Math 6520 Problem Set 11, Spring 2018

(1) (Yoneda's Lemma) Let C be any locally small category, i.e. for all objects Y, X of C, $\mathrm{Hom}_C(Y, X)$ is a set (rather than a proper class). Show that the ("Yoneda embedding") functor $C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})$ given by sending an object X of C to the (contravariant) functor $h_X(Y) = \mathrm{Hom}_C(Y, X)$ is fully faithful. Here the functor category $\mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})$ is the category whose objects are contravariant functors $C \to \mathbf{Sets}$ (i.e., covariant functors from the opposite category C^{op} to $\mathbf{Sets}$), and whose morphisms are natural transformations of functors. Convince yourself that your argument shows more generally that for any functor $F\colon C^{\mathrm{op}} \to \mathbf{Sets}$, there are natural bijections

$$\mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})}(h_X, F) \cong F(X).$$

(2) Use the representability of cohomology ($\widetilde{H}(X; \pi) \cong [X, K(\pi, n)]_*$ for based CW complexes X) and the previous exercise to construct, for any abelian groups π and π', maps $K(\pi, m) \wedge K(\pi', n) \to K(\pi \otimes \pi', m + n)$ that induce the cup-product after applying $[X, \cdot]_*$. (You can also give a direct homotopical construction of these "product" maps on the $K(\pi, m)$'s; but then you need to show the result does indeed induce our familiar cup-product.)

(3) Generalize the relative cup-product to the following setting: if A and B are open subsets of a space X, then ordinary cup-product induces a pairing $H^p(X, A; R) \times H^q(X, B; R) \xrightarrow{\cup} H^{p+q}(X, A \cup B; R)$.

(4) Use the previous exercise to show that cup-products vanish on suspensions: if $p, q > 0$, then $H^p(\Sigma X; R) \times H^q(\Sigma X; R) \xrightarrow{\cup} H^{p+q}(\Sigma X; R)$ is the zero map. (Hint: judiciously write ΣX as a union $A \cup B$).

(5) Show that any map $f\colon \mathbb{CP}^n \to \mathbb{CP}^n$ inducing an isomorphism on $H_2(\mathbb{CP}^n; \mathbb{Z})$ induces an isomorphism on all homology and cohomology groups; and if n is even, f moreover is orientation-preserving (maps a generator of $H_{2n}(\mathbb{CP}^n; \mathbb{Z})$ to itself). Show this last statement need not be the case if n is odd.

(6) (a) Compute the cohomology ring $H^*(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}/2)$.
  (b) Suppose $\mathbb{R}^n$ can be given the structure of a real division algebra via a map $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Here we only require that μ is left and right distributive ($\mu(a + b, c) = \mu(a, c) + \mu(b, c)$ and $\mu(a, b + c) = \mu(a, b) + \mu(a, c)$, and that for all $a \neq 0$, the linear maps $x \mapsto \mu(a, x)$ and $x \mapsto \mu(x, a)$ are surjective. (We don't require that μ be commutative, associative, or unital.) Show that n must be a power of 2. (Hint: check that μ induces a map $\mu\colon \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$; show that if we write $\alpha_1, \alpha_2, \alpha$ for generators of the cohomology rings (with $\mathbb{Z}/2$ coefficients) of these three copies of $\mathbb{RP}^{n-1}$, then we have $\mu^*(\alpha) = \alpha_1 + \alpha_2$; then use the known structure of the cohomology ring to conclude that n must be a power of 2.)

(7) In this question we write $\chi(M)$ for the Euler characteristic of M.
  (a) Let M be a connected compact manifold of odd dimension. Show that $\chi(M) = 0$.
  (b) Let M be a connected compact manifold of dimension $n \equiv 2 \pmod 4$. Moreover assume that M is orientable. Show that $\chi(M)$ is even. Show this need not hold if we do not assume M is orientable.

(8) The Poincaré Conjecture (now a theorem of Perelman) states that every simply-connected compact 3-manifold X is homeomorphic to S^3.
  (a) Show that any such X at least has the same homology groups as S^3 (one says that X is a "homology 3-sphere").

(b) Consider the quotient space $X = \mathrm{SO}(3)/I$, where I is the icosahedral group (the rotational symmetries of a regular icosahedron in $\mathbb{R}^3$). Show that X has homology groups isomorphic to those of S^3 but is not homotopy equivalent (let alone homeomorphic) to S^3. (So not every "homology 3-sphere" is homeomorphic to S^3.)