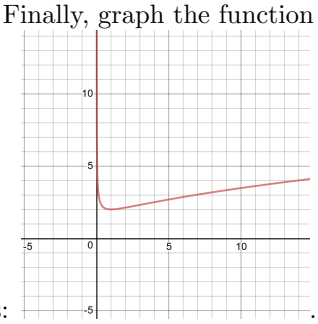
Math 1210-021, Midterm 2 Review Solutions

In class we went over problems 3, 4, 5, so I won't redo those here. Also note: when a problem requires determining intervals on which a function is increasing or decreasing, we won't be picky about the distinction between open/closed intervals.

(1) (a) $f'(x) = 3x^2 - 6x - 15 = 3(x^2 - 2x - 5)$. The critical points on $[0, 4]$ are thus $0, 4$ (the endpoints) and $1 + \sqrt{6}$ (stationary points found by the quadratic formula; note that $1 - \sqrt{6}$, also a zero of $f'(x)$, is not in the interval $[0, 4]$). Checking the sign of the derivative, we see that f attains a local and global minimum (on $[0, 4]$) at $1 + \sqrt{6}$,[1] with minimum value $f(1 + \sqrt{6})$, and since $f(0) = 1$ and $f(4) = -43$, the global maximum is at $f(0) = 1$.

(b) $f(x) = -2\cos(x) - x^2$. $f'(x) = 2\sin(x) - 2x = 2(\sin(x) - x)$. This is zero only for $x = 0$: to see this, note that $\sin(x) - x$ is decreasing for $x > 0$ and increasing for $x < 0$ (by looking at the derivative). The critical points on $[-\frac{\pi}{2}, \pi]$ are thus $-\frac{\pi}{2}, 0, \pi$, and to find the extreme values we just evaluate $f(x)$ at each critical point: $f(\pi) = 2 - \pi^2$, $f(0) = -2$, $f(-\frac{\pi}{2}) = -\frac{\pi^2}{4}$, so the maximum is $f(0) = -2$, and the minimum is $f(\pi) = 2 - \pi^2$.

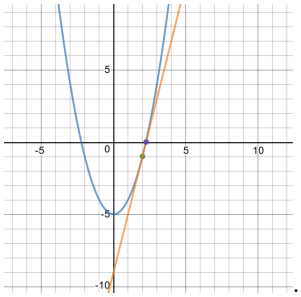
(2) $f(x) = \frac{1+x}{\sqrt{x}} = x^{-\frac{1}{2}} + x^{\frac{1}{2}}$. Domain: $x > 0$. Asymptotes: there is a vertical asymptote as $x \to 0^+$. Critical points: compute $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{3}{2}}(-1 + x)$. The only critical point is therefore $x = 1$, and (by looking at the sign of f') it is a local minimum. Inflection points: compute $f''(x) = \frac{3}{4}x^{-\frac{5}{2}} - \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{4}x^{-\frac{5}{2}}(3 - x)$, so the concavity changes from up to down at $x = 3$, the sole inflection point. Finally, graph the function by combining the above with the fact that there are no x or y-intercepts: 

. (Note that the concavity change appears rather subtle in the picture–nevertheless, it is there.)

(3) Done in class.

(4) Done in class.

(5) Done in class.

(6) Newton's method recursion: $x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$. Starting with $x_0 = 2$, we get $x_1 = 2 - \frac{-1}{4} = \frac{9}{4}$, $x_2 = \frac{9}{4} - \frac{81/16 - 5}{9/2} = \frac{9}{4} - \frac{1}{72} = \frac{161}{72}$. Here's the picture of the first step: the green point is $(x_0, f(x_0)$, and the purple point is the x-intercept of the corresponding tangent line. 

. (What your picture should indicate is that you are taking tangent lines and finding x-intercepts, then repeating.)

(7) (a) $\frac{3}{10}x^{\frac{10}{3}} - \frac{3}{2}x^6 + 2x + C$

[1] In principle you should evaluate this, but here it is a bit of a mess, so I won't bother–on the exam the calculations won't be onerous.

(b) The general antiderivative is $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$. the particular antiderivative with $F(\sqrt{15})=1$ is $\frac{1}{3}(x^2+1)^{\frac{3}{2}}-\frac{61}{3}$ (just plug in and solve for C).

(c) $\frac{(x\tan(x))^4}{4}+C$

(8) Use separation of variables: $ydy=\sqrt[3]{x}dx$ integrates to $y^2/2=\frac{3}{4}x^{\frac{4}{3}}+C$, so possible solutions are $y=\sqrt{\frac{3}{2}x^{\frac{4}{3}}+C}$ (I've relabeled $2C$ as C just for notational simplicity), or the negative of this function. A solution–it is not unique! but we are just asked to find one solution–with $y(8)=0$ is $C=-24$, i.e. $y=\sqrt{\frac{3}{2}x^{\frac{4}{3}}-24}$. Since the domain of the square-root function is nonnegative reals, the domain of this y is $x\geq 8$ and $x\leq -8$ (solving to make the quantity inside the square-root nonnegative). This function y satisfies the given differential equation on $x>8$ and $x<-8$, but not at $x=\pm 8$, where it is not differentiable. Finally, while not part of the question, let us observe that the situation in this problem is actually rather subtle. A continuous solution to this equation is determined on the interval $[8,\infty)$ by whether we take the solution y just given or its negative, $-y$. But all sorts of things can be taken for y when x is less than 8. In fact, we could take any value of C in the above equation to describe the function on $x<8$ (restricting, given our choice of C, to those x for which the resulting function is defined). Eg, we could take $y=\sqrt{\frac{3}{2}x^{\frac{4}{3}}}$ for $x<8$ and $y=\sqrt{\frac{3}{2}x^{\frac{4}{3}}-24}$ for $x\geq 8$. the resulting function is not continuous at $x=8$ and not differentiable at $x=0$, but everywhere else it is differentiable and satisfies the given equations. The point of this problem was not to get into these subtleties, but do take it as a cautionary tale.

(9) The definite integral computes the area of the region between the x-axis and the graph $y=x^2$, in the strip from $x=1$ to $x=3$. Here is the calculation using right endpoint approximations:

$$R_n=\sum_{i=1}^{n}\frac{2}{n}(1+\frac{2i}{n})^2=\frac{2}{n}\sum_{i=1}^{n}(1+\frac{4i}{n}+\frac{4i^2}{n^2})=\frac{2}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2.$$

Now using the summation formulas, we find

$$R_n=\frac{2}{n}\cdot n+\frac{8}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}.$$

Then

$$\lim_{n\to\infty}R_n=2+4+\frac{16}{6}=\frac{26}{3}.$$