Midterm 1 (/Pset 7) (Grade these as 3pts each)

① By a greedy algorithm, $\frac{9}{13} = \frac{1}{2} + \frac{5}{26} = \boxed{\frac{1}{2} + \frac{1}{6} + \frac{1}{39}}$

(Note: the Egyptian fraction decomposition is not unique: any rational number $x>0$ has infinitely many representations $x = \sum_{i=1}^{N} \frac{1}{n_i}$ with $n_i \in \mathbb{Z}$ and all n_i distinct. Having all n_i distinct is preferable, but still give credit for a little repetition, eg $\frac{1}{2}+\frac{1}{13}+\frac{1}{13}+\frac{1}{26}$. Other answers students gave: $\frac{2}{3}+\frac{1}{39}$, $\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{208}$, $\frac{1}{2}+\frac{1}{6}+\frac{1}{52}+\frac{1}{56}$ - - -)

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② $x^2+y^2=d^2 \Rightarrow \left(\frac{d}{y}\right)^2 - \left(\frac{x}{y}\right)^2 = 1$. Set $u=\frac{d}{y}$, $v=\frac{x}{y}$, so $u^2-v^2=1$.

We are given $u+v=13$, so $(u+v)(u-v)=1$ implies $u-v=\frac{1}{13}$

$\rightsquigarrow$ $2u = \frac{170}{13} \Rightarrow u = \frac{85}{13} \Rightarrow v = \frac{84}{13} \Rightarrow$ a solution is given by $\boxed{x=84, y=13, d=85}$

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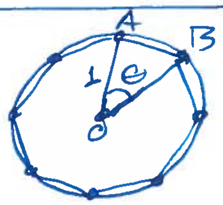
③ 

We have to compute the area of an octagon.

$\theta = 45^\circ \Rightarrow$ the area of triangle OAB is (radius $=1$)

$\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}\sin(\theta) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$8 \cdot \text{Area } \Delta OAB = 8 \cdot \frac{\sqrt{2}}{4} = \boxed{2\sqrt{2}}$ is then the area of the octagon.

[many students will solve this citing Liu Hui's doubling formula].

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$c_4 = \sqrt{2} \xRightarrow[\text{Liu Hui}]{} \text{area}(8\text{-gon}) = \frac{1}{2}\cdot 4 \cdot \sqrt{2} = 2\sqrt{2}$

$S_{2n} = \frac{1}{2} n r c_n$)

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④ Solve $\begin{cases} N \equiv 5 \mod 17 \\ N \equiv 8 \mod 19 \end{cases}$ To solve $17x+5 = 19y+8$, ie $17x - 19y = 3$, we first solve

$17x - 19y = \gcd(17,19) = 1$ via Euclid's algorithm, and find $x=9$, $y=8$.

Then $17(27) - 19(24) = 3$ (multiplying the previous solution by 3).

One N is then $17(27)+5 = 19(24)+8 = 464$, and all solutions are given by $N \equiv 464 \mod (17\cdot 19)$, ie

$$\boxed{N \equiv 141 \mod 323}$$