PSet 8

$\mathcal{P}: \{y = x^2\}$

(1) The focus F lies on the axis of $\mathcal{P}$ (the line $x=0$), and the directrix is perpendicular to the axis, so for some $p \in \mathbb{R}$, $F = (0,p)$ and the directrix is the line $y=-p$ (applying the focus-directrix defn to the point $(0,0)$ of $\mathcal{P}$).

To compute p, apply the focus-directrix defn to the point $(1,1)$ of $\mathcal{P}$ (or any other point) to find $1+p = \sqrt{1^2+(1-p)^2}$, hence $p^2+2p+1 = p^2-2p+2$.

We conclude that $p = 1/4$, so
$$\boxed{\begin{array}{l} F = (0, 1/4) \\ \text{directrix} = \text{line } y = -1/4 \end{array}}$$

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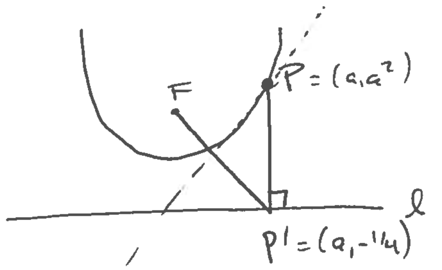
(2)



Let $P = (a,a^2)$ be any point of $\mathcal{P}$. The equation of the tangent line to $\mathcal{P}$ at P is
$$y - a^2 = 2a(x-a)$$

The equation of the line $\overleftrightarrow{FP'}$ is $y - \frac{1}{4} = -\frac{1}{2a}\cdot x$ (recall $F = (0, 1/4)$)

• Since these slopes ($2a$ and $-\frac{1}{2a}$) are negative reciprocals, the two lines are perpendicular; it remains to check the tangent bisects $\overline{FP'}$.

The two lines intersect when $2a(x-a) + a^2 = -\frac{1}{2a}x + 1/4$

$\iff 2ax - a^2 = -\frac{1}{2a}x + 1/4 \iff x\left(2a + \frac{1}{2a}\right) = a^2 + 1/4 \iff x\left(\frac{4a^2+1}{2a}\right) = \frac{4a^2+1}{4}$

$\iff x = \frac{a}{2}$ (and then $y=0$). Since $(\frac{a}{2}, 0)$ is the midpoint of $\overline{FP'}$, we win.

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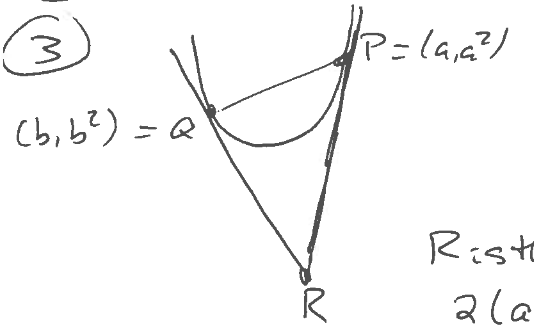
(3)



The equation of the tangents at $P = (a,a^2)$ and $Q = (b,b^2)$ are
$$y - a^2 = 2a(x-a) \qquad \text{ie} \qquad y = 2ax - a^2$$
$$y - b^2 = 2b(x-b), \qquad\qquad y = 2bx - b^2$$

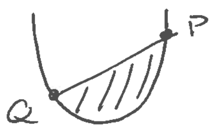
R is the intersection, where $2ax - a^2 = 2bx - b^2$, ie (note $a \neq b$)
$2(\cancel{a-b})x = (\cancel{a-b})(a+b)$, ie $\boxed{x = \frac{a+b}{2}}$, and

$$\boxed{R = \left(\frac{a+b}{2}, ab\right)}$$

($y = 2a\left(\frac{a+b}{2}\right) - a^2$
$= ab$

(PSet 8, continued)

(4) Compute area of the parabolic segment $\overline{PQ}$. $(P=(a,a^2),\ Q=(b,b^2))$



The line $\overline{PQ}$ has equation $y-a^2=\dfrac{a^2-b^2}{a-b}(x-a)$ (we may assume $b<a$)

$$= (a+b)(x-a)$$

ie $y=(a+b)x-ab$. The desired area is then

$$\int_b^a (a+b)x-ab-x^2\,dx = \left[(a+b)\frac{x^2}{2}-abx-\frac{x^3}{3}\right]_b^a$$

$$= (a+b)\frac{a^2}{2}-a^2b-\frac{a^3}{3}-\left[\frac{(a+b)b^2}{2}-ab^2-\frac{b^3}{3}\right]$$

$$\boxed{= \frac{a^3}{6}-\frac{a^2b}{2}+\frac{ab^2}{2}-\frac{b^3}{6}}\ \left(=\boxed{\frac{1}{6}(a-b)^3}\right)$$

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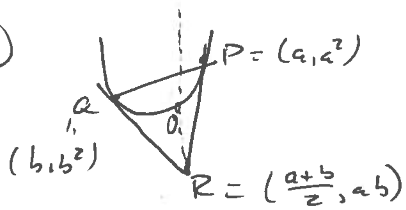
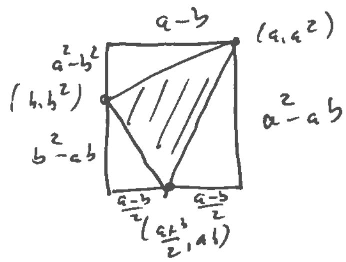
(5)





Triangle area is

$$(a^2-ab)(a-b)-\frac{1}{2}\Big[(a-b)(a^2-b^2)+(b^2-ab)\frac{(a-b)}{2}+(a^2-ab)\frac{(a-b)}{2}\Big]$$

$$=(a-b)^2\left[a-\frac{1}{2}\left(a+b-\frac{b}{2}+\frac{a}{2}\right)\right] = \boxed{\frac{(a-b)^3}{4}}$$

O is defined by taking RO parallel to the axis of P

This result implies A's thm, which said $\begin{pmatrix}\text{area}\\ \text{parabolic}\\ \text{segment}\end{pmatrix} = \frac{4}{3}\Delta PQO$

$$= \frac{2}{3}\Delta PQR$$

$$\left(\text{ie}\quad \frac{1}{6}(a-b)^3=\frac{2}{3}\cdot\frac{1}{4}(a-b)^3\right)$$

doesn't have to be included in the solution.