MATH 1210 Lab 4 (9/21/2017)

Student #:_____ Instructor & Section:_____

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated. No calculators or electronics of any kind are allowed, unless stated otherwise.

1. The chart below gives the distance between a hurricane and Windy City on the morning of September 21st.

Time (a.m.)	Distance (miles)
8 : 45	117
9 : 10	109
9 : 20	104
9 : 50	90
10 : 15	78
10 : 30	72

(a) What is the average velocity of the hurricane between 9 : 10 and 9 : 50? Give your answer in miles per hour.

(b) Find the average velocity of the hurricane between 9 : 10 and 9 : 20.

(c) Estimate, to the best of your ability, the velocity of the hurricane at 10 : 30, and use this to predict the time that it will strike Windy City.

Solution:

(a) $\frac{90-109\text{miles}}{40\text{min}} = \frac{-19\text{miles}}{2/3\text{hr}} = -28.5\text{mi/hr}$

(b) $\frac{104-109\text{miles}}{10\text{min}} = \frac{-5\text{miles}}{1/6\text{hr}} = -30\text{mi/hr}$

(c) Estimate the velocity at 10 : 30 by the average velocity between 10 : 15 and 10 : 30, which is

$$\frac{72-78\text{miles}}{15\text{min}} = \frac{-6\text{miles}}{1/4\text{hr}} = -24\text{mi/hr}$$

If it keeps traveling at 24 miles per hour, then it will take 3 hours to go 72 miles, and so you would estimate it to strike Windy City at 1:30p.m.

2. There are different ways of computing the derivative of a function $f(x)$ as a limit. Below are four different limits, each of which is equal to $f'(x)$. The first one is the one we use most often. The second is discussed in your text, but the third and fourth are not.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \tag{2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \tag{3}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \tag{4}$$

(a) For $f(x) = x^2$, find $f'(x)$ using

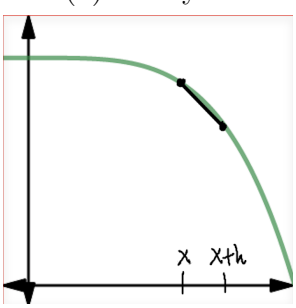
(i) limit (1):

(ii) limit (2):

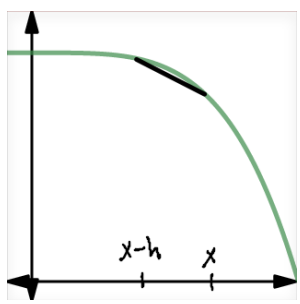
(iii) limit (3):

(iv) limit (4):

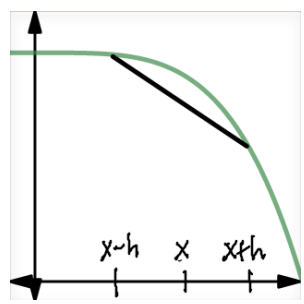
(b) Graphically, $\frac{f(x+h)-f(x)}{h}$ is the slope of the secant line connecting $(x, f(x))$ to $(x+h, f(x+h))$. See the first figure below. In the limit as $h \to 0$, this slope should approach the slope of the tangent line at x. Sketch the analogous pictures of the secant lines related to the expressions on the right-hand side of limits (3) and (4). For your sketch, assume $h > 0$.



$\frac{f(x+h)-f(x)}{h}$



$\frac{f(x)-f(x-h)}{h}$



$\frac{f(x+h)-f(x-h)}{2h}$

(c) Show that you can obtain (3) from limit (1) by replacing h with $-h$.

(d) Show that you can obtain (4) by adding (1) and (3) and then dividing by two.

Solution:

(a) (i)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) = 2x. \end{aligned}$$

(ii)

$$\begin{aligned} f'(x) &= \lim_{y\to x} \frac{y^2 - x^2}{y - x} \\ &= \lim_{y\to x} \frac{(y+x)(y-x)}{y - x} \\ &= \lim_{y\to x} (y + x) = 2x. \end{aligned}$$

(iii)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x^2 - (x-h)^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 - (x^2 - 2xh + h^2)}{h} \\ &= \lim_{h\to 0} \frac{2xh - h^2}{h} \\ &= \lim_{h\to 0} (2x - h) = 2x. \end{aligned}$$

(iv)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - (x-h)^2}{2h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - (x^2 - 2xh + h^2)}{2h} \\ &= \lim_{h\to 0} \frac{4xh}{2h} \\ &= \lim_{h\to 0} (2x) = 2x. \end{aligned}$$

(b) Drawn above.

(c) . If you set $k = -h$, then $h = -k$ and as $h \to 0$, $k \to 0$ as well.

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{k\to 0} \frac{f(x-k)-f(x)}{-k} \\ &= \lim_{k\to 0} \frac{f(x)-f(x-k)}{k}\end{aligned}$$

which is just (3) with k in place of h.

(d)

$$\begin{aligned}f'(x) = \frac{1}{2}\left(f'(x)+f'(x)\right) &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}\right) \\ &= \frac{1}{2}\left(\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right)\right) \\ &= \frac{1}{2}\left(\lim_{h\to 0}\left(\frac{f(x+h)-f(x-h)}{h}\right)\right) \\ &\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}\end{aligned}$$

3. Evaluate the following limits.

(a) $\lim_{t\to 0} \frac{\tan(2t)}{t}$

Answer:_____

(b) $\lim_{x\to \frac{\pi}{4}} \frac{\cos x - \sin x}{1-\tan^2 x}$

Answer:_____

(c) $\lim_{t\to 0} \frac{\csc^2 t - \cot^2 t}{\cos\left(\frac{t}{5}\right)}$

Answer:_____

(d) $\lim_{h\to 0} \frac{(x+h)^{1/4}-x^{1/4}}{h}$

Answer:_____

Solution:

(a) $\lim_{t\to 0} \frac{\tan(2t)}{t} = \lim_{t\to 0} \frac{\sin(2t)}{tcos(2t)} = \lim_{t\to 0} \frac{2sin(2t)}{2tcos(2t)} = 2\lim_{t\to 0}(\frac{sin(2t)}{2t})(\frac{1}{cos(2t)}) = 2$

(b) $lim_{x\to\pi/4} \frac{cosx-sinx}{1-\frac{sin^2x}{cos^2x}} = \lim_{x\to\pi/4} \frac{cosx-sinx}{\frac{cos^2x-sin^2x}{cos^2x}} = \lim_{x\to\pi/4} \frac{(cosx-sinx)cos^2x}{(cosx-sinx)(cosx+sinx)}$

$= \lim_{x\to\pi/4} \frac{cos^2x}{cosx+sinx} = \frac{1}{2\sqrt{2}}$

(c) $\lim_{t\to 0} \frac{1}{cos(\frac{t}{5})} = 1$, because $csc^2t - cot^2t = 1$

(d) $\lim_{h\to 0} \frac{(x+h)^{1/4}-x^{1/4}}{h}\left(\frac{(x+h)^{1/4}+x^{1/4}}{(x+h)^{1/4}+x^{1/4}}\right) = \lim_{h\to 0} \frac{(x+h)^{1/2}-x^{1/2}}{h[(x+h)^{1/4}+x^{1/4}]}$

$= \lim_{h\to 0} \frac{(x+h)^{1/2}-x^{1/2}}{h[(x+h)^{1/4}+x^{1/4}]}\left(\frac{(x+h)^{1/2}+x^{1/2}}{(x+h)^{1/2}+x^{1/2}}\right) = \lim_{h\to 0} \frac{x+h-x}{h[(x+h)^{1/4}+x^{1/4}][(x+h)^{1/2}+x^{1/2}]}$

$= \lim_{h\to 0} \frac{1}{[(x+h)^{1/4}+x^{1/4}][(x+h)^{1/2}+x^{1/2}]} = \frac{1}{4x^{3/4}}$