Math 3010 PSet 5

6 Points (3 each) **1**

(a)
$$1 - \tfrac{11}{19}\cdot 1 = \tfrac{8}{19}$$
$$\tfrac{11}{19} - \tfrac{8}{19}\cdot 1 = \tfrac{3}{19}$$
$$\tfrac{8}{19} - \tfrac{3}{19}\cdot 2 = \tfrac{2}{19}$$
$$\tfrac{3}{19} - \tfrac{2}{19}\cdot 1 = \tfrac{1}{19}$$
$$\tfrac{2}{19} - \tfrac{1}{19}\cdot 2 = 0$$

(I have abbreviated the successive subtractions as "division." It would also be ok to write, eg, $\frac{8}{19} - \frac{3}{19} = \frac{5}{19}$, $\frac{5}{19} - \frac{3}{19} = \frac{2}{19}$ in line 3).

$$\tfrac{3}{5} - \tfrac{1}{3}\cdot 1 = \tfrac{4}{15}$$
$$\tfrac{1}{3} - \tfrac{4}{15}\cdot 1 = \tfrac{1}{15}$$
$$\tfrac{4}{15} - \tfrac{1}{15}\cdot 4 = 0$$

In both cases, the fact that the algorithm terminates at zero attests to commensurability

(b)
$$\sqrt{3} - 1\cdot 1 = \sqrt{3} - 1$$
$$1 - (\sqrt{3}-1)\cdot 1 = 2 - \sqrt{3}$$
$$(\sqrt{3}-1) - (2-\sqrt{3})\cdot 2 = -5 + 3\sqrt{3}$$
$$(2-\sqrt{3}) - (-5+3\sqrt{3})\cdot 1 = 7 - 4\sqrt{3}$$
$$(-5+3\sqrt{3}) - (7-4\sqrt{3})\cdot 2 = -19 + 11\sqrt{3}$$
$$(7-4\sqrt{3}) - (-19+11\sqrt{3})\cdot 1 = 26 - 15\sqrt{3}$$
$$(-19+11\sqrt{3}) - (26-15\sqrt{3})\cdot 2 = -71 + 41\sqrt{3}$$
$$\vdots$$

They must say this for full credit

Irrationality of $\sqrt{3}$ is equivalent to the fact that the algorithm does not terminate. In fact, the quotients (the column circled in red) go 1, 1, 2, 1, 2, 1, 2, --- alternating 1 & 2 ad infinitum.

Not needed for credit

What this really says — and the way you can prove it — is that

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}}}$$

, which can be verified by showing $\sqrt{3}$ is a solution to

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + (x-1)}}$$

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(2) Aristotle summarizes an argument by contradiction that $\sqrt{2}$ = the diagonal of a square of side 1 is irrational (ie, the diagonal and the side are <u>incommensurate</u>). The argument assumes $\sqrt{2}$ is rational, and indicates that this will force some integer to be <u>both</u> even and odd ("odd numbers come out equal to evens"). Namely, in modern symbolism, if $\sqrt{2} = p/q$ with p and q relatively prime integers, then $2q^2 = p^2$. By assumption, at most one of p, q is even; the equation forces p to be even, since 2 divides p^2. Set $p = 2p_0$ for $p_0 \in \mathbb{Z}$. Then $q^2 = 2p_0^2$, forcing q to be even, whereas our assumption has already implied q is odd. A contradiction results, since q cannot be both even and odd.