Math 6520 Problem Set 8, Spring 2018

More homology, and some homological algebra

(1) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. Show that T extends to the one-point compactification of $\mathbb{R}^n$, and thus gives a map $\overline{T}\colon S^n \to S^n$. Show that the degree of $\overline{T}$ is the sign of $\det(T)$.

(2) This problem addresses how to compute the homology groups $H_*(X, A; G)$ of a pair (X, A) with coefficients in any abelian group G starting from the ordinary homology groups with integer coefficients $H_*(X, A) = H_*(X, A; \mathbb{Z})$.

  (a) Let $(C_\bullet, \partial)$ be a chain complex of *free* abelian groups (for example, the singular chains $C_\bullet(X, A)$). There is a short exact sequence of chain complexes

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

  where $Z_n = \ker(\partial)$ and $B_{n-1} = \operatorname{im}(\partial)$. The differentials on Z_n and B_{n-1} are induced by ∂ on C_n, and they are clearly both zero. Show that there is a long exact sequence

$$\cdots \to B_n \otimes_{\mathbb{Z}} G \to Z_n \otimes_{\mathbb{Z}} G \to H_n(C_\bullet \otimes_{\mathbb{Z}} G) \to B_{n-1} \otimes_{\mathbb{Z}} G \to Z_{n-1} \otimes_{\mathbb{Z}} G \to \cdots,$$

  where the various maps are induced by the obvious inclusions or by $\partial\colon C_n \to B_{n-1}$. Pause and reflect on why $H_n(C_\bullet \otimes_{\mathbb{Z}} G) \to B_{n-1} \otimes_{\mathbb{Z}} G$ is not necessarily the zero map!

  (b) Break up the long exact sequence into short exact sequences, and show that for each n there is an exact sequence (natural in $C_\bullet$)

$$0 \to H_n(C_\bullet) \otimes_{\mathbb{Z}} G \to H_n(C_\bullet \otimes_{\mathbb{Z}} G) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet, G) \to 0.$$

  (Hint: $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$ is a projective resolution of $H_{n-1}(C_\bullet)$.) Taking $C_\bullet$ to be the singular chains on (X, A), we obtain the SES

$$0 \to H_n(X, A) \otimes_{\mathbb{Z}} G \to H_n(X, A; G) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), G) \to 0.$$

  (c) Show that the SES in (b) is split. (Hint: choose a splitting of $Z_n \subset C_n$.)

  (d) Redo Problem Set 7.5b using this problem.

  (e) For any abelian group π, compute (for all q) the homology groups

$$H_q(M(\mathbb{Z}/m, n); \pi),$$

  where m and n are positive integers, and $M(\mathbb{Z}/m, n)$ is the Moore space from Problem Set 7.6.

  (f) For any space X, show that $\widetilde{H}_n(X; \mathbb{Z}) = 0$ for all n if and only if $\widetilde{H}_n(X; \mathbb{Q}) = 0$ and $\widetilde{H}_n(X; \mathbb{F}_p) = 0$ for all n and all primes p. Deduce that a map $f\colon X \to Y$ induces isomorphisms on integer homology in all degrees if and only if it does so on homology with $\mathbb{Q}$ and $\mathbb{F}_p$ (for all p) coefficients. (Note: as we will soon see, there are non-contractible spaces all of whose homology groups are trivial.)

(3) Compute $H_*(\mathbb{RP}^2; G)$ for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2$ (you already did this using simplicial homology; now do it from the basic properties–excision, etc.–of singular homology). Show that the map $\mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{RP}^1 \cong S^2$ (here $\mathbb{RP}^1$ is the 1-skeleton of $\mathbb{RP}^2$) induces an isomorphism $H_2(\mathbb{RP}^2; \mathbb{Z}/2) \to H_2(S^2; \mathbb{Z}/2)$, and deduce that the splitting in Problem 2c cannot be natural (i.e., we can't define the splitting for all spaces X in such a way that for maps $f\colon X \to Y$ the respective splittings are compatible with f_*).

(4) (a) Find a chain complex C of free abelian groups such that$H_1(C) = \mathbb{Z}/n$, $H_0(C) = \mathbb{Z}/n$, and $H_q(C) = 0$ for all $q \neq 0, 1$.

(b) Find a chain complex C of free abelian groups such that$H_q(C) = \mathbb{Z}/n$ for all $q \geq 0$, and $H_q(C) = 0$ for all $q < 0$.

(c) For any abelian group G, compute $\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

(5) Find an example of a map $f: A \to B$ of chain complexes of abelian groups such that $H_*(f) = 0$ (i.e. $H_n(f): H_n(A) \to H_n(B)$ is zero for all n), but f is not null-homotopic (i.e. chain-homotopic to the zero map).