Math 3010 Pset 10

(1) Katz 10.31. a) Show $F_{n+1}F_{n-1}=F_n^2-(-1)^n$.

- We prove this by induction on n, noting it is true for $n=1$:

$$F_2 \cdot F_0 = F_1^2-(-1) \qquad \left(2 \cdot 1 = 1^2-(-1)\right).$$

Assume known for n. We then compute

$$F_{n+2}F_n - F_{n+1}^2 = (F_{n+1}+F_n)F_n - (F_n+F_{n-1})^2$$
$$= F_{n+1}F_n - 2F_nF_{n-1} - F_{n-1}^2$$
$$= F_n(F_{n+1}-F_{n-1}) - F_{n-1}(F_n+F_{n-1})$$
$$= F_n^2 - F_{n-1}F_{n+1} \underset{\text{by induction hypothesis}}{=} -(-1)^n = (-1)^{n+1}. \quad \blacksquare$$

b) Show $\lim_{n\to\infty} F_n/F_{n-1} = \frac{1+\sqrt{5}}{2}$

Pf: By the defining property, $F_{n+1}=F_n+F_{n-1}$, hence

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$ Take $\lim_{n\to\infty}$ of both sides to find

$$L = 1+\frac{1}{L} \quad \text{where} \quad L=\lim_{n\to\infty} F_n/F_{n-1}.$$

Then $L^2-L-1=0$, so $\boxed{L=\frac{1+\sqrt{5}}{2}}$

(To be precise, one must first show that L exists; this follows from part (a).
Namely, $\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}} = \frac{(-1)^{n+1}}{F_nF_{n-1}}$, so $\frac{F_{n+m}}{F_{n+m-1}} - \frac{F_n}{F_{n-1}} = \sum_{i=1}^{m} \frac{(-1)^{n+i}}{F_{n+i-1}F_{n+i-2}}$

The series $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{F_jF_{j-1}}$ converges by the alternating series test, so we win. Students do NOT need to include this)

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(2) Katz 10.34. Solve $\begin{cases} x+y=9 \\ xy+x-y=21 \end{cases}$. Substitute: $x(9-x)+x-(9-x)=21$

$\rightsquigarrow 9x-x^2+x-9+x=21$

$\rightsquigarrow x^2-11x+30=0 \Rightarrow (x-5)(x-6)=0 \Rightarrow x=5$ or 6.

$\Rightarrow \boxed{(x,y) = (5,4) \text{ or } (6,3)}$

③ Katz 10.41

The rectangular areas form a geometric series $\sum_{n\geq 1} \frac{1}{2^n} = 1$, and the trapezoidal areas form a geometric series $\sum_{n\geq 3} \frac{3}{2^n} = \frac{3}{4}$, for a total area (=distance) of $1+\frac{3}{4} = \boxed{\frac{7}{4}}$.

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④ Katz 10.42. $\boxed{\text{Pf 1}}$

$$1 \geq 1/2$$
$$\frac{1}{2} \geq 1/2$$
$$\frac{1}{3}+\frac{1}{4} \geq 1/2$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \geq 1/2$$
$$\vdots$$
$$\frac{1}{2^n+1}+\frac{1}{2^n+2}+\dots+\frac{1}{2^{n+1}} \geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$
$$\vdots$$

$\Rightarrow \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{1}{2}\cdot(n+1)$

$\Rightarrow \sum_{i=1}^{\infty} \frac{1}{i}$ diverges.

$\boxed{\text{Pf 2}}$ $$\sum_{i=1}^{N} \frac{1}{i} > \int_1^{N+1} \frac{1}{x}\,dx = \log(x)\Big|_1^{N+1} = \log(N+1)$$

Since $\log(x) \to \infty$ as $x \to \infty$, the integral, and thus the series, must diverge.
