Math 3010 Homework Due 19 February, 2016
Incommensurability

(1) One way of understanding the "commensurability" of any two rational numbers is a repeated subtraction technique much in the spirit of the Euclidean algorithm: given (positive) numbers $a > b$, successively subtract off multiples of b from a until the result–call it a_2–is smaller than b; then subtract off multiples of a_2 from b until the result–call it b_2–is smaller than a_2. Then repeat this procedure: if a and b are both rational, at some point, the procedure will terminate (leave a remainder of zero). For example, let $a = 1$ and $b = 13/17$. Then

$$a - b = 4/17 = a_2 < b;$$
$$b - a_2 = 9/17; 9/17 - a_2 = 5/17; 5/17 - a_2 = 1/17 = b_2 < a_2;$$
$$a_2 - b_2 = 3/17; 3/17 - b_2 = 2/17; 2/17 - b_2 = 1/17; 1/17 - b_2 = 0.$$

Of course, the successive subtractions can be abbreviated as a "division," eg $b - 3a_2 = 1/17 = b_2 < a_2$. Here is your question:

(a) Carry out this iterated subtraction (which you may abbreviate as division) to show the "commensurability" of the pairs $(a, b) = (1, \frac{11}{19})$ and $(a, b) = (3/5, 1/3)$.

(b) What happens when you carry out this iterated division for the pair $(a, b) = (\sqrt{3}, 1)$? Look for a pattern! (Hint: at each stage–that is, each line in the above sample calculation–keep track of how many subtractions you have to carry out before moving on to the next line.)

(2) This question addresses how the Pythagoreans most likely proved irrationality of $\sqrt{2}$. Our best guess is the argument given by Aristotle in Book I, Chapter 23 of the *Prior Analytics*: this is one of his foundational texts on logic, and in I.23 he discusses the use of syllogism for logical demonstration, and in particular the method of proof known as *reductio ad absurdum* (or *reductio ad impossible*), which proceeds by assuming the contrary of the desired conclusion, and then deriving a contradiction with some known true statement. As prime example of *reductio ad absurdum*, Aristotle adduces a proof of the incommensurability of the diagonal of a square with its side. Here is your question:

- Read section 23 of Book I of the *Prior Analytics*, which can be found here:
  `https://ebooks.adelaide.edu.au/a/aristotle/a8pra/book1.html`
  This is a very short passage, but do not expect it to be easy: you may have to reread it several times. Read especially carefully the last paragraph of this section, and explain the "proof" of irrationality of $\sqrt{2}$ that it is describing; by "explain," I mean give a careful argument in modern language that your classmates would find entirely convincing. You also must explain how your argument corresponds to Aristotle's: reference specifically the language of the *Prior Analytics*.