Math 1210-021, Midterm 1 Solutions

(1) (a) $\lim_{x\to 0} \frac{\tan(\frac{x}{3})}{2x} = \lim_{x\to 0} \frac{1}{6}\frac{\sin(\frac{x}{3})}{\frac{x}{3}\cos(\frac{x}{3})} = \boxed{\frac{1}{6}}$.

(b) $\lim_{x\to 0^-} \cos(1/x)$ $\boxed{\text{DNE}}$ (oscillates wildly).

(c) $\lim_{x\to 1} x\lfloor x\rfloor$ $\boxed{\text{DNE}}$ (the left-hand limit is 0, while the right-hand limit is 1).

(d) $\lim_{x\to -\infty} \frac{3x^3-7x+12}{5x^3+9x^2} = \boxed{\frac{3}{5}}$.

(e) $\lim_{x\to\infty} \frac{\tan(x)}{x}$ $\boxed{\text{DNE}}$ (on every interval $(\frac{\pi}{2}+n\pi, \frac{\pi}{2}+(n+1)\pi)$, the function hits all real numbers)

(f) $\lim_{x\to 1} \frac{1}{(x-1)^3}$ $\boxed{\text{DNE}}$ ($-\infty$ from the left and $+\infty$ from the right)

(g) $\lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\cos(y+\frac{\pi}{2})}{y} = \lim_{y\to 0} \frac{-\sin(y)}{y} = \boxed{-1}$.

(2) (a) $f'(a) = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$.

(b) $f'(a) = \lim_{x\to a} \frac{1/x-1/a}{x-a} = \lim_{x\to a} \frac{-1}{ax} = \boxed{-\frac{1}{a^2}}$.

(3) (a) Since $f'(a) = 3a^2-3$, the equation of the tangent line at $x=a$ is $\boxed{y-(a^3-3a-1) = (3a^2-3)(x-a)}$.

(b) The tangent line is horizontal when $3a^2-3=0$, i.e. when $\boxed{a=1 \text{ or } a=-1}$.

(c) Intermediate Value Theorem: Let f be a continuous function on a closed interval $[a,b]$, and let α be any real number strictly between $f(a)$ and $f(b)$. Then there exists a $c\in(a,b)$ such that $f(c)=\alpha$.
To show that $f(x)=0$ has three distinct real solutions, note that f is everywhere continuous, so the IVT applies on any closed interval. Since $f(-2)<0$, $f(-1)>0$, $f(1)<0$, and $f(2)>0$, the IVT shows that $f(c)=0$ for some $c\in(-2,-1)$, some $c\in(-1,1)$, and some $c\in(1,2)$.

(4) (a) $\frac{dy}{dx} = \frac{(x^2+1)(x\cos(x)+\sin(x)) - x\sin(x)2x}{(x^2+1)^2} = \boxed{\frac{(x^3+x)\cos(x)+(-x^2+1)\sin(x)}{(x^2+1)^2}}$.

(b) $g'(t) = \boxed{-t^{-\frac{1}{4}}\csc^2(t) - \frac{1}{4}t^{-\frac{5}{4}}\cot(t)}$.

(c) The two points in question are $(\frac{3}{5},\frac{1}{5})$ and $(-\frac{3}{5},\frac{1}{5})$. The general formula for the derivative is $2x+32y\frac{dy}{dx}=0$, i.e. $\frac{dy}{dx} = -\frac{x}{16y}$. Subbing in, we find that the slopes of the tangent lines at $(\frac{3}{5},\frac{1}{5})$ and $(-\frac{3}{5},\frac{1}{5})$, respectively, are $\boxed{-\frac{3}{16} \text{ and } \frac{3}{16}}$.

(5) (a) Recall that $f'(x)=\sec^2(x)$. Starting from $\frac{\pi}{6}$, the approximation is $f(\frac{\pi}{5}) \approx f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(\frac{\pi}{5}-\frac{\pi}{6}) = \frac{1}{\sqrt{3}}+\frac{4}{3}\frac{\pi}{30} = \boxed{\frac{1}{\sqrt{3}}+\frac{2\pi}{45}}$. Starting from $\frac{\pi}{4}$, the approximation is $f(\frac{\pi}{5}) \approx f(\frac{\pi}{4})+f'(\frac{\pi}{4})(\frac{\pi}{5}-\frac{\pi}{4}) = \boxed{1-\frac{\pi}{10}}$.

(b) By inspection of the graph of the tangent function, for any $x\in(0,\frac{\pi}{2})$, the tangent line to f at x lies below the graph of $y=f(x)$ (as long as one stays in the interval $(0,\frac{\pi}{2})$. Since we are approximating $f(\frac{\pi}{5})$ by taking the values at $\frac{\pi}{5}$ of the tangent lines to $y=f(x)$ at $\frac{\pi}{4}$ and $\frac{\pi}{6}$, in both cases we get $\boxed{\text{underestimates}}$.

(6) Let $a(t)$ be the distance traveled by person A, let $b(t)$ be the distance traveled by person B, and let $d(t)$ be the distance between them. By the law of cosines,

$$d(t)^2 = a(t)^2 + b(t)^2 - 2a(t)b(t)\cos(\frac{3\pi}{4}) = a(t)^2 + b(t)^2 + \sqrt{2}a(t)b(t).$$

(If you forget the law of cosines, just drop the perpendicular from person B to the axis where person A is traveling, and apply the Pythagorean theorem to that right triangle.) Differentiating, we get

$$2d(t)d'(t) = 2a(t)a'(t) + 2b(t)b'(t) + \sqrt{2}(a(t)b'(t) + a'(t)b(t)).$$

When $a = 1$ and $b = \sqrt{2}$, the first formula shows that $d = \sqrt{5}$, so when $a' = 1$ and $b' = 3\sqrt{2}$, we can plug everything in to the derivative formula to get (at this moment t_0 of interest, and where we've already divided by 2)

$$\sqrt{5}d'(t_0) = 1 + \sqrt{2} \cdot 3\sqrt{2} + \frac{\sqrt{2}}{2}(3\sqrt{2} + \sqrt{2}),$$

i.e. $\boxed{d'(t_0) = \frac{11}{\sqrt{5}} \text{ leagues/hour}}$.

(7) In order from top to bottom, the correct labeling is $\boxed{\text{D, B, A, E, C}}$.