Math 6520 Problem Set 9, Spring 2018

(1) Let \( p \) be an odd prime, and let \( \mu_p \) be the group of \( p \)-th roots of 1 in \( \mathbb{C}^\times \). Regard \( S^{2n-1} \) as the unit sphere in \( \mathbb{C}^n \), and let \( \mu_p \) act on \( S^{2n-1} \) by \( \zeta \cdot (z_1, \ldots, z_n) = (\zeta z_1, \ldots, \zeta z_n) \). This is a free group action, and we let \( L_{n,p} \) be the quotient space \( S^{2n-1}/\mu_p \) (\( L_{n,p} \) is known as a “lens space” and can be thought of as a \( p \)-primary analogue of real projective space). Describe a CW structure on \( L_{n,p} \) and use it to compute the integral homology and the homology with \( \mathbb{Z}/\ell \) coefficients for every prime \( \ell \) (including the case \( \ell = p \)).

(2) Let \( C_* \) be a chain complex of finitely-generated \( \mathbb{Z} \)-modules such that \( C_i = 0 \) for all \( |i| > 0 \) (i.e., the complex \( C_* \) is bounded). Define the Euler characteristic of \( C \) to be

\[
\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}(C_i).
\]

(a) Let \( X \) be a finite CW complex, i.e. one having finitely many cells in total. Define \( \chi(X) \) to be \( \chi(C^{\text{CW}}_*(X)) \). Show that

\[
\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}(H_i(X)).
\]

(b) Let \( F \) be a field. Does \( \sum_{i \in \mathbb{Z}} \text{dim}_F(H_i(X; F)) \) also calculate \( \chi(X) \)?

(c) Deduce Euler’s formula \( F - E + V = 2 \), where \( V, E, \) and \( F \) are the number of vertices, edges, and faces in an arbitrary triangulation of the 2-sphere.

(3) Prove the algebraic K"unneth formula, as stated in class, for the homology of a tensor product of chain complexes of free modules over a PID.

(4) Let \( p: E \to B \) be a covering map, where \( E \) and \( B \) are connected and locally path-connected. Assume that the fibers of \( p \) have finite order \( d \) (the “degree” of \( p \)).

(a) Show that \( p: \text{Map}(\Delta^n, E) \to \text{Map}(\Delta^n, B) \) is surjective for all \( n \geq 0 \), and that the preimage of any \( n \)-simplex in \( B \) has exactly \( d \) elements.

(b) Define a group homomorphism \( p^*: C_*(B) \to C_*(E) \) by sending an \( n \)-simplex to the sum of its \( d \) preimages in \( \text{Map}(\Delta^n, E) \). Show that \( p_\ast \circ p^* \) is multiplication by \( d \). Also check that \( p^* \) induces a map \( H_*(B) \to H_*(E) \).

(c) Now work with homology with coefficients in a field \( F \). If \( d \) is invertible in \( F \), show that \( p_\ast: H_*(E; F) \to H_*(B; F) \) is surjective.

(d) Give an example with \( d \) not invertible in \( F \) where \( p_\ast \) fails to be surjective.

(e) If we do not assume \( p \) has finite fibers, need \( p_\ast: H_*(E; \mathbb{Q}) \to H_*(B; \mathbb{Q}) \) be surjective?

(5) (a) If a space \( X \) is \( n \)-connected, show that the pair \( (CX, X) \) is \( (n + 1) \)-connected.

(b) Check exactness of the end of the relative homotopy sequence of a pair:

\[
\pi_1(X, x_0) \to \pi_1(X, A, x_0) \to \pi_0(A) \to \pi_0(X).
\]

(Here \( \pi_0(A) \) and \( \pi_0(X) \) are pointed sets with base-point given by the component containing \( x_0 \in A \subset X \).)

(c) See Hatcher Chapter 1.3, Exercise 7 for the definition of the “quasi-circle” (sometimes known as the “Polish circle”). Show that the quasi-circle has trivial homotopy groups but is not contractible.

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