Math 6520 Problem Set 7, Spring 2018
Homology

(1) Show that a pushout of a cofibration is a cofibration.

(2) Let $f : X \to Y$ be a map of topological spaces, and let $M_f = (X \times I) \cup_f Y$ be the mapping cylinder of $f$ (that is, glue $X \times \{0\}$ to $Y$). The map $f$ factors as the composite

$$
\begin{array}{c}
M_f \\
\downarrow j \\
X \\
\downarrow f \\
Y \\
\downarrow r
\end{array}
$$

where $j(x) = (x, 1)$, and $r(y) = y$ (for $y \in Y$) and $r(x, t) = f(x)$ (for $(x, t) \in X \times I$). Show that $j$ is a cofibration, and $r$ is a homotopy equivalence. (“Up to homotopy, any map can be replaced by a cofibration.”)

(3) Using our calculation of $H_*(S^n)$, prove the higher-dimensional version of the Brouwer fixed-point theorem: any (continuous) map $f : D^{n+1} \to D^{n+1}$ has a fixed point.

(4) (The Mayer-Vietoris sequence) Let $(X; A, B)$ be an excisive triad: that is, $A$ and $B$ are subspaces of $X$ whose interiors cover $X$. Name the various inclusions as follows:

$$
\begin{array}{c}
A \cap B \xrightarrow{j_A} A \\
\downarrow j_B \\
B \xrightarrow{i_B} X \\
\downarrow i_A \\
A \xrightarrow{} X
\end{array}
$$

Then there is a long-exact (Mayer-Vietoris) sequence

$$
\cdots \to H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{(j_A, -i_B)} H_n(A) \oplus H_n(B) \xrightarrow{(i_A, -j_B)} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \xrightarrow{} \cdots ,
$$

where the boundary map $\partial : H_n(X) \to H_{n-1}(A \cap B)$ is constructed as the composite

$$
H_n(X) \to H_n(X, B) \xleftarrow{i_B} H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B).
$$

(a) Apply the Mayer-Vietoris sequence to give another (not essentially different) calculation of $H_*(S^n)$, for all $n \geq 1$.

(b) Establish exactness of the Mayer-Vietoris sequence in the following two steps:

(i) Use excision to show that $H_n(A, A \cap B) \oplus H_n(B, A \cap B) \to H_n(X, A \cap B)$ (induced by inclusions of pairs of spaces) is an isomorphism.

(ii) Check that $\partial : H_n(X) \to H_{n-1}(A \cap B)$ is the negative of the analogously defined composite

$$
H_n(X) \to H_n(X, A) \xleftarrow{i_A} H_n(B, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),
$$

and is equal to the composite

$$
H_n(X) \to H_n(X, A) \xrightarrow{i_A} H_n(B, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B).
$$

(iii) Conclude the proof of exactness of the Mayer-Vietoris sequence.

(5) (a) Show that the Klein bottle $K$ can be decomposed as the union of two Möbius bands glued along their boundary circles. Then apply the Mayer-Vietoris sequence to compute the homology groups $H_*(K)$.  

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(b) As in Problem Set 6, Problem 2b, all of our work so far with singular homology could likewise be carried out with an arbitrary abelian group $G$ of coefficients: that is, redefine the singular chain complex with coefficients in $G$ to be $C_q(X; G) = \oplus_{\sigma: \Delta^q \to X} G[\sigma]$, formally take the same boundary operator, and let $H_q(X; G)$ be the homology of the resulting chain complex. Our whole theory (homotopy invariance, the long exact sequence of a pair, excision) works verbatim in this setting (convince yourself of this). Now take $G = \mathbb{Z}/2$ and recompute the homology groups $H_*(K; \mathbb{Z}/2)$.

(6) Let $G$ be a finitely-generated abelian group, and let $n \geq 1$ be an integer. Show that there is a connected space $M(G, n)$ (a so-called “Moore space”) such that $H_q(M(G, n))$ is $G$ for $q = n$ and is 0 for $q \neq 0, n$. 