Math 6520 Problem Set 5, Spring 2018
Some elementary homotopy theory

(1) Show that \( \pi_k(S^n, \ast) = 0 \) for any \( k < n \).

(2) Last week for any group \( G \) you constructed a space \( BG \) with fundamental group \( G \) and contractible universal cover. Show that \( \mathbb{R}P^n = \text{colim}_n \mathbb{R}P^n \) (the inclusions \( \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n \) are induced by \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \) by adding a zero in the last coordinate) is another example of such a space in the case \( G = \mathbb{Z}/2 \) (in fact, \( \mathbb{R}P^n \) and the space \( B(\mathbb{Z}/2) \) you constructed last week are homotopy equivalent). The key part of this proof will be showing that the infinite-dimensional sphere \( S^\infty \) is contractible.

(3) Give a careful proof that \( \Omega S^1 \) is homotopy-equivalent to the discrete space \( \mathbb{Z} \). (Interpretation: \( \Omega B\mathbb{Z} \cong \mathbb{Z} \).

(4) Dual to the notion of fibration is a similarly central notion of cofibration: a map \( i: A \to X \) is a cofibration if it satisfies the homotopy extension property (HEP): for any map \( f: X \to Y \) and homotopy \( h: A \times I \to Y \) such that \( h_0 = f \), there exists an extension of \( h \) to \( \tilde{h}: X \times I \to Y \).

Diagrammatically,

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\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (Y) at (2,0) {Y};
  \node (X) at (0,-2) {X};
  \node (A_times_I) at (2,-2) {A \times I};
  \node (X_times_I) at (0,-4) {X \times I};
  \draw[->] (A) -- node[above] {$i_0$} (A_times_I);
  \draw[->] (A) -- node[below] {$i$} (X);
  \draw[->] (X) -- node[below] {$i_0$} (X_times_I);
  \draw[->] (Y) -- node[left] {$h$} (A_times_I);
  \draw[->] (Y) -- node[right] {$\exists \tilde{h}$} (X_times_I);
  \draw[->] (A_times_I) -- node[above] {$i \times \text{id}$} (X_times_I);
  \draw[->] (X) -- node[above] {$f$} (X_times_I);

diagram completed
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(a) Show that \( i: A \to X \) is a cofibration if and only if any diagram

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\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (Y) at (2,0) {Y};
  \node (X) at (0,-2) {X};
  \node (Y_I) at (2,-2) {Y^I};
  \node (X_times_I) at (0,-4) {X \times I};
  \draw[->] (A) -- node[above] {$h$} (Y_I);
  \draw[->] (A) -- node[below] {$i$} (X);
  \draw[->] (X) -- node[below] {$f$} (X_times_I);
  \draw[->] (X_times_I) -- node[above] {$p_0$} (Y_I);

diagram completed
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where we write \( p_0: Y^I = \text{Map}(I, Y) \to Y \) for evaluation at 0, can be completed by a dotted arrow as indicated.

(b) Define the mapping cylinder \( M_i \) of \( i: A \to X \) as the pushout of \( A \times I \leftarrow i \to A \to X \). Show that \( i \) is a cofibration if the HEP is satisfies for \( Y = M_i \) and \( f \) and \( h \) the canonical maps. Use this to deduce that a cofibration is always a homeomorphism onto its image.

(c) Show that if \( i: A \to X \) is a cofibration, then \( X \times \{0\} \cup A \times I \) is a retract of \( X \times I \). Show that if \( i: A \to X \) is moreover closed, then the converse holds. (Actually, you don’t need to assume \( i \) is closed . . .)

(d) Establish the most important non-obvious example of a cofibration: for any \( n \geq 1 \), \( S^{n-1} \to D^n \) (inclusion of the boundary) is a cofibration.

(5) (a) Show that if \( i: A \to X \) is a cofibration, and \( A \) is contractible, then \( X \to X/A \) is a homotopy equivalence.

(b) A pointed space \( (X, x_0) \in \text{Top}_* \) is said to be non-degenerately based (or well-pointed) if the inclusion \( i: \{x_0\} \to X \) is a cofibration. Show that for such a space \( (X, x_0) \), the quotient map \( SX \to \Sigma X \) from the unreduced to the reduced suspension is a homotopy equivalence.

(c) Exhibit homeomorphisms \( S(S^{n-1}) \cong S^n \) and \( \Sigma S^{n-1} \cong S^n \).