent and $s : Z \to Y$, so u is equivalent to $(sg)g^{-1}u : X \to Y$. Conversely, if $f, f' : X \to Y$ in $K(\mathcal{A})$ become identified in $D(\mathcal{A})$ then $tf = tf'$ for some quasi-isomorphism $t : Y \to Z$, hence $f = stf = stf' = f'$.

Proof. of (*)
$I \xrightarrow{s} X$ a quasi-isomorphisms with $I \in K^+\mathcal{I}$ and $X \in D^+\mathcal{A}$. Take the cone, so

$$I \xrightarrow{s} X \xrightarrow{i(s)} C(s) \xrightarrow{\pi(s)} I[1]$$

is a distinguished triangle in $K^+\mathcal{A}$. Since s is a quasi-isomorphism, $C(s)$ is acyclic (i.e., $H^n(C(s)) = 0$ for all n) by the long exact sequence in H^n.

Apply $\mathrm{Hom}(\cdot, I)$ to this distinguished triangle to get the following long exact sequence

$$\cdots \to \mathrm{Hom}(C(s), I) \to \mathrm{Hom}(X, I) \to \mathrm{Hom}(I, I) \to \mathrm{Hom}(C(s)[-1], I) \to \cdots$$

(this long exact sequence of Hom exists for all distinguished triangles in a triangulated category).

To get (*), it suffices to show $\mathrm{Hom}(C(s)[-1], I) = 0$.

We claim that if C is any acyclic bounded below complex and $I \in K^+\mathcal{I}$, then $\mathrm{Hom}_{K(\mathcal{A})}(C, I) = 0$.

Since C, I are bounded below, up to a shift we may assume that both are zero in degree < 0, so we have

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r, "d_C^0"] \arrow[d, "\pi^0"'] & C^1 \arrow[r, "d_C^1"] \arrow[d, "\pi^1"] \arrow[dl, dashed, "\exists k^0"] & C^2 \arrow[r] \arrow[d, "\pi^2"] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

since C is acyclic, then d_C^0 is injective so there exists k^0 since I^0 is injective.

To get the map $k^1 : C^2 \to I^1$, note $(\pi^1 - d_I^0 \underbrace{k^0) \circ d_C^0}_{\pi^0} : C^0 \to I^1$ is zero since π is a map of complexes. So

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}(d_C^0) \arrow[r, "d_C^1"] \arrow[d, "\pi^1 - d_I^0 k^0"'] & C^2 \arrow[dl, dashed, "\exists k^1"] \\
& I^1 &
\end{tikzcd}$$

such that $k^1 \circ d_C^1 = \pi^1 - d_I^0 \circ k^0$.

Iterate this construction, and the resulting chain-homotopy k shows that π is nullhomotopic.

□

19. Derived Functors (notes by Michael Zhao)

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. We will always implicitly assume F is also additive. Then F induces a functor $F : C(\mathcal{A}) \to C(\mathcal{B})$. Since F is is additive, it further induces a functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$. How can we construct a functor $D(A) \to D(B)$? The naïve construction would be to try term-by-term application of F. However, for $F(f)$ to be a quasi-isomorphism for all quasi-isomorphisms f, F would have to be exact.

We would like to be able to have a derived functor $D(A) \to D(B)$ for non-exact functors, e.g.

$$\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathsf{Ab},$$

where Ab is the category of abelian groups. So we will generalize to left or right-exact functors. Let's assume that F is left-exact. We want to construct a derived functor $RF : D^+\mathcal{A} \to D^+\mathcal{B}$, which satisfies the

Definition 19.1. A **derived functor** of $F : \mathcal{A} \to \mathcal{B}$ is a pair $(RF : D^+\mathcal{A} \to D^+B, s)$, where RF is **exact** (i.e. takes distinguished triangles to distinguished triangles and commutes with the shift [1]) and s is a natural transformation $s : Q \circ K^+F \to RF \circ Q$ (where $Q : K^+\mathcal{A} \to D^+\mathcal{A}$), which is **initial** among all such pairs, i.e. for all pairs $(G : D^+\mathcal{A} \to D^+\mathcal{B}, t : Q \circ K^+F \to G \circ Q)$, there is a unique natural transformation $\eta : RF \to G$ such that for all $X \in K^+\mathcal{A}$, the following diagram commutes.

$$\begin{tikzcd}
& RF \circ Q(X) \arrow[dr, "\eta"] & \\
Q \circ K^+F(X) \arrow[ur, "s_X"] \arrow[rr, "t_X"] & & G \circ Q(X)
\end{tikzcd}$$

19.1. **Construction of Derived Functors.** How to construct derived functors?

Definition 19.2. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is **adapted to** F if

1) $\mathcal{R}$ is closed under finite direct sums
2) If $X \in C^+(\mathcal{R})$ is acyclic, then $F(X)$ is acyclic.
3) For all $X \in \mathrm{ob}(\mathcal{A})$, there exists a monomorphism $X \hookrightarrow I$ for some $I \in \mathcal{R}$

The typical example is if $\mathcal{A}$ has enough injectives. Then $\mathcal{R}$ can be the class of injective objects (Clearly, 1 and 3 hold. For 2, if $I \in K^+\mathcal{I}$ is acyclic, then I is isomorphic in $K^+\mathcal{I}$ to 0).

As with our recent arguments with injectives, for any such $\mathcal{R}$, we get an equivalence of categories $K^+(\mathcal{R})[S^{-1}] \simeq D^+\mathcal{A}$, where $K^+(\mathcal{R})[S^{-1}]$ denotes the localization of $K^+(\mathcal{R})$ with respect to the class S of quasi-isomorphisms.

Then we can construct RF as follows. Consider the composite

$$K^+(\mathcal{R}) \to K^+\mathcal{A} \xrightarrow{K^+F} K^+\mathcal{B} \xrightarrow{Q} D^+\mathcal{B},$$

which takes quasi-isomorphisms to isomorphisms in $D^+\mathcal{B}$. By the universal property of localization, this defines a map $K^+(\mathcal{R})[S^{-1}] \to D^+\mathcal{B}$. By choosing an inverse to the equivalence of categories in the previous paragraph, we have a functor $D^+\mathcal{A} \to D^+\mathcal{B}$, and RF is defined to be this functor. (In sum, the construction is to take $\mathcal{R}$-resolutions and apply F term by term.)

Lemma 19.3 (Exercise)**.** *RF is exact and satisfies the universal property in definition 19.1.*

A reason not to always take $\mathcal{R}$ to be the class of injective objects is given by the

Lemma 19.4 (Grothendieck-Leray "spectral sequence")**.** *Given left-exact functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ and $\mathcal{R}_F, \mathcal{R}_G$ classes of objects adapted to F, G respectively, such that $F(\mathcal{R}_F) \subseteq \mathcal{R}_G$, there is a natural isomorphism of functors $R(G \circ F) \xrightarrow{\sim} RG \circ RF$.*

Proof. There is always a map $R(G \circ F) \to RG \circ RF$. Via the universal property of $R(G \circ F)$, we have an isomorphism

$$\mathrm{Hom}(R(G \circ F), RG \circ RF) \xrightarrow{\sim} \mathrm{Hom}(Q \circ K^+(G \circ F), (RG \circ RF) \circ Q),$$

which sends, for all pairs $(RG \circ RF, t)$, $\mathrm{Hom}(R(G \circ F), RF \circ RG) \ni \eta \mapsto t \in \mathrm{Hom}(Q \circ K^+(G \circ F), (G \circ F) \circ Q)$, where η, t, s are as in the diagram in definition 19.1. However, the right-hand side is non-empty, as it contains

$$Q \circ K^+G \circ K^+F \xrightarrow{s_G \circ K^+F} RG \circ Q \circ K^+F \xrightarrow{RG \circ s_F} RG \circ RF \circ Q.$$

Now, in the setting of the lemma, $\mathcal{R}_F$ is adapted to $G \circ F$, so $R(G \circ F)$ is computed by

(i) resolve by $K^+\mathcal{R}_F$.

(ii) apply $K^+(G \circ F) = K^+G \circ K^+F$; the left-hand side computes $R(G \circ F)$, while the right-hand side computes $RG \circ RF$.

□

Definition 19.5. For a derived functor $RF : D^+\mathcal{A} \to D^+\mathcal{B}$, write R^iF for the functor $H^i \circ RF : D^+\mathcal{A} \to \mathcal{B}$.

Example 19.6. Suppose $\mathcal{A}$ is abelian with enough injectives, and let X be an object of $\mathcal{A}$. Then we have the left-exact functor $Y \mapsto \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Then we get the derived functor $\mathrm{RHom}(X, -) : D^+\mathcal{A} \to D^+(\mathsf{Ab})$. Then we have the

Lemma 19.7. *There is an isomorphism of functors $\mathrm{Ext}^i_{\mathcal{A}}(X, -) \xrightarrow{\sim} R^i\,\mathrm{Hom}(X, -)$.*

We will actually give a more general construction, similar to this one, which allows for $X \in D\mathcal{A}$. Define a functor

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(\ ,\ ) : C(\mathcal{A})^{\mathrm{op}} \times C^+(\mathcal{A}) \to C(\mathsf{Ab})$$

by

$$\mathrm{Hom}^n_{\mathcal{A}}(X,Y) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+n}),$$

with

$$d^n((f^i : X^i \to Y^{i+n})_{i\in\mathbb{Z}}) = d_Y \circ f^i + (-1)^{n+1} f^{i+1} \circ d_X,$$

which is a map from $X^i \to Y^{i+n+1}$. (Check that $d^2 = 0$.)

A key property of $\mathrm{Hom}^\bullet_{\mathcal{A}}$ is that it computes $\mathrm{Hom}_{K(\mathcal{A})}$:

$$Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X,Y)) = \{(f^i : X^i \to Y^i) \mid d_Y f = f d_X\} = \mathrm{Hom}_{C(\mathcal{A})}(X,Y),$$

and

$$B^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X,Y)) = \{(g_i : X^i \to Y^i)_{i\in\mathbb{Z}} \mid \exists f_i : X^i \to Y^{i-1}, g = d_Y f + f d_X\},$$

which is the set of null-homotopic chain maps. Then $H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X,Y)) = \mathrm{Hom}_{K(\mathcal{A})}(X,Y)$.

Proposition 19.8. *There exists a bifunctor* $\mathrm{RHom} : D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathsf{Ab})$ *extending the previously defined* $\mathrm{RHom} : \mathcal{A}^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathsf{Ab})$*, given by*

$$\mathrm{RHom}(X,Y) := \mathrm{Hom}^\bullet(X,I),$$

for a given quasi-isomorphism $Y \xrightarrow{\sim} I$*, where* I *is a complex of injectives. Moreover, for any* X, Y*,*

$$H^i(\mathrm{RHom}(X,Y)) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]).$$

Proof. We need to check that RHom is well-defined and functorial. Note that for any $Y \xrightarrow{f} Z$ (in $C^+(\mathcal{A})$) and injective resolutions $\alpha : Y \xrightarrow{\sim} I$, $\Psi : Z \xrightarrow{\sim} J$, there exists a unique f' in $K(\mathcal{A})$ such that the following diagram commutes in $K(\mathcal{A})$ (not in $C(\mathcal{A})$!).

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\alpha"] & Z \arrow[d, "\Psi"] \\ I \arrow[r, dotted, "f'"] & J \end{tikzcd}$$

To see this, apply $\mathrm{Hom}_{K(\mathcal{A})}(-,J)$ to the distinguished triangle

$$Y \xrightarrow{\alpha} I \to C(\alpha) \to Y[1].$$

Note that $C(\alpha)$ is acyclic, and we showed that $\mathrm{Hom}_{K^+(\mathcal{A})}(M,N) = 0$ if M is acyclic and N is a complex of injectives. Thus we have a bijection

$$\mathrm{Hom}_{K(\mathcal{A})}(I,J) \xrightarrow{\sim} \mathrm{Hom}_{K(\mathcal{A})}(Y,J),$$

The desired $f' \colon I \to J$ is then the preimage of $\Psi \circ f$ under this isomorphism. By post-composition with f' we get the map $\mathrm{RHom}(X,Y) \to \mathrm{RHom}(X,Z)$.

Exercise 19.9. If you change f' by something null-homotopic, then the induced map

$$\mathrm{Hom}^\bullet(X,I) \to \mathrm{Hom}^\bullet(X,J)$$

changes by something null-homotopic.

Now we check that this extends the previous RHom : $\mathcal{A}^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathsf{Ab})$. Previously, for $X \in \mathcal{A}, Y \in D^+(\mathcal{A})$ and $\alpha : Y \xrightarrow{\sim} I$ an injective resolution, we had $\mathrm{RHom}(X, Y)_{\mathrm{old}} = \mathrm{Hom}_{\mathcal{A}}(X, I^\bullet)$. On the other hand, the version just defined is

$$\mathrm{RHom}(X, Y)_{\mathrm{new}} = \mathrm{Hom}^\bullet(X, I),$$

which equals

$$\prod_{i \in \mathbb{Z}} \mathrm{Hom}(X^i, I^{i+n}) = \mathrm{Hom}(X, I^n),$$

in degree n. There is an obvious quasi-isomorphism between these two versions of RHom.

Finally, we show that

$$H^0(\mathrm{RHom}(X, Y)) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y). \tag{1}$$

Let $\alpha : Y \xrightarrow{\sim} I$ be an injective resolution. By definition,

$$H^0(\mathrm{RHom}(X, Y)) = H^0(\mathrm{Hom}^\bullet(X, I)) = \mathrm{Hom}_{K(\mathcal{A})}(X, I) = \mathrm{Hom}_{D(\mathcal{A})}(X, I) \simeq \mathrm{Hom}_{D(\mathcal{A})}(X, Y),$$

where the second to last equality follows relies on I being a complex of injectives (Lemma 18.31).

□

20. Sheaf Cohomology (notes by Michael Zhao)

Everything we discuss will have an analog in étale topology, but we will work in the setting of classical topology where everything is more down-to-earth. Our main aim is to develop the formalism of the "six operations" between derived categories of sheaves on topological spaces.

Let X be a topological space. Let R be a commutative ring. Let $\mathsf{Sh}_R(X)$ denote the category of sheaves of R-modules. We may also leave R implicit, and just write $\mathsf{Sh}(X)$. Let $D(X) := D(\mathsf{Sh}(X))$ denote the derived category of $\mathsf{Sh}(X)$, and likewise for its bounded variants. We are interested in the induced maps on or between $D(X)$ by maps of topological spaces.

Example 20.1 (Direct and Inverse Image)**.** Let $f : X \to Y$ be a map of topological spaces. Let $\mathcal{F} \in \mathsf{Sh}(X)$. Define $f_*\mathcal{F} \in \mathsf{Sh}(Y)$ by $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$. This provides a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$.

Define, for $\mathcal{G} \in \mathsf{Sh}(Y)$, $f^{-1}\mathcal{G} \in \mathsf{Sh}(X)$ as the sheafification of the presheaf

$$(f^{-1}\mathcal{G})^\#(V) = \lim_{U \supset f(V)} \mathcal{G}(U).$$

Reminder. Sheafification and the forgetful functor which takes a sheaf and returns the underlying presheaf are left and right adjoints, respectively.

Lemma 20.2 (Exercise)**.** *f^{-1} is left adjoint to f_* for all $f : X \to Y$.*

Whenever we have F right adjoint to G, there are natural transformations

$$\mathbb{1} \xrightarrow{\epsilon} F \circ G$$
$$G \circ F \xrightarrow{\eta} \mathbb{1},$$

which are respectively called the unit and counit of the adjunction.

Exercise 20.3. For f^{-1} and f_*, write down ϵ, η.

Lemma 20.4 (Exercise)**.**

(1) All left-adjoints are right-exact.

(2) All right-adjoints are left-exact.

(3) f^{-1} is exact.

(4) f_* preserves injectives, since it has an exact left-adjoint.

Since f^{-1} is exact, we get an induced functor $f^{-1} : D(Y) \to D(X)$ by the naïve definition. Since f_* is left-exact and $\mathsf{Sh}(X)$ has enough injectives, get $Rf_* : D^+(X) \to D^+(Y)$.

Corollary 20.5.

(1) There exists an adjunction isomorphism $\mathrm{RHom}(f^{-1}K, L) \simeq \mathrm{RHom}(K, Rf_*L)$ *for $K \in D^+(Y), L \in D^+(X)$.*

(2) Apply H^0 to (1) to get the adjunction

$$\mathrm{Hom}_{D^+(X)}(f^{-1}K, L) \simeq \mathrm{Hom}_{D^+(Y)}(K, Rf_*L).$$

Remark 20.6. This is the first adjoint pair of the six operations.

Proof. (2) comes from the formula (1) applied to part (1) of the corollary, so we just need to show (1). Let $L \overrightarrow{\sim} I$ be an injective resolution. Then $Rf_*L = f_*I$ and

$$\mathrm{RHom}(K, f_*I) = \mathrm{Hom}^\bullet(K, f_*I) = \mathrm{Hom}^\bullet(f^{-1}K, I) = \mathrm{RHom}(f^{-1}K, I) = \mathrm{RHom}(f^{-1}K, L).$$

□

20.1. **Internal Hom.** We can enrich these adjuctions using the **internal hom** on $\mathsf{Sh}(X)$, which is a functor

$$\mathcal{H}om : \mathsf{Sh}(X)^{\mathrm{op}} \times \mathsf{Sh}(X) \to \mathsf{Sh}(X),$$

given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathsf{Sh}(U)}(\mathcal{F}\big|_U, \mathcal{G}\big|_U) \in \mathsf{Ab}.$$

(Check that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$) is a sheaf.) Then $\mathcal{H}om(\mathcal{F}, -)$ is a left-exact functor $\mathsf{Sh}(X) \to \mathsf{Sh}(X)$, so we get its derived functor

$$R\mathcal{H}om(\mathcal{F}, -) : D^+(X) \to D^+(X).$$

As with our extension of RHom from $\mathcal{A}^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathsf{Ab})$ to $D(\mathcal{A}) \times D^+(\mathcal{A}) \to D(\mathsf{Ab})$, we can extend $R\mathcal{H}om$ to a bifunctor $R\mathcal{H}om : D(X)^{\mathrm{op}} \times D^+(X) \to D(X)$. As before, define $\mathcal{H}om^\bullet(K, L)$ by

$$\mathcal{H}om^n(K, L) = \prod_{i \in \mathbb{Z}} \mathcal{H}om(K^i, L^{i+n}),$$

and proceed as with RHom.

Theorem 20.7 (Formula for Global Sections Derived Functor). *For all $K \in D^-(X)$, $L \in D^+(X)$, there are natural isomorphisms*

$$R\Gamma(R\mathcal{H}om(K, L)) \xrightarrow{\sim} \mathrm{RHom}(K, L),$$

where $R\Gamma : D^+(X) \to D^+(\mathsf{Ab})$ is the derived functor of global sections $\Gamma : \mathsf{Sh}(X) \to \mathsf{Ab}$ (alternatively, $R\Gamma = R\pi_*$ for the terminal map $\pi : X \to *$).

Proof. Let $L \xrightarrow{\sim} I$ be an injective resolution. Then $R\mathcal{H}om(K, L) = \mathcal{H}om^\bullet(K, I)$. Note that this complex is bounded below. The terms of $\mathcal{H}om^\bullet(K, I)$ need not be injective, but they still belong to a class of sheaves adapted to Γ, namely the flasque sheaves.

Definition 20.8. $\mathcal{F} \in \mathsf{Sh}(X)$ is **flasque** or **flabby** if for all opens $U \supset V$, $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$.

Lemma 20.9.

(1) The class of flasque sheaves is adapted to the left-exact functor Γ.

(2) For any $\mathcal{F}$ and for an injective $I \in \mathsf{Sh}(X)$, $\mathcal{H}om(\mathcal{F}, I)$ is flasque.

(3) Injective sheaves are flasque.

Granted this lemma, we then have

$$R\Gamma(R\mathcal{H}om(K, L)) = \Gamma(X, \mathcal{H}om^\bullet(K, I)) = \mathrm{Hom}^\bullet(K, I) = \mathrm{RHom}(K, L).$$

(Given the lemma, this is the spectral sequence for composite derived functors.) □

Remark 20.10 (On (1) of Lemma 20.9). Need to know that

(a) Flasque sheaves are stable under direct sums.

(b) Every sheaf injects into a flasque one: given $\mathcal{F} \in \mathsf{Sh}(X)$, form $J \in \mathsf{Sh}(X)$ with $J(U) = \prod_{x \in U} \mathcal{F}_x$. Then $\mathcal{F} \hookrightarrow J$ and J is flasque.

(c) $\Gamma(X, N)$ is acyclic for $N \in C^+$(flasque sheaves). The main point is to show that if there is the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$ flasque, then $\mathcal{G}(X) \twoheadrightarrow \mathcal{H}(X)$.

21. Perverse Sheaves (Christian Klevdal)

The original reference for this topic the paper by Beilinson-Bernstein-Deligne (BBD) "Faisceux pervers". The name perverse sheaf is somewhat misleading since they are not honest sheaves, rather they live in $D^b_c(X)$, the bounded derived category of constructible sheaves on a topological space. In what follows, we will either take X to be a variety over the complex numbers in the classical (analytic) topology, or a variety over any perfect field k, in the étale topology.

Perverse sheaves are the heart of a non-standard t-structure on $D^b_c(X)$. Recall that for an abelian category $\mathcal{A}$, the standard t-structure on $D(\mathcal{A})$ takes $D^{\leq 0}$, respectively $D^{\geq 0}$ to be the complexes K with $H^i(K) = 0$ for $i > 0$, respectively $i < 0$. Before getting down to the specific case of perverse sheaves, we recall the definition and some generalities about t-structures.

Definition 21.1. A t-structure on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories where (with the notation $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ and likewise for $\mathcal{D}^{\leq n}$) we have:

(1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.

(2) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.

(3) For all $X \in \mathcal{D}$, there exists a distinguished triangle
$$Y \to X \to Z \to Y[1]$$
with $Y \in \mathcal{D}^{\leq 0}$ and $Z \in \mathcal{D}^{\geq 1}$.

Recall that in the case of $D(\mathcal{A})$ and the standard t-structure we had truncation functors
$$\begin{aligned} &\tau^{\leq n} \colon D(\mathcal{A}) \to D(\mathcal{A})^{\leq n} \qquad && K \mapsto \quad \cdots \to K^{n-1} \to \ker(d^n) \to 0 \to \cdots \\ &\tau^{\geq n} \colon D(\mathcal{A}) \to D(\mathcal{A})^{\geq n} \qquad && K \mapsto \quad \cdots \to 0 \to \mathrm{coker}(d^n) \to K^{n+1} \to \cdots \end{aligned}$$
and we used these functors to show the existence of the distinguished triangle in axiom 3 of a t-structure. Our next step is to show that given any t-structure, such truncation functors always exist.

Proposition 21.2. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{D}$. Then*

(1) There exists a left adjoint $\tau^{\geq n}$ for the inclusion $\mathcal{D}^{\geq n} \to \mathcal{D}$ and a right adjoint $\tau^{\leq n}$ for the inclusion $\mathcal{D}^{\leq n} \to \mathcal{D}$.

(2) For all $X \in \mathcal{D}$ there exists a unique $\delta_X \colon \tau^{\geq n+1} X \to \tau^{\leq n} X[1]$ such that
$$\tau^{\leq n} X \to X \to \tau^{\geq n} X \xrightarrow{\delta_X} \tau^{\leq n} X[1]$$
is a distinguished triangle (the maps to and from X are the canonical maps from adjunction).

Proof. First, we may assume $n = 0$. By (3) from the definition of a t-structure, there exists some distinguished triangle
$$X_{\leq 0} \to X \to X_{\geq 1} \to X_{\leq 0}[1]$$
with $X_{\leq 0} \in \mathcal{D}^{\leq 0}$ and $X_{\geq 1} \in \mathcal{D}^{\geq 1}$. If Y is in $\mathcal{D}^{\leq 0}$ we apply $\mathrm{Hom}(Y, \cdot)$ to the above exact triangle and get an exact sequence
$$\mathrm{Hom}(Y, X_{\geq 1}[-1]) \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, X_{\leq 0}) \to \mathrm{Hom}(Y, X_{\geq 1})$$
But we know that the first and last terms of the sequence are zero from properties (1), (2) of a t-structure. Defining $\tau^{\leq 0} X = X_{\leq 0}$ we get the required isomorphism. However, we need to check that this definition of $\tau^{\leq 0}$ gives a well defined functor, and that $\tau^{\leq 0}$ is the right adjoint of inclusion (i.e. that the isomorphisms given above are functorial). It is sufficient to show for each map $X \to X'$ in $\mathcal{D}$ that there is a unique factorization

$$\begin{tikzcd}
\tau^{\leq 0}X \arrow[r, dotted, "\exists!"] \arrow[d] & \tau^{\leq 0}X' \arrow[d] \\
X \arrow[r] & X'
\end{tikzcd}$$

To see the unique factorization, consider the exact triangle $\tau^{\leq 0}X' \to X' \to X'_{\geq 1} \to (\tau^{\leq 0}X')[1]$. Applying $\mathrm{Hom}(\tau^{\leq 0}X, \cdot)$ we get an exact sequence

$$\mathrm{Hom}(\tau^{\leq 0}X, X'_{\geq 1}[-1]) \to \mathrm{Hom}(\tau^{\leq 0}X, X') \to \mathrm{Hom}(\tau^{\leq 0}X, \tau^{\leq 0}X') \to \mathrm{Hom}(\tau^{\leq 0}X, X'_{\geq 1})$$

Again the first and last terms vanish. Thus the middle arrow is an isomorphism and we get a unique factorization as required. If we define $\tau^{\geq 1}X = X_{\geq 1}$ a similar proof shows that this is a well defined functor that is left adjoint to inclusion.

The second claim follows from the following lemma. □

Lemma 21.3. *Suppose we are given maps δ_1, δ_2 such that $A \to X \to B \xrightarrow{\delta_i} A[1]$ is an exact triangle for $i = 1, 2$. Then $\delta_1 = \delta_2$ if* $\mathrm{Hom}(A[1], B) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "id"] & X \arrow[r, "b"] \arrow[d, "id"] & B \arrow[r, "\delta_1"] \arrow[d, dotted, "f"] & A[1] \arrow[d, "id"] \\
A \arrow[r, "a"] & X \arrow[r, "b"] & B \arrow[r, "\delta_2"] & A[1]
\end{tikzcd}$$

The existence of the dotted arrow making this a morphism of triangles follows from the axioms of a triangulated category. Our aim is to show that $f = id_B$. Now we have that $fb = b$ or equivalently $(id_B - f)b = 0$. Applying $\mathrm{Hom}(\cdot, B)$ to the top row we get an exact sequence

$$0 = \mathrm{Hom}(A[1], B) \to \mathrm{Hom}(B, B) \to \mathrm{Hom}(X, B)$$

Clearly $f - id_B$ maps to zero in the last Hom set. Since the map is injective, we conclude that $f = id_B$ as required. □

Definition 21.4. Let $\mathcal{D}$ be a triangluated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t-structure on $\mathcal{D}$. The heart of this t-structure is $\mathcal{D}^{\heartsuit} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

We can define cohomology a cohomology functor $H^0 \colon \mathcal{D} \to \mathcal{D}^{\heartsuit}$ with respect to this t-structure to be the composition $\tau^{\leq 0} \circ \tau^{\geq 0} (\simeq \tau^{\geq 0} \circ \tau^{\leq 0}$. Likewise, we define $H^n = H^0 \circ [n]$.

Example 21.5. If $\mathcal{A}$ is an abelian category we can give $D(A)$ the standard t-structure (where $\mathcal{D}^{\leq 0}$, resp. $\mathcal{D}^{\geq 0}$, consists of those complexes with zero cohomology in degree $i > 0$, resp. $i < 0$). In this case, the heart of this t-structure can be naturally identified with $\mathcal{A}$ (by consider objects as chain complexes concentrated in degree 0); and the cohomology functor is just the usual cohomology of a chain complex.

Note that in the above example, the heart of the t-structure is an abelian category. Surprisingly, this turns out to be true in general.

Theorem 21.6. *(BBD) For any t-structure on a triangulated category $\mathcal{D}$, the heart $\mathcal{D}^\heartsuit$ is an abelian category.*

The next natural question is what can we say about $\mathcal{D}^\heartsuit$? One thing that holds is that for $A, B \in \mathcal{D}^\heartsuit$ one has $\mathrm{Ext}^1_{\mathcal{D}^\heartsuit}(A,B) \cong \mathrm{Hom}(A, B[1])$. We caution that the analogue does not hold for higher exts.

Example 21.7. Suppose X is a (reasonable) topological space, and let

$$\mathcal{D} = D_{\mathrm{loc}}(X) = \{K \in D(X) \colon H^i(K) \text{ is locally constant for all } i\}.$$

The standard t-structure on $D(X)$ induces a t-structure on $\mathcal{D}$. If X is connected and simply connected, then locally constant sheaves are constant, hence $\mathcal{D}^\heartsuit$ is equivalent to the category Ab of abelian groups. Thus if $n > 1$

$$\mathrm{Ext}^n_{\mathcal{D}^\heartsuit}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \mathrm{Ext}^n_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}) = 0$$

However,

$$\mathrm{Hom}_{\mathcal{D}}(\underline{\mathbb{Z}}_X, \underline{\mathbb{Z}}_X[n]) = H^0\mathrm{RHom}(\underline{\mathbb{Z}}_X, \underline{\mathbb{Z}}_X[n]) = H^0\mathrm{R}\Gamma(X, \underline{\mathbb{Z}}_X[n]) = H^n(X, \mathbb{Z})$$

So if X is simply connected but has higher cohomology (eg, $X = S^2$), we see that $D(\mathcal{D}^\heartsuit)$ is not equivalent to $D_{\mathrm{loc}}(X)$.

However, in the cases we are interested in, we have the following

Theorem 21.8. *(Nori) If X/k is a quasi-projective variety, then $D^b_c(X)$ is equivalent to the bounded derived category of the abelian category of constructible sheaves on X. (Beilinson) These are equivalent to the bounded derived category of the abelian category of perverse sheaves on X.*

Both of these assertions have geometric content!

Before defining the perverse t-structure, we make a couple more general definitions. First recall (excercise):

Theorem 21.9. *For any t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{D}$, the cohomology functor $H^0 \colon \mathcal{D} \to \mathcal{D}^\heartsuit$ is a cohomological functor (meaning it takes distinguished triangles to long exact sequences).*

Definition 21.10. Lef $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ be exact functors of triangulated categories (meaning that F commutes with the shift and preserves distinguished triangles). Suppose we have t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ on $\mathcal{D}_i$ for $i = 1, 2$. Then we say that F is t-left exact if $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$ and that F is t-right exact if $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$. We say that F is t-exact if it is both t-left and t-right exact.

In the setting of the definition (with out any assumptions on the t-exactness of F), we define a functor $F^\heartsuit \colon \mathcal{D}_1^\heartsuit \to \mathcal{D}_2^\heartsuit$ by the composition

$$\mathcal{D}_1^\heartsuit \subset \mathcal{D}_1 \xrightarrow{F} \mathcal{D}_2 \xrightarrow{H^0} \mathcal{D}_2^\heartsuit$$

As a remark, in BBD, $F^\heartsuit$ is denoted ${}^pF$. We may later lapse into this notation.

Before defining the perverse t-structure (whose heart will be the category of perverse sheaves) we begin with a little motivation. The source of the theory is intersection cohomology

$IH^*(X)$ which for a variety X is defined as $H^*(X, IC_X)$ where IC_X is a particular perverse sheaf on X called the intersection complex. Here are some of the good features of $IH^*(X)$.

(1) $IH^*(X) = H^*(X)$ if X is smooth.

(2) $IH^*(X)$ always satisfies Poincare duality (if X is not projective, you need the compact supports version–we'll see this later).

(3) For any projective X, $IH^m(X)$ is pure of weight m, either in the sense of Hodge theory ($k = \mathbb{C}$) or Frobenius eigenvalues ($k = \mathbb{F}_q$).

Let's elaborate on this last point. In the case that $X/\mathbb{C}$ is smooth and projective, $H^m(X, \mathbb{Q})$ is naturally endowed with a pure Hodge structure of weight m. However if X is projective but not smooth, $H^m(X, \mathbb{Q})$ has a mixed Hodge structure of weights $\leq m$. In the case that $X/\mathbb{F}_q$ is smooth and projective, then the action of Frobenius on $H^m_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is pure of weight m, meaning all eigenvalues are algebraic numbers with archimedean absolute values $q^{m/2}$. If we don't assume X is smooth, then the étale cohomology is mixed with weights $\leq m$. What the final point above is saying is that when X is projective, but possibly singular, $IH^m(X)$ is pure of weight m in the appropriate sense.

We now define the perverse t-structure. In this setting, let k be a perfect field, X/k a separated scheme of finite type over k (which we may assume is reduced). We write $D(X)$ for $D^b_c(X)$ the subcategory of the bounded derived category of sheaves on X with constructible cohomology.

Definition 21.11. We define full subcategories of $D(X)$ by

$$^pD^{\leq 0}(X) = \{K \in D(X) \colon \dim(\mathrm{supp} H^i K) \leq -i, \forall i\}$$

$$^pD^{\geq 0}(X) = \{K \in D(X) \colon \dim(\mathrm{supp} H^i(\mathbb{D}_X K)) \leq -i, \forall i\}$$

(Recall that $\mathbb{D}_X K = \mathrm{R}\mathcal{H}om(K, \pi^!\mathbb{Q}_\ell)$ for $\pi \colon X \to \mathrm{Spec} k$).

This is the "self-dual" (for reasons we will see) or "middle" perversity. There are variants where one changes the numerics of the above definition, but this is the version of central importance in algebraic geometry.

Example 21.12. If X is smooth, dimension d and $\mathcal{F}$ a local system on X then $\mathcal{F}[d] \in {}^pD^{\geq 0} \cap {}^pD^{\leq 0}$. To see that it is in ${}^pD^{\leq 0}$ note that $H^i(\mathcal{F}[d])$ is non zero only in degree d, where it has full support. The dual statement will follow from Lemma 21.16

Definition 21.13. The category of perverse sheaves on X is $\mathrm{Perv}(X) = {}^pD^{\geq 0} \cap {}^pD^{\leq 0}$.

Theorem 21.14. *$({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ is a t-structure on $D(X)$ with heart* $\mathrm{Perv}(X)$*.*

This theorem is proven by induction on the dimension of X, using the BBD technique of gluing abstract t-structures. In addition to this formalism, we will have to know what happens in "the smooth case." We define $D_{\mathrm{loc}}(X) = \{K \in D(X) \colon H^i(X) \text{ is locally constant } \forall i\}$, and we next show that when X is smooth, the perverse t-structure on $D_{\mathrm{loc}}(X)$ is just a translate of the standard t-structure.

Proposition 21.15. *If U is smooth of dimension d then*

$$({}^pD^{\leq 0}(U) \cap D_{loc}(U), {}^pD^{\geq 0}(U) \cap D_{loc}(U)) = (D^{\leq -d}_{loc}(U), D^{\geq -d}_{loc}(U))$$

(a shift of the standard t-structure).

Proof. For $K \in D_{\mathrm{loc}}(U)$ if $H^{-i}K \neq 0$ then $\mathrm{supp} H^{-i}K = U$ since the cohomology is a local system. Thus the condition $K \in {}^pD^{\leq 0}(U) \cap D_{\mathrm{loc}}(U)$ is equivalent to $H^iK = 0$ for all $i > -d$. Hence ${}^pD^{\leq 0}(U) \cap D_{\mathrm{loc}}(U) = D^{\leq -d}_{\mathrm{loc}}(U)$. On the other hand $K \in {}^pD^{\geq 0}(U)$ if and only if $\mathbb{D}_U K \in {}^pD^{\leq 0}(U)$ which is equivalant to saying $H^i(\mathbb{D}_U K) = 0$ for all $i > -d$. The following lemma shows that this is equivalent to having $H^jK = 0$ for all $j < -d$. $\square$

Lemma 21.16. *If U is smooth of dimension d and $K \in D_{loc}(U)$ then for all i*

$$H^i(\mathbb{D}_U K) \cong H^{-i-2d}(K)^\vee(d)$$

(where the (d) is a Tate twist in case we are working with étale cohomology).

Proof. Proceed by induction on $|\{i\colon H^iK \neq 0\}|$. Given K let n be the maximal index such that $H^nK \neq 0$. We get a distinguished triangle

$$\tau^{\leq n-1}K \to K \to \tau^{\geq n}K \xrightarrow{+1}$$

Since n is maximal, we know that $\tau^{\geq n}K \cong H^nK[-n]$. Write $\mathcal{F} = H^nK$. Now we apply $R\mathcal{H}om(\cdot, \mathcal{K}_U) = \mathbb{D}_U$ and take the long exact sequence of the resulting distinguished triangle. Since U is smooth we have $\mathcal{K}_U = \mathbb{Q}_\ell[2d](d)$ and the long exact sequence is

$$\cdots \to H^i(R\mathcal{H}om(\mathcal{F}[-n], \mathbb{Q}_\ell[2d])) \to H^i(R\mathcal{H}om(K, \mathbb{Q}_\ell[2d])) \to H^i(R\mathcal{H}om(\tau^{\leq n-1}K, \mathbb{Q}_\ell[2d])) \to \cdots$$

Note that the first term can be can be computed as

$$H^{i+n+2d}R\mathcal{H}om(\mathcal{F}, \mathbb{Q}_\ell) = \begin{cases} 0 & \text{if } i \neq -n-2d \\ \mathcal{F}^\vee & \text{if } i = -n-2d \end{cases}$$

since $\mathcal{F}$ is locally constant (if $\mathcal{F}$ is constant, i.e. $\mathbb{Q}_\ell$, then $\mathcal{H}om(\mathbb{Q}_\ell, \cdot)$ is the identity functor so there are no higher derived functors; since $\mathcal{F}$ is locally constant we can reduce to this case). This implies that

$$H^iR\mathcal{H}om(K, \mathbb{Q}_\ell[2d]) = \begin{cases} \mathcal{F}^\vee & \text{if } i = -2d-n \\ H^i(\mathbb{D}_U\tau^{\leq n-1}K) & \text{if } i > -2d-n \\ 0 & \text{if } i < -2d-n \end{cases}$$

(Check the details as an exercise. For instance,

$$H^iR\mathcal{H}om(\tau^{\leq n-1}K, \mathbb{Q}_\ell[2d]) = H^0R\mathcal{H}om(\tau^{\leq n-1}K, \mathbb{Q}_\ell[2d+i]),$$

so if $i \leq -2d-n$, i.e. $-2d-i \geq n$, then this group vanishes for degree reasons.) The lemma now follows by applying the induction hypothesis to $\tau^{\leq n-1}K$. $\square$

The following important lemma gives a way to describe ${}^pD^{\leq 0}(X)$ and ${}^pD^{\geq 0}(X)$ in terms of restricting to a decomposition $X = U \sqcup Z$. It motivates the abstract glueing procedure for t-structures to be described in the next section.

Lemma 21.17. *Let X/k be separated of finite type. Let $j\colon U \to X$ be an open imbedding with closed complement $i\colon Z \to X$, and let $K \in D(X)$. then*

- *$K \in {}^pD^{\leq 0}(X)$ if and only if $j^*K \in {}^pD^{\leq 0}(U)$ and $i^*K \in {}^pD^{\leq 0}(Z)$; and*
- *$K \in {}^pD^{\geq 0}(X)$ if and only if $j^*K \in {}^pD^{\geq 0}(U)$ and $i^!K \in {}^pD^{\geq 0}(Z)$.*

Proof. Since i^* and j^* are exact for the standard t-structure, $\mathrm{Supp}(H^nK) = \mathrm{Supp}(j^*H^n(K)) \sqcup \mathrm{Supp}(i^*H^n(K))$. The first item follows. For the second, $K \in {}^pD^{\geq 0}(X) \iff \mathbb{D}_X(K) \in {}^pD^{\leq 0}(X)$, which holds (by the first part) if and only if $j^*\mathbb{D}_X(K) \in {}^pD^{\leq 0}(U)$ and $i^*\mathbb{D}_X(K) \in {}^pD^{\leq 0}(Z)$. Since $j^*\mathbb{D}_X \simeq \mathbb{D}_U j^!$ and $i^*\mathbb{D}_X \simeq \mathbb{D}_Z i^!$, the second item follows from the definitions of the perverse "t-structures" (not yet) on U and Z. □

22. t-structures and gluing (Notes by Sabine Lang)

In this lecture we explain how to glue t-structures. Following BBD, we axiomatize the situation. Let $\mathcal{D}$, $\mathcal{D}_U$, and $\mathcal{D}_Z$ be triangulated categories equipped with exact (i.e. preserving shifts and distinguished triangles) functors $i_* \colon \mathcal{D}_Z \to \mathcal{D}$ and $j^* \colon \mathcal{D} \to \mathcal{D}_U$. Set (for convenience) $i_! = i_*$ and $j^! = j^*$. Assume the following "glueing axioms" hold:

(G1) i_* admits a left adjoint i^* and a right adjoint $i^!$. j^* admits a left adjoint $j_!$ and a right adjoint j_*.

(G2) $j^*i_* = 0$. (By adjunction and Yoneda, we deduce from this $i^*j_! = 0$ and $i^!j_* = 0$; for instance,
$$\mathrm{Hom}(i^*j_!K, L) = \mathrm{Hom}(j_!K, i_*L) = \mathrm{Hom}(K, j^*i_*L) = 0.)$$

(G3) For all $K \in \mathcal{D}$, there are distinguished triangles (with the adjunction maps)
$$j_!j^*K \to K \to i_*i^*K \xrightarrow{+1}$$
and
$$i_*i^!K \to K \to j_*j^*K \xrightarrow{+1} .$$

(G4) $j_!$, i_*, and j_* are fully faithful. (Exercise: deduce from this that the natural transformations $j^*j_* \to \mathrm{id}$, $\mathrm{id} \to j^*j_!$, $i^*i_* \to \mathrm{id}$, $\mathrm{id} \to i^!i_*$ are all isomorphisms.)

Before proceeding with the formalism, we check that the glueing axioms hold in the setting of sheaf theory:

Proposition 22.1. *Working in the sheaf theory setting, let $U \subset X$ be open, $Z = X \backslash U$, and denote by $j : U \to X$, $i : Z \to X$ the inclusions. Then the axioms $G1$ to $G4$ are all satisfied by the associated (derived) functors, taking $\mathcal{D} = D^b_c(X)$, $\mathcal{D}_U = D^b_c(U)$ and $\mathcal{D}_Z = D^b_c(Z)$.*

Proof.

- $G1$: We have seen it before.
- $G2$: It is clear from stalks.
- $G3$: For the first triangle, for $\mathcal{F} \in \mathsf{Sh}(X)$, we already have that
$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$
is exact, by looking at stalks. The functors $j_!$, j^*, i_* and i^* are all exacts, so for $K \in D^b_c(X)$, we can compute $Rj_!j^*K$, etc ... just by applying the functors term by term. We get a short exact sequence of complexes :
$$0 \to j_!j^*K \to K \to i_*i^*K \to 0,$$
and this gives rise to an associated distinguished triangle in $D^b_c(X)$.

For the second triangle, for all $\mathcal{F} \in \mathsf{Sh}(X)$, we have an exact sequence

$$0 \to i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}.$$

Moreover, if $\mathcal{F}$ is injective, the last map is surjective (see our discussion of cohomology with supports). So for a general $K \in D^b_c(X)$, we take an injective replacement and apply this to get the distinguished triangle

$$i_* R i^! K \to K \to R j_* j^* K \xrightarrow{+1} .$$

- $G4$: We have seen some of this. For example, for Rj_*, we have $\mathrm{Hom}_{D(X)}(Rj_* K, Rj * L) = \mathrm{Hom}_{D(X)}(j^* Rj_* K, L) = \mathrm{Hom}_{D(X)}(K, L)$.

$\square$

Theorem 22.2. *Let $\mathcal{D}$, $\mathcal{D}_U$, $\mathcal{D}_Z$, i_*, j^* be as in the gluing axiomatics. Let $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ and $(\mathcal{D}_Z^{\leq 0}, \mathcal{D}_Z^{\geq 0})$ be t-structures on $\mathcal{D}_U$ and $\mathcal{D}_Z$. Define*

- $\mathcal{D}^{\leq 0} = \{K \in \mathcal{D} \mid j^* K \in \mathcal{D}_U^{\leq 0}, i^* K \in \mathcal{D}_Z^{\leq 0}\}$,
- $\mathcal{D}^{\geq 0} = \{K \in \mathcal{D} \mid j^! K \in \mathcal{D}_U^{\geq 0}, i^! K \in \mathcal{D}_Z^{\geq 0}\}$.

Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.

Proof. We need to check three things :

(1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$,

(2) $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$,

(3) for all $X \in \mathcal{D}$, there exists $X_{\leq 0} \in \mathcal{D}^{\leq 0}$ and $X_{\geq 1} \in \mathcal{D}^{\geq 1}$ so that we have a distinguished triangle $X_{\leq 0} \to X \to X_{\geq 1} \xrightarrow{+1}$.

We use the following arguments :

(1) Clear.

(2) Let $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$. We have a distinguished triangle

$$j_! j^* X \to X \to i_* i^* X \xrightarrow{+1} .$$

Applying $\mathrm{Hom}(-, Y)$, we get the long exact sequence

$$\cdots \leftarrow \mathrm{Hom}(j_! j^* X, Y) \leftarrow \mathrm{Hom}(X, Y) \leftarrow \mathrm{Hom}(i_* i^* X, Y) \leftarrow \dots .$$

But we get $\mathrm{Hom}(j_! j^* X, Y) = \mathrm{Hom}_{\mathcal{D}_U}(j^* X, j^* Y) = 0$ by the t-structure on U since $j^* X \in \mathcal{D}_U^{\leq 0}$ and $j^* Y \in \mathcal{D}_U^{\geq 1}$. Similarly, $\mathrm{Hom}(i_* i^* X, Y) = \mathrm{Hom}_{\mathcal{D}_Z}(i^* X, i^! Y) = 0$ using the t-structure on Z and $i^* X \in \mathcal{D}_Z^{\leq 0}$, $i^! Y \in \mathcal{D}_Z^{\geq 1}$. So this shows that $\mathrm{Hom}(X, Y) = 0$.

(3) The t-structures on $\mathcal{D}_U$, $\mathcal{D}_Z$ give rise to the associated truncation functors $\tau_U^{\leq n}$, $\tau_U^{\geq n}$, $\tau_Z^{\leq n}$, $\tau_Z^{\geq n}$. For all $X \in \mathcal{D}$, we have a map $X \to j_* \tau_U^{\geq 1} j^* X$ given by the composition $X \to j_* j^* X \to j_* \tau_U^{\geq 1} j^* X$. We can complete this composite to a distinguished triangle in $\mathcal{D}$: $Y \to X \to j_* \tau_U^{\geq 1} j^* X \xrightarrow{+1}$. Likewise, we have a map $Y \to i_* \tau_Z^{\geq 1} i^* Y$ from the composition $Y \to i_* i^* Y \to i_* \tau_Z^{\geq 1} i^* Y$. We obtain a distinguished triangle $A \to Y \to i_* \tau_Z^{\geq 1} i^* Y \xrightarrow{+1}$. Also, $A \to Y \to X$ gives a distinguished triangle $A \to X \to B \xrightarrow{+1}$.

Claim: this is the triangle we need, i.e., $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

To prove this, we use the octahedral axiom to produce a fourth distinguished triangle:

$$
\begin{tikzcd}
 & & & & . & & \\
 & & & i_*\tau_Z^{\geq 1} i^*Y \arrow[ur, "+1"] \arrow[dr, dashed] & & . & \\
 & & Y \arrow[ur] \arrow[dr] & & B \arrow[ur, "+1"] \arrow[ddr, dashed] & & \\
 & & & X \arrow[ur] \arrow[drr] & & & \\
 & A \arrow[uur] \arrow[urr] & & & & j_*\tau_U^{\geq 1} j^*X \arrow[r, "+1"] \arrow[dr, dashed, "+1"'] & . \\
 & & & & & & .
\end{tikzcd}
$$

So the octahedral axiom gives us a distinguished triangle making this diagram commute (dotted lines in the picture). To establish the claim, we need to check four things.

(a) We apply j^* to the fourth triangle, and use the facts that $j^*j_* \cong id$ and $j^*i_* = 0$. We obtain $j^*B \cong \tau_U^{\geq 1} j^*X \in \mathcal{D}_U^{\geq 1}$.

(b) We apply j^* to $A \to X \to B \xrightarrow{+1}$ to get a distinguished triangle $j^*A \to j^*X \to j^*B \xrightarrow{+1}$. Since $j^*B \cong \tau_U^{\geq 1} j^*X$, we get $j^*A \cong \tau_U^{\leq 0} j^*X \in \mathcal{D}_U^{\leq 0}$.

(c) We apply i^* to $A \to Y \to i_*\tau_Z^{\geq 1} i^*Y \xrightarrow{+1}$ to get a distinguished triangle $i^*A \to i^*Y \to i^*i_*\tau_Z^{\geq 1} i^*Y \xrightarrow{+1}$. We have $i^*i_*\tau_Z^{\geq 1} i^*Y \cong \tau_Z^{\geq 1} i^*Y$ and therefore $i^*A \cong \tau_Z^{\leq 0} i^*Y \in \mathcal{D}_Z^{\leq 0}$.

(d) We apply $i^!$ to the fourth triangle and we get a distinguished triangle $i^!i_*\tau_Z^{\geq 1} i^*Y \to i^!B \to i^!j_*\tau_U^{\geq 1} j^*B \xrightarrow{+1}$. Since $i^!j_* = 0$, we have $i^!j_*\tau_U^{\geq 1} j^*B \cong 0$. Also, $i^!i_* \cong id$ gives us $i^!i_*\tau_Z^{\geq 1} i^*Y \cong \tau_Z^{\geq 1} i^*Y$. We conclude that $i^!B \cong \tau_Z^{\geq 1} i^*Y \in \mathcal{D}_Z^{\geq 1}$.

This proves the claim.

$\square$

Corollary 22.3. *Let k be a perfect field. Let X be a separated k-scheme of finite type. Then $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ is a t-structure on $D_c^b(X)$.*

Proof. For any $U \subset X$ open with $j : U \to X$ the inclusion, we set $D(X, U) = \{K \in D_c^b(X) \mid j^*K \in D_{loc}^b(U)\}$ (we will only use this construction for U smooth open dense). We may assume that X is reduced. Then, since X has an open dense smooth subscheme and by definition of a constructible sheaf, we can write $D_c^b(X) = \cup D(X, U)$, where the union is taken over all the open smooth dense U. Now, for any such U, we write $Z = X \backslash U$. We have:

(1) By induction on $\dim(X)$, $({}^pD^{\leq 0}(Z), {}^pD^{\geq 0}(Z))$ is a t-structure on $D^b_c(Z)$. (Base case: $\dim(X) = 0$, but then the perverse and standard t-structures are the same.)

(2) $({}^pD^{\leq 0}(X) \cap D_{loc}(U), {}^pD^{\geq 0}(X) \cap D_{loc}(U))$ is a t-structure on $D_{loc}(U)$, by a $\dim(X)$-translation of the standard t-structure.

(3) By (1), (2) and the gluing theorem, we deduce that $({}^pD^{\leq 0}(X) \cap D(X,U), {}^pD^{\geq 0}(X) \cap D(X,U))$ is a t-structure.

(4) Since $D^b_c(X) = \cup D(X,U)$, for U open smooth dense, we see that $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ is a t-structure (by checking the three axioms).

□

Corollary 22.4. *The intersection ${}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$ is an abelian category.*

Definition 22.5. The abelian category ${}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$ is called the category of perverse sheaves and denoted by $\mathrm{Perv}(X)$.

22.1. **How to make perverse sheaves ?**

Example 22.6. If X is smooth of dimension d, then $\mathbb{Q}_l[d]$ is perverse and $\mathcal{F}[d]$ is perverse for any local system $\mathcal{F}$.

In general, for any $U \subset X$ open smooth dense, and any local system on U, $\mathcal{F}[d] \in \mathrm{Perv}(U)$. The main method for building perverse sheaves is to start with $(U, \mathcal{F})$ and apply the intermediate extension functor $j_{!*} : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$. So constructing and studying $j_{!*}$ is our next task.

22.2. **Some preliminaries.**

We can make this construction in the abstract gluing setup. Let $\mathcal{D}$, $\mathcal{D}_U$, $\mathcal{D}_Z$, ... be as in the gluing theorem. We have t-structures on $\mathcal{D}_U$, $\mathcal{D}_Z$, and on $\mathcal{D}$ by gluing. We have perverse analogues of the standard functors between these triangulated categories.

Example 22.7. The functor $j_! : \mathcal{D}_U \to \mathcal{D}$ gives rise to a map $\mathcal{D}_U^\heartsuit \to \mathcal{D}^\heartsuit$ via the composition $\mathcal{D}_U^\heartsuit \subset \mathcal{D}_U \xrightarrow{j_!} \mathcal{D} \xrightarrow{{}^pH^0} \mathcal{D}^\heartsuit$. We call this composite $j_!^\heartsuit$ or ${}^pj_!$. We make similar definitions for the other basic functors.

Here are the general t-exactness properties of our basic functors:

Lemma 22.8. *(1) The functors i_* and j^* are t-exact and have both left and right adjoints.*

(2) The functors $j_!$ and i^* are right-t-exact and have right adjoints.

(3) The functors j_* and $i^!$ are left-t-exact and have left adjoints.

Proof. We will just give a sample argument in the case of i_*. We have $i^* i_* \mathcal{D}_Z^{\leq 0} \subset \mathcal{D}_Z^{\leq 0}$ and $j^* i_* \mathcal{D}_Z^{\leq 0} = 0 \subset \mathcal{D}_U^{\leq 0}$, so i_* is right-t-exact.

The rest of the proof works in a similar way, using adjunctions and truncations. □

With this lemma in hand, we consider the perverse analogues:

Lemma 22.9. *We have adjoints (left on top, right on bottom) :*

$$\begin{tikzcd} \mathcal{D}_U^\heartsuit \arrow[r, shift left=2, "{}^p j_!"] \arrow[r, shift right=2, "{}^p j_*"'] & \mathcal{D}^\heartsuit \arrow[l, "{}^p j^*"'] \arrow[r, shift left=2, "{}^p i^*"] \arrow[r, shift right=2, "{}^p i^!"'] & \mathcal{D}_Z^\heartsuit \arrow[l, "{}^p i_*"'] \end{tikzcd}$$

and ${}^p j^* \circ {}^p i_* = 0$.

Remark 22.10. This is a generalization of the familiar relations for the standard t-structure. Some differences : for example, ${}^p j_!$ is not necessarily t-exact.

Proof. Let $X \in \mathcal{D}_U^\heartsuit$ and $Y \in \mathcal{D}^\heartsuit$. By definition, we have

$$\mathrm{Hom}_{\mathcal{D}^\heartsuit}({}^p j_! X, Y) = \mathrm{Hom}_{\mathcal{D}}({}^p H^0(j_! X), Y).$$

We have ${}^p H^0 = {}^p\tau^{\leq 0} \circ {}^p\tau^{\geq 0}$, and $\mathrm{Hom}({}^p\tau^{\leq -1} X, Y) = 0$ since $Y \in \mathcal{D}^{\geq 0}$, so we get

$$\mathrm{Hom}_{\mathcal{D}}({}^p H^0(j_! X), Y) = \mathrm{Hom}_{\mathcal{D}}({}^p\tau^{\leq 0}(j_! X), Y).$$

Since $X \in \mathcal{D}_U^\heartsuit$, we have $j_! X \in \mathcal{D}^{\leq 0}$ and therefore $j_! X = {}^p\tau^{\leq 0} j_! X$, so

$$\mathrm{Hom}_{\mathcal{D}}({}^p\tau^{\leq 0}(j_! X), Y) = \mathrm{Hom}_{\mathcal{D}}(j_! X, Y).$$

By adjunction, we get

$$\mathrm{Hom}_{\mathcal{D}}(j_! X, Y) = \mathrm{Hom}_{\mathcal{D}_U}(X, j^* Y).$$

Finally, since j^* is t-exact, we have

$$\mathrm{Hom}_{\mathcal{D}_U}(X, j^* Y) = \mathrm{Hom}_{\mathcal{D}_U}(X, {}^p j^* Y).$$

This shows that ${}^p j_!$ is left-adjoint to ${}^p j^*$. The other cases are similar. □

Lemma 22.11. *For $X \in \mathcal{D}^\heartsuit$, there exist two exact sequences in $\mathcal{D}^\heartsuit$:*

(1)
$$0 \to {}^p i_* \, {}^p H^{-1}(i^* X) \to {}^p j_! \, {}^p j^* X \to X \to {}^p i_* \, {}^p i^* X \to 0.$$

(2)
$$0 \to {}^p i_* \, {}^p i^! X \to X \to {}^p j_* \, {}^p j^* X \to {}^p i_* \, {}^p H^1(i^! X) \to 0.$$

Proof. We leave (1) as an exercise.

Recall that we have a distinguished triangle $i_* i^! X \to X \to j_* j^* X \xrightarrow{+1}$. We take the corresponding long exact sequence in ${}^p H^n$:

$$\cdots \to {}^p H^{-1}(j_* j^* X) \to {}^p H^0(i_* i^! X) \to {}^p H^0(X) \to {}^p H^0(j_* j^* X) \to {}^p H^1(i_* i^! X) \to {}^p H^1(X) \to \ldots.$$

We have the following identifications :

- We have $j^* X \in \mathcal{D}_U^\heartsuit$, so $j_* j^* X \in \mathcal{D}^{\geq 0}$ and consequently ${}^p H^{-1}(j_* j^* X) = 0$.
- Because i_* is t-exact, we have ${}^p H^0(i_* i^! X) = {}^p i_*({}^p H^0(i^! X)) = {}^p i_* \, {}^p i^! X$.

- Since $X \in \mathcal{D}^\heartsuit$, we have ${}^pH^0(X) = X$.
- Because j^* is t-exact, ${}^pH^0(j_*j^*X) = {}^pj_* \, {}^pj^*X$.
- We have ${}^pH^1(i_*i^!X) = {}^pi_* \, {}^pH^1(i^!X)$.
- Since $X \in \mathcal{D}^\heartsuit$, we have ${}^pH^1(X) = 0$.

This gives us the exact sequence

$$0 \to {}^pi_* \, {}^pi^!X \to X \to {}^pj_* \, {}^pj^*X \to {}^pi_* \, {}^pH^1(i^!X) \to 0.$$

□

23. Intermediate extension (Allechar Serrano López)

$$j_{!*} : \mathcal{D}_U^\heartsuit \to \mathcal{D}^\heartsuit$$

For $Y \in \mathcal{D}_U$, an extension of Y to $\mathcal{D}$ is an $X \in \mathcal{D}$ and an isomorphism $j^*X \simeq Y$.

Example 23.1. $j_!Y$ and j_*Y are extensions of Y.

Definition 23.2. Let $X \in \mathcal{D}_U^\heartsuit$. There is a canonical map $j_!X \to j_*X$. This induces a map ${}^pj_!X \to {}^pj_*X$. Then we define the intermediate extension

$$j_{!*}X = \operatorname{im}({}^pj_!X \to {}^pj_!X)$$

where the image is taken in the abelian category $\mathcal{D}^\heartsuit$.

Lemma 23.3. *$j_{!*}$ is the unique extension $\overline{X}$ of X satisfying*

$$\iota^*\overline{X} \in \mathcal{D}_Z^{\leqslant -1} \text{ and } \iota^!\overline{X} \in \mathcal{D}_Z^{\geqslant 1}$$

Proof. We will show that $j_{!*}$ satisfies these properties. We have a surjection ${}^pj_!X \twoheadrightarrow j_{!*}X$. Apply ${}^p\iota^*$; this is the left adjoint to ${}^p\iota_*$, so it is right-exact, and we get a surjection

$$\underbrace{{}^p\iota^{*p}j_!X}_{=0} \twoheadrightarrow {}^p\iota^*j_{!*}X$$

so ${}^p\iota^* \underbrace{j_{!*}X}_{\in\mathcal{D}^\heartsuit} = 0$. Since ι^* is right-exact, we know that $\iota^*j_{!*}X \in \mathcal{D}_Z^{\leqslant 0}$; and we have just seen ${}^pH^0\iota^*j_{!*}X = 0$, so we must have $\iota^*j_{!*}X \in \mathcal{D}_Z^{\leqslant -1}$.

For $\iota^!j_{!*}X$, argue similarly, applying ${}^p\iota^!$ to the monomorphism $j_{!*}X \hookrightarrow {}^pj_*X$.

That shows existence. We leave uniqueness as an exercise. □

More generally:

Proposition 23.4. *For any $Y \in \mathcal{D}_U$ and $p \in \mathbb{Z}$, up to isomorphism, there exists a unique extension X such that $\iota^*X \in \mathcal{D}_Z^{\leqslant p-1}$ and $\iota^!X \in \mathcal{D}_Z^{\geqslant p+1}$.*

The main idea is to define a new glued t-structure on $\mathcal{D}$ by glueing $(\mathcal{D}_U, 0)$ and $(\mathcal{D}_Z^{\leqslant 0}, \mathcal{D}_Z^{\geqslant 0})$. The t-structure formalism then gives truncation functors ${}^Z\tau^{\leqslant 0}$, ${}^Z\tau^{\geqslant 0}$ for this glued t-structure; and the extension promised in the Proposition is $X = {}^Z\tau^{\leq p-1}(j_* Y)$.

The following result gives a third characterization of $j_{!*}$:

Lemma 23.5. *Consider $X \in \mathcal{D}_U^{\heartsuit}$, then $j_{!*}$ is the unique perverse extension of X having no non-trivial subobject or quotient in $\iota_*(\mathcal{D}_Z^{\heartsuit})$.*

Proof. $j_{!*}X$ has no quotient of this form since ${}^p\iota^* j_{!*}X = 0$ (and ${}^p\iota^*$ is right t-exact) and has no subextension because ${}^p\iota^! j_{!*}X = 0$ (and ${}^p\iota^!$ is left t-exact).

Now we check that these two properties uniquely characterize a perverse extension:

Suppose $\overline{X}$, $j^*\overline{X} \simeq X$ is an extension. We have a surjection $\overline{X} \to {}^p\iota_* {}^p\iota^* \overline{X} \to 0$. $\overline{X}$ has no nonzero quotient of this form, so ${}^p\iota_* {}^p\iota^* \overline{X} = 0$, so by 22.11 we get $\underbrace{{}^p j_! {}^p j^* \overline{X}}_{= {}^p j_! X} \twoheadrightarrow \overline{X}$. The dual argument shows $\overline{X} \hookrightarrow {}^p j_* X$, and the composite of these two maps to and from $\overline{X}$ is the canonical map, i.e., $\overline{X} = \operatorname{im}({}^p j_! X \to {}^p j_* X)$. ∎

We can use what we've shown to classify the simple objects of $\mathcal{D}^{\heartsuit}$:

Proposition 23.6. *The simple objects of $\mathcal{D}^{\heartsuit}$ are those of the form ${}^p\iota_* S_Z$ for S_Z simple in $\mathcal{D}_Z^{\heartsuit}$ or $j_{!*}S_U$ for S_U simple in $\mathcal{D}_U^{\heartsuit}$.*

Proof. Let $S \in \mathcal{D}^{\heartsuit}$ be simple. There are two cases:

(i) If ${}^p\iota^* S \neq 0$, then $S \to {}^p\iota_* {}^p\iota^* S$ (22.11) is a nonzero quotient of S. S is simple, so $S \simeq {}^p\iota_* {}^p\iota^* S$.

(ii) ${}^p\iota^* S = 0$. We may also assume ${}^p\iota^! S = 0$, else we would get a nonzero subobject of X from ${}^p\iota_* {}^p\iota^! S$ (and again since X is simple this inclusion would have to be an isomorphism).

By 23.5, we have that $S \simeq j_{!*} j^* S$, so we have to check $j^*S \in \mathcal{D}_U^{\heartsuit}$ is simple. Suppose not, so there exists a non-trivial quotient $j^*S \twoheadrightarrow Q \in \mathcal{D}_U^{\heartsuit}$. Then we have the diagram

$$\begin{tikzcd} {}^p j_! j^* S \arrow[r, two heads] \arrow[d, two heads] & {}^p j_! Q \arrow[d, two heads] \\ j_{!*} j^* S = S \arrow[r, dashed] & j_{!*} Q \end{tikzcd}$$

Since the other three arrows are surjective, we can deduce $j_{!*}j^*S \twoheadrightarrow j_{!*}Q$ is surjective; since S is simple, this is an isomorphism, hence $j^*S \to Q$ is an isomorphism.

To finish the proof, we need to check that $j_{!*}(\text{simple}) = \text{simple}$. If not, there exists a proper quotient $\alpha : j_{!*}S_U \twoheadrightarrow Q$. Since S_U is simple, $j^*(\alpha)$ is either zero or an isomorphism. If $j^*\alpha = 0$, then $j^*Q = 0$ so $Q \simeq {}^p\iota_* {}^p\iota_* Q$ which contradicts 23.5. So we may assume $j^*\alpha$ is an isomorphism, i.e., $j^*(\ker \alpha) = 0$. (We have repeatedly used that j^* is t-exact.) From the following short exact sequence

$$ {}^p j_! {}^p j^* \ker\alpha \to \ker\alpha \to {}^p\iota_* {}^p\iota^* \ker\alpha \to 0 $$

we get that $\ker\alpha \simeq {}^p\iota_* {}^p\iota^* \ker\alpha$, but $j_{!*}(\ )$ has no no-zero subobject in ${}^p\iota_*(\mathcal{D}_Z^\heartsuit)$. Thus, $\ker\alpha = 0$. $\square$

Proposition 23.7. *$j_{!*} : \mathcal{D}_U^\heartsuit \to \mathcal{D}^\heartsuit$ is fully faithful.*

Proof. Consider $A, B \in \mathcal{D}_U^\heartsuit$. We have maps

$$ \mathrm{Hom}(A,B) \xrightarrow{j_{!*}} \mathrm{Hom}(j_{!*}A, j_{!*}B) \xrightarrow{j^*} \mathrm{Hom}(A,B), $$

and $j^* j_{!*}$ is the identity on $\mathrm{Hom}(A,B)$. To show $j_{!*}$ is bijective, we need to check that $j_{!*}j^*$ is the identity on $\mathrm{Hom}(j_{!*}A, j_{!*}B)$. Since $j_{!*}B \subset {}^p j_* B$, we get

$$ \begin{aligned} \mathrm{Hom}(j_{!*}A, j_{!*}B) \subset\, & \mathrm{Hom}(j_{!*}A, {}^p j_* B) \\ &= \mathrm{Hom}({}^p j^* j_{!*}A, B) \\ &= \mathrm{Hom}(A,B). \end{aligned} $$

Now, for any $\phi \in \mathrm{Hom}(j_{!*}A, j_{!*}B)$ to show $\phi = j_{!*}j^*\phi$, it suffices to show via this injective composite that $j^*\phi = j^* j_{!*} j^* \phi$. This holds because $j_{!*}j^* \simeq \mathrm{id}$. $\square$

24. Examples. Poincaré Duality.

Now, we return to algebraic geometry. Consider X over k separated of finite type for k a perfect field. The basic example of a perverse sheaf is given U smooth of dimension d and $\mathcal{F} \in \mathrm{Loc}(U)$, consider $X = \mathcal{F}[d] \in \mathrm{Perv}(U)$. Now, use $j_{!*}$.

Definition 24.1. For any X of (pure) dimension d, let U be a smooth open dense subscheme, and let $\mathcal{F} \in \mathrm{Loc}(U)$, so $j_{!*}\mathcal{F}[d] \in \mathrm{Perv}(X)$. We define $IC_X(\mathcal{F}) = j_{!*}\mathcal{F}[d] \in \mathrm{Perv}(X)$; in particular, the intersection cohomology complex of X is by definition $IC_X = j_{!*}\mathbb{Q}_l[d]$.

Definition 24.2. The intersection cohomology groups of X with coefficients in $\mathcal{L} \in \mathrm{Loc}(U)$ (some smooth open dense $U \subset X$) are by definition $IH^n(X,\mathcal{L}) = H^{n-d}(X, IC_X(\mathcal{L})) = H^{n-d}(R\Gamma(X, IC_X(\mathcal{L})))$. This $n-d$ normalization fixes things so that for X smooth we have $IH^n = H^n$.

Example 24.3. Assume $U = X \smallsetminus \{x\} \hookrightarrow X$ is smooth. Then $IC_X = \tau^{\leqslant -1}(Rj_*\mathbb{Q}_l[d])$.

Proof. Recall that, in general, IC_X is characterized as the unique extension of $\mathbb{Q}_l[d]|_U$ such that

$$ \iota^* IC_X \in {}^pD^{\leqslant -1}(Z) \text{ and } \iota^! IC_X \in {}^pD^{\geqslant 1}(Z) $$

where $j : U \hookrightarrow X$ and $\iota : Z = X \smallsetminus U \hookrightarrow X$

Since i^* is exact for the standard t-structure, $i^*\tau^{\leq -1}(Rj_*\mathbb{Q}_l[d]) = \tau^{\leq -1}i^*Rj_*\mathbb{Q}_l[d] \in \mathcal{D}^{\leq -1}_{\{x\}} = {}^p\mathcal{D}^{\leq -1}_{\{x\}}$, since $\dim\{x\} = 0$. Similarly, to compute

$$\iota^!(\tau^{\leqslant -1}Rj_*\mathbb{Q}_l[d])$$

we need to look at the distinguished triangle

$$\iota^!\tau^{\leqslant -1}(Rj_*\mathbb{Q}_l[d]) \to \iota^!Rj_*\mathbb{Q}_l[d] \to \iota^!\tau^{\geqslant 0}(Rj_*\mathbb{Q}_l[d]) \xrightarrow{+1} .$$

Since $i^!Rj_* = 0$, we find

$$\underbrace{\underbrace{\underbrace{\iota^!}_{\text{left-exact}}\ \tau^{\geqslant 0}(Rj_*\mathbb{Q}_l[d])}_{\mathcal{D}^{\geqslant 0}(\{x\})}[-1]}_{\mathcal{D}^{\geqslant 1}(\{x\})={}^pD^{\geqslant 1}(\{x\})} \xrightarrow{\sim} \iota^!\tau^{\leqslant -1}(Rj_*\mathbb{Q}_l[d])$$

, so by uniqueness we find $IC_X \equiv \tau^{\leq -1}(Rj_*\mathbb{Q}_l[d])$. $\square$

Remark 24.4. This formula generalizes to Deligne's successive truncation (coming from a smooth stratification of X) formula for $j_{!*}$

Example 24.5. Continue in the setting of the last example: $j : U = X \setminus \{x\} \overset{\circ}{\hookrightarrow} X$ and $\iota : Z = \{x\} \hookrightarrow X$, with U smooth. We aim to compute the intersection cohomology $IH^\bullet(X) = H^\bullet(X, IC_X[-d])$. Since $IC_X = \tau^{\leqslant -1}(Rj_*\mathbb{Q}_l[d])$, $IC_X[-d] = \tau^{\leqslant d-1}(Rj_*\mathbb{Q}_l)$ so we have the following distinguished triangle

$$IC_X[-d] \to Rj_*\mathbb{Q}_l \to \tau^{\geqslant d}Rj_*\mathbb{Q}_l \xrightarrow{+1}$$

Taking the long exact sequence in hypercohomology (i.e., apply $R\Gamma(X, \cdot)$ and then take the LES in cohomology), we obtain

$$\cdots \to IH^r(X) \to H^r(U) \to H^r(X, \tau^{\geqslant d}Rj_*\mathbb{Q}_l) \to IH^{r+1}(X) \to \cdots ,$$

where we have rewritten the $H^r(U)$ term using the Leray spectral sequence for $U \to X \to \mathrm{Spec}(k)$.

The calculation splits into three cases, namely:

1) $r < d$
2) $r = d$
3) $r > d$

We will expand on each of these cases.

1) ($r < d$) We have $IH^r(X) \xrightarrow{\sim} H^r(X \smallsetminus \{x\})$. Indeed, there is a hypercohomology spectral sequence

$$E_2^{a,b} = H^a(X, H^b(K)) \implies H^{a+b}(X, K)$$

for $K \in D^b(X)$. Apply this to $K = \tau^{\geqslant d} Rj_*\mathbb{Q}_l$. For $b < d$, $H^b(\tau^{\geqslant d} Rj_*\mathbb{Q}_l) = 0$. So for $r < d$, all relevant E_2 terms are already zero, and we find $H^r(X, \tau^{\geqslant d} Rj_*\mathbb{Q}_l) = 0$.

2) and 3) ($r \geq d$) We have the distinguished triangles

$$\begin{tikzcd}
\tau^{\leqslant 0} Rj_*\mathbb{Q}_l = \mathbb{Q}_l \arrow[r] \arrow[d] & Rj_*\mathbb{Q}_l \arrow[r] \arrow[d, equal] & \tau^{\geqslant 1} Rj_*\mathbb{Q}_l \arrow[r, "+1"] \arrow[d, dashed, "\exists"] & \cdots \\
\tau^{\leqslant d-1} Rj_*\mathbb{Q}_l = IC_X[-d] \arrow[r] & Rj_*\mathbb{Q}_l \arrow[r] & \tau^{\geqslant d} Rj_*\mathbb{Q}_l \arrow[r, "+1"] & \cdots
\end{tikzcd}$$

and we get

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(X) \arrow[r] \arrow[d] & H^r(U) \arrow[r] \arrow[d, equal] & H^r(X, \tau^{\geqslant 1} Rj_*\mathbb{Q}_l) \arrow[r] \arrow[d] & H^{r+1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & IH^r(X) \arrow[r] & H^r(U) \arrow[r] & H^r(X, \tau^{\geqslant d} Rj_*\mathbb{Q}_l) \arrow[r] & IH^{r+1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By 1), for $r = d$ we have

$$0 \to IH^d(X) \to H^d(U)$$

so we deduce $IH^d(X) \simeq \mathrm{im}(H^d(X) \to H^d(X \smallsetminus x))$.

For 3), we claim that $H^r(X) \simeq IH^r(X)$ for $r \geqslant d+1$. Indeed, the top triangle is isomorphic to the (adjunction) distinguished triangle

$$\mathbb{Q}_l \to Rj_*\mathbb{Q}_l \to \iota_* R\iota^!\mathbb{Q}_l[1] \xrightarrow{+1}$$

so the cohomology groups in the third column above are $H^r(X, \iota_* R\iota^!\mathbb{Q}_l[1]) = H^r(\{x\}, R\iota^!\mathbb{Q}_l[1])$ and $H^r(X, \tau^{\geqslant d}\iota_* R\iota^!\mathbb{Q}_l[1]) = H^r(\{x\}, \tau^{\geqslant d} R\iota^!\mathbb{Q}_l[1])$.

In the corresponding hypercohomology spectral sequences, only $E_2^{0,r}$ terms could be nonzero; but for $r \geq d$, $H^r(R\iota^!\mathbb{Q}_l[1]) = H^r(\tau^{\geqslant d} R\iota^!\mathbb{Q}_l[1])$. A diagram chase then shows that for $r \geq d+1$, the maps $H^r(X) \xrightarrow{\sim} IH^r(X)$ are themselves isomorphisms.

Example 24.6. Let X be a pinched torus (one nodal singularity). Then by the above calculation,

$$IH^r(X) = \begin{cases} H^0(U) = \mathbb{Q}, & r = 0 \\ \mathrm{im}(H^1(X) \to H^1(U)) = 0, & r = 1 \\ H^2(X) = \mathbb{Q}, & r = 2 \end{cases}$$

In contrast, $H^1(X) = \mathbb{Q}$.

Example 24.7. Let X be the union of two lines in $\mathbb{P}^2$. Topologically, $X = S^2 \vee S^2$. Poincaré duality fails since we have

$$H^r(X) = \begin{cases} \mathbb{Q}, & r = 0 \\ 0, & r = 1 \\ \mathbb{Q} \oplus \mathbb{Q}, & r = 2 \end{cases}$$

In contrast,

$$IH^r(X) = \begin{cases} H^0(U) = \mathbb{Q} \oplus \mathbb{Q}, & r = 0 \\ 0, & r = 1 \\ H^2(X) = \mathbb{Q} \oplus \mathbb{Q}, & r = 2 \end{cases}$$

so at least at the level of Betti numbers, Poincaré duality is restored.

Indeed, one of the original motivations for Goresky and Macpherson to define intersection cohomology was to find a version of Poincaré duality for singular spaces. We'll now see that this works in general for intersection cohomology.

Theorem 24.8. *Let X be any variety of dimension d. Then $IH^r(X) \xrightarrow{\sim} \underbrace{IH_c^{2d-r}(X)^\vee}_{H^{2d-r}(R\Gamma_c(X, IC_X[-d]))}$.*

More generally, for $j : U \hookrightarrow X$,

$$\mathbb{D}_X \circ j_{!*} \simeq j_{!*} \circ \mathbb{D}_U$$

on Perv(U).

Proof. Let $K \in \text{Perv}(U)$. Set $L = \mathbb{D}_X \circ j_{!*}K$. Then $j^*L = j^*\mathbb{D}_X j_{!*}K = \mathbb{D}_U j^* j_{!*}K = \mathbb{D}_U K$ so L extends $\mathbb{D}_U K$.

To show $L \cong j_{!*}\mathbb{D}_U K$ we have to check the support and co-support conditions on $\iota^* L$ and $\iota^! L$. We have

$$\begin{aligned} \iota^* L &= \iota^* \mathbb{D}_X j_{!*} K \\ &= \mathbb{D}_Z \underbrace{\iota^! j_{!*} K}_{\in {}^pD^{\geqslant 1}(Z)} \in {}^pD^{\leqslant -1}(Z) \end{aligned}$$

and likewise,

$$\begin{aligned} \iota^! L &= \iota^! \mathbb{D}_X j_{!*} K \\ &= \mathbb{D}_Z \underbrace{\iota^* j_{!*} K}_{\in {}^pD^{\leqslant -1}} \in {}^pD^{\geqslant 1}(Z) \end{aligned}$$

For Poincaré duality, take U smooth open dense in X; applying the above, and our previous calculation of $\mathbb{D}_U$ on lisse sheaves (for smooth U), we find $\mathbb{D}_X(IC_X) \simeq IC_X$ so $R\Gamma(X, IC_X) \simeq R\Gamma(X, \mathbb{D}_X IC_X)$. Letting $\pi \colon X \to \mathrm{Spec}(k)$ be the structure map, the relation $\mathbb{D}_{\mathrm{Spec}(k)} R\pi_! \equiv R\pi_* \mathbb{D}_X$ translates to an isomorphism $R\Gamma(X, \mathbb{D}_X IC_X) \simeq R\Gamma_c(X, IC_X)^\vee$. Taking cohomology, we find $H^n(X, IC_X) \simeq H_c^{-n}(X, IC_X)^\vee$. i.e., $IH^{d+n}(X) \simeq IH_c^{d-n}(X)^\vee$. ☐

25. Statements of a few more basic facts about perverse sheaves in algebraic geometry

We conclude with a few important facts about perverse sheaves that we won't have time to prove. See BBD for details.

1. Let X be a variety.
    a) Classification of simple objects in $\mathrm{Perv}(X)$: all have the form $\iota_* j_{!*} \mathcal{L}[\dim Z]$ for a closed immersion $\iota : Z \hookrightarrow X$ and a local system $\mathcal{L}$ on a smooth dense open $W \subset Z$. (This is very easy to prove from Proposition 23.6.)
    b) $\mathrm{Perv}(X)$ is artinian and noetherian (again, easy at this point).
2. Beilinson showed $\mathrm{Perv}(X) \subset D_c^b(X)$ extends to an equivalence

$$D^b(\mathrm{Perv}X) \xrightarrow{\sim} D_c^b(X).$$

   This statement is not at all obvious; the key geometric point, which has many uses, is:

3) If $f : X \to Y$ is affine, then Rf_* is t-right exact and $Rf_!$ is t-left exact.

The following is a substantial theorem. It is proven in BBD by reduction to the case of finite fields, where the argument rests on Deligne's relative version of the Weil conjectures (Weil II). Over the complex numbers there are also proofs (Saito, DeCataldo-Migliorini) via Hodge theory.

Theorem 25.1 (The Decomposition Theorem)**.** *Let $f : X \to Y$ be a proper map of varieties over k. Then there exists a non-canonical isomorphism in $D_c^b(Y)$*

$$Rf_* IC_X \simeq \oplus_{i \in \mathbb{Z}} {}^p H^i(Rf_* IC_X)[-i],$$

and for each i there exists a canonical decomposition

$${}^p H^i(Rf_* IC_X) = \oplus_\alpha IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha[\dim Y_\alpha])$$

where $Y = \sqcup_\alpha Y_\alpha$ is a stratification into smooth locally-closed subschemes, and each $\mathcal{L}_\alpha$ is a $\underbrace{\textit{geometrically semi-simple}}_{\substack{\textit{i.e., semi-simple as a} \\ \pi_1^{\acute{e}t}(Y_{\alpha,\overline{k}})\textit{-representation}}}$ *local system on Y_α.*

To give a sense for what this theorem means, here is one of its classical precursors. If $f : X \to Y$ is a smooth projective (or proper) map of smooth varieties, then the Leray spectral sequence

$$E_2^{a,b} = H^a(Y, R^b f_* \mathbb{Q}_l) \implies H^{a+b}(X, \mathbb{Q}_l)$$

degenerates at the E_2 page, and the local systems $R^b f_* \mathbb{Q}_l$ are geometrically semi-simple (Deligne). Even better, we have a splitting

$$Rf_* \mathbb{Q}_l \xrightarrow{\sim} \oplus_{i \in \mathbb{Z}} R^i f_* \mathbb{Q}_l[-i]$$

Example 25.2. Note that in general the Leray spectral sequence does not degenerate at E_2: consider any resolution of singularities $f : X \to Y$ where $H^n(Y)$ is not pure (in the sense of Hodge theory or the Frobenius action over finite fields) for some n.

So one perspective on the decomposition theorem is that it generalizes Deligne's Leray degeneration theorem to arbitrary proper maps.

Time's up! But now we at least have the necessary foundations for geometric Satake.

References