Math 6520 Problem Set 10, Spring 2018

The homework for March 30 is to catch up on the old homework, and to make sure you understand how to solve the midterm problems. During the week of April 1-8 you should complete, together, as a class, the following problems, and post the solutions to the Overleaf page.

(1) Let R be a (commutative) ring, and let $(C_\bullet, d_\bullet)$ be a chain complex of R-modules. Define the dual cochain complex $(C^\bullet, d^\bullet)$ by

$$C^q = \mathrm{Hom}_R(C_q, R)$$

with $d^q\colon C^q \to C^{q+1}$ given by $\varphi \mapsto (-1)^q \varphi \circ d_{q+1}$. (The sign is conventional and suitable for certain generalizations; it has no real significance here.) More generally, for any R-module M, form the complex $\mathrm{Hom}_R(C_\bullet, M)$ (with the differential defined similarly) and define the cohomology of $C_\bullet$ with coefficients in M to be

$$H^n(C_\bullet; M) = \ker(d^q)/\operatorname{im}(d^{q-1}).$$

Now let X be a topological space, and define the cohomology of X with coefficients in M, $H^*(X; M$, to be the cohomology of the singular chain complex $C_\bullet(X; R)$ with coefficients in the R-module M. Similarly, for a pair (X, A), define the relative cohomology by defining $C_\bullet(X, A; R)$ via the short exact sequence of complexes of *free* R-modules

$$0 \to C_\bullet(A; R) \to C_\bullet(X; R) \to C_\bullet(X; R)/C_\bullet(A; R) \to 0,$$

and letting $H^*(X, A; M)$ be the cohomology of $C_\bullet(X; R)/C_\bullet(A; R)$ with coefficients in M (that is, apply $\mathrm{Hom}_R(\cdot, M)$ and take cohomology).

(2) Give an explicit description of $H^0(X; M)$ for any space X and R-module M.

(3) Establish the following *universal coefficient theorem* relating cohomology to homology: for any chain complex $C_\bullet$ of free R-modules over a PID R, there is a natural short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X; R), M) \to H^n(X; M) \to \mathrm{Hom}_R(H_n(X; R), M) \to 0.$$

Show moreover that the sequence is split (it is not naturally split).

(4) As with homology, there is a small collection of basic properties of cohomology which yield all the basic calculations, and which in fact (although we have not yet proven this) characterize the theory. These are known as the Eilenberg-Steenrod axioms. They can be phrased in several equivalent ways, depending on whether one works with (a) cohomology of pairs of spaces; (b) reduced cohomology of (well-pointed) spaces; (c) cohomology of CW pairs; or (d) reduced cohomology of (pointed) CW complexes. See Chapters 14 (for homology) and 19 (for cohomology) of May for a discussion of the equivalence (using CW approximation!) of these four versions of the Eilenberg-Steenrod axioms. Here we give the axioms for cohomology of pairs of spaces, and your task in this problem is to establish these axioms for singular cohomology. Fix a (commutative) ring R. Then for all integers $q \geq 0$, there exist *contravariant* functors $H^q(X, A; R)$ from the category of pairs (X, A) of spaces (that is, A is a subspace of X) to the category of R-modules, and natural transformations $\delta\colon H^q(A; R) \to H^{q+1}(X, A; R)$ satisfying the following

- (homotopy invariance) $H^q(\cdot, \cdot; R)$ factors through the homotopy category of pairs of spaces, i.e. if f and g are homotopic maps $(X, A) \to (Y, B)$, then $f^* = g^*$ on $H^q(Y, B; R)$.
- (dimension) If X is a point, then $H^0(X; R) = R$, and $H^q(X; R) = 0$ for $q > 0$.

- (exactness) The sequence
$$\cdots \to H^q(X,A;R) \to H^q(X;R) \to H^q(A;R) \xrightarrow{\delta} H^{q+1}(X,A;R) \to \cdots$$
is exact, where the unlabeled maps are the obvious ones coming from the contravariant functoriality.
- (excision) If X is the union of the interiors of subspaces A and B (i.e. $(X;A,B)$ is an excisive triad), then the inclusion $(A, A \cap B) \to (X, B)$ induces isomorphisms $H^q(X,B;R) \to H^q(A, A\cap B;R)$.
- (additivity) If (X,A) is a disjoint union $\coprod_{i\in I}(X_i, A_i)$, then the inclusions $(X_i, A_i) \subset (X,A)$ induce isomorphisms $H^q(X,A;R) \to \prod_{i\in I} H^q(X_i, A_i;R)$.
- We haven't yet proven the following result for homology, so you don't need to do this part; but do note that it is also true:
(weak equivalence) If $f\colon (X,A) \to (Y,B)$ is a weak equivalence, then $f^*\colon H^q(Y,B;R) \to H^q(X,A;R)$ is an isomorphism.[1]

(5) Imitating our discussion for homology, define the CW cohomology of a CW complex, and show that it is isomorphic to the singular cohomology.

(6) Compute the cohomology (with $\mathbb{Z}$ coefficients) of the following spaces:
- S^n
- $\mathbb{RP}^n$, $\mathbb{RP}^\infty$, $\mathbb{CP}^n$, $\mathbb{CP}^\infty$
- an orientable surface of genus g
- BC_p, i.e. a $K(C_p, 1)$, where C_p is a cyclic group of order p. Do this example with $\mathbb{Z}$ coefficients and $\mathbb{Z}/n$ coefficients for any integer n.

[1] Here the map of pairs f is said to be a weak equivalence if the two maps $f\colon X \to Y$ and $f\colon A \to B$ are both weak equivalences.