Problems for Math 7890, Spring 2017

As always, if a problem or hint is incorrectly stated, the problem becomes to formulate it correctly, and then solve the corrected version.

1. Basic structure and representation theory of reductive groups

(1) If you do not have prior experience with representations of semisimple Lie algebras (or compact Lie groups), study Chapters 11-14 of Fulton & Harris (or equivalent; feel free to skip the "geometric plethysm" subsections). Make sure you understand the $\mathfrak{sl}_2$ theory well! Try some exercises as well (eg, 11.9, 13.9).

(2) (a) For the compact Lie group $G = \mathrm{U}(n)$, interpret the fact that every $g \in G$ is conjugate into a fixed maximal torus as a result you already know from linear algebra.
   (b) Show that every real $n \times n$ matrix A can be written as the product of an upper-triangular and an orthogonal matrix. When $A \in \mathrm{GL}_n(\mathbb{R})$, how unique is this decomposition? *After* doing this exercise, look up "Iwasawa decomposition."

(3) Over any field k (or even over $\mathbb{Z}$) we have the multiplicative group $\mathbf{G}_m$ defined by $\mathbf{G}_m(R) = R^\times$ (under multiplication) and the additive group $\mathbf{G}_a$ defined by $\mathbf{G}_a(R) = R$ (under addition). Write down explicitly the comultiplication, coinverse, and counits on the corresponding Hopf algebras. We classified representations of $\mathbf{G}_m$ in class; now classify representations of $\mathbf{G}_a$.

(4) Show that SL_n is semi-simple (in any characteristic). Show that SL_n is not linearly reductive in characteristic $p > 0$ (for this part you may want to start with SL_2 to see what's going on).

(5) Compute the root data of the groups (over an algebraically closed field) GL_n, Sp_{2n}, and SO_{2n+1} (you can choose which bilinear pairings to use to define the latter two; some choices will be more convenient than others). Verify that the Langlands dual groups are $\mathrm{GL}_n^\vee \cong \mathrm{GL}_n$ and $\mathrm{Sp}_{2n}^\vee \cong \mathrm{SO}_{2n+1}$.

(6) If $(X, \Phi, X^\vee, \Phi^\vee)$ is the root datum of a connected reductive group G, write down a root datum for the derived group of G.

(7) Let $B \subset \mathrm{GL}_2$ be the stabilizer of a line L inside a two-dimensional vector space $V = k^2$. Work out the example (or some other you prefer) discussed in class of a k-algebra R such that the action map $g \mapsto g(L)$ does not induce a surjection $\mathrm{GL}_2(R)/B(R) \to \mathbb{P}(V)(R)$.

(8) Let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-planes in an n-dimensional vector space. Find a subgroup $P \subset \mathrm{GL}_n$ and an isomorphism $\mathrm{GL}_n/P \xrightarrow{\sim} \mathrm{Gr}(k, n)$. Can you find a parametrization of the orbits $B\backslash \mathrm{GL}_n/P$ analogous to the Bruhat decomposition?

(9) Use row and column operations to establish by hand the Bruhat decomposition for GL_n.

(10) The Borel-Weil theorem admits a generalization (Borel-Weil-Bott) that also describes the G-representations given by the higher cohomology groups $H^i(G/B, L(\lambda))$ of the equivariant line bundles $L(\lambda)$ associated to (not necessarily dominant!) weights $\lambda \in X^\bullet(T)$. Work out this story for $G = \mathrm{SL}_2$, i.e. calculate the SL_2-representations $H^1(\mathbb{P}^1, \mathcal{O}(n))$ for all integers n.

(11) Formulate precisely what it should mean for two finite groups to have "the same character table," and find two non-isomorphic finite groups that have the same character table. Show explicitly (by decomposing tensor products of representations) that your two groups have inequivalent (abelian tensor) categories of representations.

2. Smooth representations of reductive groups over local fields