Math 1210-021, Midterm 2 Review Solutions

In class we went over problems 3, 4, 5, so I won’t redo those here. Also note: when a problem requires determining intervals on which a function is increasing or decreasing, we won’t be picky about the distinction between open/closed intervals.

(1) (a) $f'(x) = 3x^2 - 6x - 15 = 3(x^2 - 2x - 5)$. The critical points on $[0, 4]$ are thus 0, 4 (the endpoints) and $1 + \sqrt{6}$ (stationary points found by the quadratic formula; note that $1 - \sqrt{6}$, also a zero of $f'(x)$, is not in the interval $[0, 4]$). Checking the sign of the derivative, we see that $f$ attains a local and global minimum (on $[0, 4]$) at $1 + \sqrt{6}$, with minimum value $f(1 + \sqrt{6})$, and since $f(0) = 1$ and $f(4) = -43$, the global maximum is at $f(0) = 1$.

(b) $f(x) = -2\cos(x) - x^2$. $f'(x) = 2\sin(x) - 2x = 2(\sin(x) - x)$. This is zero only for $x = 0$: to see this, note that $\sin(x) - x$ is decreasing for $x > 0$ and increasing for $x < 0$ (by looking at the derivative). The critical points on $[-\frac{\pi}{2}, \pi]$ are thus $-\frac{\pi}{2}, 0, \pi$, and to find the extreme values we just evaluate $f(x)$ at each critical point: $f(\pi) = 2 - \frac{\pi}{2}$, $f(0) = -2$, $f(-\frac{\pi}{2}) = -\frac{\pi^2}{4}$, so the maximum is $f(0) = -2$, and the minimum is $f(\pi) = 2 - \frac{\pi}{2}$.

(2) $f(x) = 1 + x\sqrt{x} = x^{\frac{3}{2}} + x^{\frac{1}{2}}$. Domain: $x > 0$. Asymptotes: there is a vertical asymptote as $x \to 0^+$. Critical points: compute $f'(x) = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{3}{2}}(-1 + x)$. The only critical point is therefore $x = 1$, and (by looking at the sign of $f'$) it is a local minimum. Inflection points: compute $f''(x) = \frac{3}{4}x^{-\frac{5}{2}} - \frac{1}{4}x^{-\frac{1}{2}} = \frac{1}{4}x^{-\frac{3}{2}}(3 - x)$, so the concavity changes from up to down at $x = 3$, the sole inflection point. Finally, graph the function by combining the above with the fact that there are no $x$ or $y$-intercepts: $f(0) = 1$. (Note that the concavity change appears rather subtle in the picture—nevertheless, it is there.)

(3) Done in class.
(4) Done in class.
(5) Done in class.

(6) Newton’s method recursion: $x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$. Starting with $x_0 = 2$, we get $x_1 = 2 - \frac{1}{4} = \frac{9}{4}$, $x_2 = \frac{9}{4} - \frac{81/16 - 5}{9/4} = \frac{9}{4} - \frac{1}{4} = \frac{10}{2}$. Here’s the picture of the first step: the green point is $(x_0, f(x_0))$, and the purple point is the $x$-intercept of the corresponding tangent line.

(What your picture should indicate is that you are taking tangent lines and finding $x$-intercepts, then repeating.)

(7) (a) $\frac{3}{16}x^{\frac{4}{3}} - \frac{3}{2}x^2 + 2x + C$
(b) The general antiderivative is \( \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + C \). The particular antiderivative with \( F(\sqrt{15}) = 1 \) is \\
\( \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} - \frac{61}{3} \) (just plug in and solve for \( C \)).

(c) \( \frac{1}{4} \tan(x)^4 + C \)

(8) Use separation of variables: \( ydy = \sqrt{x}dx \) integrates to \( y^2/2 = \frac{3}{4}x^{\frac{4}{3}} + C \), so possible solutions are \\
y = \sqrt[3]{\frac{3}{2}x^{\frac{4}{3}} + C} (I’ve relabeled 2C as C just for notational simplicity), or the negative of this function. A solution—it is not unique! but we are just asked to find one solution—with \( y(8) = 0 \) is \( C = -24 \), i.e. \( y = \sqrt[3]{\frac{3}{2}x^{\frac{4}{3}} - 24} \). Since the domain of the square-root function is nonnegative reals, the domain of this \( y \) is \( x \geq 8 \) and \( x \leq -8 \) (solving to make the quantity inside the square-root nonnegative). This function \( y \) satisfies the given differential equation on \( x > 8 \) and \( x < -8 \), but not at \( x = \pm 8 \), where it is not differentiable. Finally, while not part of the question, let us observe that the situation in this problem is actually rather subtle. A continuous solution to this equation is determined on the interval \([8, \infty)\) by whether we take the solution \( y \) just given or its negative, \( -y \). But all sorts of things can be taken for \( y \) when \( x \) is less than 8. In fact, we could take any value of \( C \) in the above equation to describe the function on \( x < 8 \) (restricting, given our choice of \( C \), to those \( x \) for which the resulting function is defined). Eg, we could take \( y = \sqrt[3]{\frac{3}{2}x^{\frac{4}{3}}} \) for \( x < 8 \) and \( y = \sqrt[3]{\frac{3}{2}x^{\frac{4}{3}} - 24} \) for \( x \geq 8 \). The resulting function is not continuous at \( x = 8 \) and not differentiable at \( x = 0 \), but everywhere else it is differentiable and satisfies the given equations. The point of this problem was not to get into these subtleties, but do take it as a cautionary tale.

(9) The definite integral computes the area of the region between the \( x \)-axis and the graph \( y = x^2 \), in the strip from \( x = 1 \) to \( x = 3 \). Here is the calculation using right endpoint approximations:

\[
R_n = \sum_{i=1}^{n} \frac{2}{n} \left(1 + \frac{2i}{n}\right)^2 = \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) = \frac{2}{n} \sum_{i=1}^{n} 1 + \frac{8}{n^2} \sum_{i=1}^{n} i + \frac{8}{n^3} \sum_{i=1}^{n} i^2.
\]

Now using the summation formulas, we find

\[
R_n = \frac{2}{n} \cdot n + \frac{8}{n^2} \frac{n(n+1)}{2} + \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}.
\]

Then

\[
\lim_{n \to \infty} R_n = 2 + 4 + \frac{16}{6} = \frac{26}{3}.
\]