Math 5310 Fall 2015 Midterm 1

"Don't Panic"–*The Hitchhiker's Guide to the Galaxy*

Solve the following problems, explaining your reasoning clearly and concisely. If you get stuck somewhere, you may want to jump to a different problem.

1. (a) (1 pt) Define what it means for elements x and y of a group G to be *conjugate* in G.

   (b) (3 pts) Fix an element g of a group G. Show that the map 'conjugation by g' is an isomorphism from G to itself.

   (c) (4 pts) For any integer $n \geq 1$, let S_n denote the symmetric group on n elements. Show that any two transpositions in S_n are conjugate to each other. [If you are stuck, you may do the case $n = 3$ for partial credit.]

2. (a) (6 pts) For any integer n, consider the function $f_n \colon \mathbb{C}^\times \to \mathbb{C}^\times$ given by $f_n(z) = z^n$. Show that f_n is a homomorphism, and identify its kernel, its image, and the quotient group $\mathbb{C}^\times / \ker(f_n)$. Exhibit an isomorphism from $\ker(f_n)$ to a more familiar group (one whose 'name we already know').

   (b) (4 pts) For which integers n is the map $g_n \colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ given by $g_n(A) = A^n$ a homomorphism? (Prove it.)

3. (a) (4 pts) State Lagrange's Theorem. Prove that it implies that for any finite group G, and any element $g \in G$, $g^{|G|} = 1$.

   (b) (4 pts) Define what it means for a subgroup H of a group G to be *normal*. Which of the following are normal subgroups? (Don't just answer 'yes' or 'no': prove or disprove!)

      i. The subgroup $S_2 < S_3$ of permutations fixing the element $3 \in \{1, 2, 3\}$.

      ii. The subgroup of S_4 generated by the cycle $(1, 2, 3, 4)$.

      iii. The subgroup B of $\mathrm{GL}_2(\mathbb{C})$ consisting of upper-triangular matrices.