Homework 4 Comments

1. (2.5.6) The center of $\mathrm{GL}_n(\mathbb{R})$ is the subgroup of scalar matrices, i.e. $\{r \cdot 1 : r \in \mathbb{R}^\times\}$.

2. (2.6.10)

   (a) A homomorphism from a cyclic group to any group is uniquely determined by the image of a generator, so any hom $f\colon \mathbb{Z}/N \to \mathbb{Z}/N$ is determined by $f(1)$. Such an f is an isomorphism if and only if $f(1)$ is a unit mod N, since we need (for surjectivity) to be able to solve the equation $f(1) \cdot x \equiv a \pmod{N}$ for any $a \in \mathbb{Z}/N$ (similarly for injectivity–but once you know surjectivity, injectivity is immediate, since the source and target of f have the same number of elements). We deduce an isomorphism (of groups!)

   $$\begin{aligned} \mathrm{Aut}(\mathbb{Z}/N) &\xrightarrow{\sim} (\mathbb{Z}/N)^\times \\ f &\mapsto f(1). \end{aligned}$$

   (b) The homomorphism $c\colon S_3 \to \mathrm{Aut}(S_3)$, $g \mapsto c_g$ (conjugation by g), is an isomorphism. To see this, check that the center of S_3 is trivial, so that c is injective. The order of $\mathrm{Aut}(S_3)$ can be at most 6, however, since any automorphism must map $(1,2)$ to a transposition (3 choices) and $(1,2,3)$ to a 3-cycle (2 choices). Thus c must be surjective as well.

3. (2.8.3) If $|G| = p^a$ with $a \geq 1$, consider any non-trivial element g of G. By (the corollary to) Lagrange's Theorem, the order of g divides $|G|$, hence the order of g is p^b for some $1 \leq b \leq a$. The element $g^{p^{b-1}}$ then has order p.

4. (2.8.6) Let K be the kernel of φ. The order of K divides 18, and the order of G/K $(= 18/|K|)$ divides 15. Since K is non-trivial, $|K|$ must be 6.

5. (2.11.4)

   (a) The multiplication map $H \times K \to G$ is an isomorphism.

   (b) G is not isomorphic to the product: $H \times K$ is abelian, whereas G is not abelian.

   (c) The multiplication map $H \times K \to G$ is an isomorphism.

6. (2.12.2) I leave it to you to check that H is a subgroup of $\mathrm{GL}_3(\mathbb{R})$, and that K is normal in H. To identify the quotient, note that the map

   $$\begin{aligned} H &\to \mathbb{R} \times \mathbb{R} \\ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} &\mapsto (a, c) \end{aligned}$$

is a surjective group homomorphism (check!) with kernel equal to K. The first isomorphism theorem thus implies that H/K is isomorphic to $\mathbb{R} \times \mathbb{R}$. The center of H is K, as checked directly by matrix multiplication. (Aside: H is often called the 'Heisenberg group' because of its appearance in quantum mechanics.)

7. (2.12.4) The homomorphism $G \to G$ given by $z \mapsto z^4$ induces (by the first isomorphism theorem) an isomorphism $G/H \xrightarrow{\sim} G$.