Math 6520 Problem Set 5, Spring 2018

Some elementary homotopy theory

(1) Show that $\pi_k(S^n, *) = 0$ for any $k < n$.

(2) Last week for any group G you constructed a space BG with fundamental group G and contractible universal cover. Show that $\mathbb{RP}^\infty = \operatorname{colim}_n \mathbb{RP}^n$ (the inclusions $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ are induced by $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by adding a zero in the last coordinate) is another example of such a space in the case $G = \mathbb{Z}/2$ (in fact, $\mathbb{RP}^\infty$ and the space $B(\mathbb{Z}/2)$ you constructed last week are homotopy equivalent). The key part of this proof will be showing that the infinite-dimensional sphere S^∞ is contractible.

(3) Give a careful proof that ΩS^1 is homotopy-equivalent to the discrete space $\mathbb{Z}$. (Interpretation: $\Omega B\mathbb{Z} \simeq \mathbb{Z}$.)

(4) Dual to the notion of fibration is a similarly central notion of *cofibration*: a map $i\colon A \to X$ is a cofibration if it satisfies the *homotopy extension property* (HEP): for any map $f\colon X \to Y$ and homotopy $h\colon A \times I \to Y$ such that $h_0 = f$, there exists an extension of h to $\tilde{h}\colon X \times I \to Y$. Diagrammatically,

$$\begin{tikzcd}
A \arrow[rr, "\iota_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i\times\mathrm{id}"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "\iota_0"] & & X \times I \arrow[ul, dashed, "\exists\tilde{h}"']
\end{tikzcd}$$

(a) Show that $i\colon A \to X$ is a cofibration if and only if any diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p_0"] \\
X \arrow[r, "f"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where we write $p_0\colon Y^I = \mathrm{Map}(I, Y) \to Y$ for evaluation at 0, can be completed by a dotted arrow as indicated.

(b) Define the mapping cylinder M_i of $i\colon A \to X$ as the pushout of $A \times I \xleftarrow{\iota_0} A \xrightarrow{i} X$. Show that i is a cofibration if the HEP is satisfies for $Y = M_i$ and f and h the canonical maps. Use this to deduce that a cofibration is always a homeomorphism onto its image.

(c) Show that if $i\colon A \to X$ is a cofibration, then $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Show that if $i\colon A \to X$ is moreover closed, then the converse holds. (Actually, you don't need to assume i is closed ...)

(d) Establish the most important non-obvious example of a cofibration: for any $n \geq 1$, $S^{n-1} \to D^n$ (inclusion of the boundary) is a cofibration.

(5) (a) Show that if $i\colon A \to X$ is a cofibration, and A is contractible, then $X \to X/A$ is a homotopy equivalence.

(b) A pointed space $(X, x_0) \in \mathrm{Top}_*$ is said to be *non-degenerately based* (or *well-pointed*) if the inclusion $i\colon \{x_0\} \to X$ is a cofibration. Show that for such a space (X, x_0), the quotient map $\mathrm{S}X \to \Sigma X$ from the unreduced to the reduced suspension is a homotopy equivalence.

(c) Exhibit homeomorphisms $\mathrm{S}(S^{n-1}) \cong S^n$ and $\Sigma S^{n-1} \cong S^n$.