1. Consider the function

\[ f(x) = \frac{x^2 + 1}{x^3 - 1} \]

Compute the following. Answers may be numbers, \(+\infty\), or \(-\infty\).

(a) \( \lim_{x \to \infty} f(x) = \)

Answer: 

(b) \( \lim_{x \to -\infty} f(x) = \)

Answer: 

(c) \( \lim_{x \to 1^-} f(x) = \)

Answer: 

(d) \( \lim_{x \to 1^+} f(x) = \)

Answer: 

(e) Use the information you found above to determine which graph below is the graph of \( y = f(x) \). Circle your answer.
Solution:

(a) 0
(b) 0
(c) $-\infty$
(d) $\infty$
(e) The first one.
2. In this problem, we’ll use limits and the idea that regular polygons give better and better approximations to a circle as the number of sides increase.

(a) Suppose an \( n \)-sided regular polygon is inscribed in a circle of radius \( r \). The polygon is composed of \( n \) triangles (see Figure 1 below). If we zoom in on one of the triangles (Figure 2 below), what is the measure (in radians) of the angle labeled \( \theta \)? Keep \( n \) arbitrary.

(b) Find the height of the triangle, labeled \( h \) in Figure 2, and use this to find the area of \( \Delta PQR \).

(c) Find the area of the inscribed polygon. Remember, it is made of \( n \) triangles.

(d) Take the limit as \( n \to \infty \) of your answer in (c) and show that you get \( \pi r^2 \), the area of the circle of radius \( r \). \textbf{Hint:} It may be helpful to make a change of variables \( \theta = \frac{2\pi}{n} \) and let \( \theta \to 0 \) instead.
(e) Now compute the length of the line segment $PR$ as a function of $n$. **Hint:** You’ll have to remember the Law of Cosines.

(f) What is the perimeter of the inscribed polygon? Again, remember that it is made up of $n$ triangles.

(g) Now take the limit as $n \to \infty$ of your answer in part (e) and show that you get $2\pi r$, the circumference of the circle of radius $r$. **Hint:** Again, the change of variables $\theta = \frac{2\pi}{n}$ might be helpful and you’ll also need to use the fact that

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$
Solution:

(a) \( \theta = \frac{2\pi}{n} \)

(b) \( h = r \sin \theta = r \sin \left( \frac{2\pi}{n} \right) \) and so \( A = \frac{1}{2} rh = \frac{r^2}{2} \sin \left( \frac{2\pi}{n} \right) \).

(c) \( \frac{r^2 \theta}{2} \sin \left( \frac{2\pi}{n} \right) \).

(d) If \( \theta = \frac{2\pi}{n}, n = \frac{2\pi}{\theta} \)

and,

\[
\frac{r^2 n}{2} \sin \left( \frac{2\pi}{n} \right) = \frac{2\pi r^2}{2\theta} \sin \theta = (\pi r^2) \sin \theta.
\]

Since,

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \to 1
\]

we have \( \pi r^2 \cdot 1 = \pi r^2 \).

(e) The Law of Cosines gives that

\[
|PR|^2 = 2r^2 - 2r^2 \cos \left( \frac{2\pi}{n} \right),
\]

and so

\[
|PR| = \sqrt{2r} \sqrt{1 - \cos \left( \frac{2\pi}{n} \right)}.
\]

(f) \( \sqrt{2rn} \sqrt{1 - \cos \left( \frac{2\pi}{n} \right)} \).

(g) If \( \theta = \frac{2\pi}{n} \), then

\[
\sqrt{2rn} \sqrt{1 - \cos \left( \frac{2\pi}{n} \right)} = \sqrt{2r} \left( \frac{2\pi}{\theta} \right) \sqrt{1 - \cos \theta} = 2\sqrt{2}\pi r \sqrt{\frac{1 - \cos \theta}{\theta^2}}.
\]

Then

\[
\lim_{\theta \to 0} 2\sqrt{2}\pi r \sqrt{\frac{1 - \cos \theta}{\theta^2}} = 2\sqrt{2}\pi r \sqrt{\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2}} = 2\sqrt{2}\pi r \sqrt{\frac{1}{2}} = 2\pi r.
\]
3. In this problem, you will see that you can’t reliably do arithmetic with infinities. For each part, find two functions \( f(x) \) and \( g(x) \) with \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to \infty} g(x) = \infty \) satisfying the specified limit. **Note:** You could do all of these by choosing either \( x \) or \( x^2 \) for \( f(x) \) and/or \( g(x) \).

(a) \((\infty - \infty = ?) \lim_{x \to \infty} (f(x) - g(x)) = \infty \)

\[
f(x) = x^2 \quad g(x) = x
\]

(b) \((\infty - \infty = ?) \lim_{x \to \infty} (f(x) - g(x)) = 0 \)

\[
f(x) = x \quad g(x) = x
\]

(c) \((\infty - \infty = ?) \lim_{x \to \infty} (f(x) - g(x)) = -\infty \)

\[
f(x) = x \quad g(x) = x^2
\]

(d) \((\infty = ?) \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \)

\[
f(x) = x^2 \quad g(x) = x
\]

(e) \((\infty = ?) \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \)

\[
f(x) = x \quad g(x) = x
\]

(f) \((\infty = ?) \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \)

\[
f(x) = x \quad g(x) = x^2
\]