Math 6520 Problem Set 2, Spring 2018

First properties of the fundamental group

(1) (a) Dually to Problem Set 1, #2b, show that for any set $\{X_i\}_{i\in I}$ of topological spaces (I is the indexing set), and for any topological space Z, there is a natural isomorphism

$$\mathrm{Map}(Z, \prod_{i\in I} X_i) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Map}(Z, X_i),$$

where $\prod_{i\in I} X_i$ is equipped with the usual product topology.

(b) Deduce that for any two pointed spaces (X, x) and (Y, y) there is an isomorphism $\pi_1(X\times Y, (x, y)) \xrightarrow{\sim} \pi_1(X, x) \times \pi_1(Y, y)$.

(2) Compute the fundamental groups of the following spaces. You may use the van Kampen theorem, even though the proof has not yet been completed in class.

(a) The torus $T = S^1 \times S^1$.

(b) The Möbius band $M = I \times I/((0, t) \sim (1, 1 - t)\forall t \in I))$.

(c) The Klein bottle $K = I \times I/((0, t) \sim (1, 1 - t), (s, 0) \sim (s, 1)\forall s, t \in I)$.

(d) The complement $\mathbb{R}^2 \setminus S$ of a finite set of points $S \subset \mathbb{R}^2$.

(e) The complement $\mathbb{R}^3 \setminus L$ of a union L of a finite number of lines through the origin.

(3) Recall that the real projective space $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$, topologized as the quotient $\mathbb{R}^{n+1} \setminus \{0\}/\sim$, where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} \setminus 0$, $\lambda \in \mathbb{R}^\times$. Equivalently, $\mathbb{RP}^n$ is the quotient of the unit sphere S^n in $\mathbb{R}^{n+1}$ by its antipodal map $\iota(x) = -x$. Using the covering map $S^n \to \mathbb{RP}^n$, mimic the argument we used to show $\pi_1(S^1) \cong \mathbb{Z}$ to compute $\pi_1(\mathbb{RP}^n)$ for all $n \geq 2$. What about the case $n = 1$?

(4) (a) Let X be a set equipped with two binary operations, $*$ and $\bullet$. Assume that there are (two-sided) units e_* and $e_\bullet$ for $*$ and $\bullet$, and that the operations satisfy the mutual commutativity relation

$$(w * x) \bullet (y * z) = (w \bullet y) * (x \bullet z).$$

Show that $*$ and $\bullet$ coincide, and that they are both commutative and associative.

(b) Let G be a topological group with identity e. Show that the two obvious composition laws on $\pi_1(G, e)$ coincide, and that $\pi_1(G, e)$ is abelian (this generalizes the observation from class in the case $G = S^1$).

(5) Show that S^n and $\mathbb{R}^{n+1} \setminus \{0\}$ are homotopy equivalent.

(6) Recall that #2 on Problem Set 1 characterized certain constructions on topological spaces via universal mapping properties. This problem asks you to make analogous constructions, which is to say satisfying the same universal mapping properties, in different contexts. Specifically, in #2b and #2c we gave universal mapping properties that characterize the *coproduct* of a set $\{X_i\}$ of spaces; and that characterize the *pushout* of a diagram $X \leftarrow A \rightarrow Y$ of spaces.

(a) Let $\{A_i\}_{i\in I}$ be a set of abelian groups. Show that there is an abelian group "$\coprod_i A_i$" satisfying the universal property of the coproduct.

(b) Let $X \leftarrow A \rightarrow Y$ be homomorphisms of abelian groups. Show that there is an abelian group "$X \cup_A Y$" satisfying the universal property of the pushout.

(c) Redo parts (a) and (b) for *groups* rather than abelian groups.

(d) Redo parts (a) and (b) for *pointed topological spaces* rather than topological spaces. (Note that you may already have more familiar names for some of these constructions.)