Math 6520 Problem Set 7, Spring 2018

Homology

(1) Show that a pushout of a cofibration is a cofibration.

(2) Let $f\colon X \to Y$ be a map of topological spaces, and let $M_f = (X \times I) \cup_f Y$ be the mapping cylinder of f (that is, glue $X \times \{0\}$ to Y). The map f factors as the composite

$$\begin{tikzcd} & M_f \arrow[dr, "r"] & \\ X \arrow[ur, "j"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $j(x) = (x, 1)$, and $r(y) = y$ (for $y \in Y$) and $r(x, t) = f(x)$ (for $(x, t) \in X \times I$). Show that j is a cofibration, and r is a homotopy equivalence. ("Up to homotopy, any map can be replaced by a cofibration.")

(3) Using our calculation of $H_*(S^n)$, prove the higher-dimensional version of the Brouwer fixed-point theorem: any (continuous) map $f\colon D^{n+1} \to D^{n+1}$ has a fixed point.

(4) (The Mayer-Vietoris sequence) Let $(X; A, B)$ be an excisive triad: that is, A and B are subspaces of X whose interiors cover X. Name the various inclusions as follows:

$$\begin{tikzcd} A \cap B \arrow[r, "j_A"] \arrow[d, "j_B"'] & A \arrow[d, "\iota_A"] \\ B \arrow[r, "\iota_B"] & X \end{tikzcd}$$

Then there is a long-exact (Mayer-Vietoris) sequence

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{(j_{A,*}, j_{B,*})} H_n(A) \oplus H_n(B) \xrightarrow{(\iota_{A,*} - \iota_{B,*})} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots,$$

where the boundary map $\partial\colon H_n(X) \to H_{n-1}(A \cap B)$ is constructed as the composite

$$H_n(X) \to H_n(X, B) \xleftarrow{\sim} H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B).$$

(a) Apply the Mayer-Vietoris sequence to give another (not essentially different) calculation of $H_*(S^n)$, for all $n \geq 1$.

(b) Establish exactness of the Mayer-Vietoris sequence in the following two steps:

(i) Use excision to show that $H_n(A, A \cap B) \oplus H_n(B, A \cap B) \to H_n(X, A \cap B)$ (induced by inclusions of pairs of spaces) is an isomorphism.

(ii) Check that $\partial\colon H_n(X) \to H_{n-1}(A \cap B)$ is the negative of the analogously defined composite

$$H_n(X) \to H_n(X, A) \xleftarrow{\sim} H_n(B, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),$$

and is equal to the composite

$$H_n(X) \to H_n(X, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B).$$

(iii) Conclude the proof of exactness of the Mayer-Vietoris sequence.

(5) (a) Show that the Klein bottle K can be decomposed as the union of two Möbius bands glued along their boundary circles. Then apply the Mayer-Vietoris sequence to compute the homology groups $H_*(K)$.

(b) As in Problem Set 6, Problem 2b, all of our work so far with singular homology could likewise be carried out with an arbitrary abelian group G of coefficients: that is, redefine the singular chain complex with coefficients in G to be $C_q(X;G) = \oplus_{\sigma\colon \Delta^q \to X} G[\sigma]$, formally take the same boundary operator, and let $H_q(X;G)$ be the homology of the resulting chain complex. Our whole theory (homotopy invariance, the long exact sequence of a pair, excision) works verbatim in this setting (convince yourself of this). Now take $G = \mathbb{Z}/2$ and recompute the homology groups $H_*(K;\mathbb{Z}/2)$.

(6) Let G be a finitely-generated abelian group, and let $n \geq 1$ be an integer. Show that there is a connected space $M(G,n)$ (a so-called "Moore space") such that $H_q(M(G,n))$ is G for $q = n$ and is 0 for $q \neq 0, n$.