MATH 1210 Lab 13 (11/16/2017)

Student #:_____ Instructor & Section:_____

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated. No calculators or electronics of any kind are allowed, unless stated otherwise.

1. Compute the following definite integrals using symmetry and/or properties of the definite integral.

   (a) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5 x \; dx$

   (b) $\displaystyle\int_{-2}^{2} |x|^3 \; dx$

   (c) $\displaystyle\int_{0}^{4} |3x^2 - 18x + 24| \; dx$

   (d) $\displaystyle\int_{3\pi}^{6\pi} |\sin x| \; dx$

Solution:

(a) $\tan^5 x$ is an odd function, so $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5 x \; dx = 0$.

(b) $|x|^3$ is even, so

$$\int_{-2}^{2} |x|^3 \; dx = 2\int_{0}^{2} |x|^3 \; dx = 2\int_{0}^{2} x^3 \; dx = 2\left(\frac{1}{4}x^4\Big|_0^2\right. = 8.$$

(c) Note that $3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$, and so $3x^2 - 18x + 24 < 0$ when $2 < x < 4$. Therefore

$$\begin{aligned}\int_{0}^{4} |3x^2 - 18x + 24| \; dx &= \int_{0}^{2} (3x^2 - 18x + 24) \; dx - \int_{2}^{4} (3x^2 - 18x + 24) \; dx \\ &\left(x^3 - 9x^2 + 24x\Big|_0^2 - \left(x^3 - 9x^2 + 24x\Big|_2^4\right.\right. \\ &= (20 - 0) - (16 - 20) = 24.\end{aligned}$$

(d) Note that $|\sin x|$ is periodic with period π. The interval of integration includes three periods. From 0 to π, $|\sin x| = \sin x$ since $\sin x$ is positive. So

$$\int_{3\pi}^{6\pi} |\sin x| \; dx = 3\int_{0}^{\pi} |\sin x| \; dx = 2\int_{0}^{\pi} \sin x \; dx = 2\left(-\cos x\Big|_0^{\pi}\right. = 4.$$

2. In this problem, we will show that $\displaystyle\int_0^{2\pi} \sin^2 x \; dx = \pi$.

(a) First show that $\displaystyle\int_0^{2\pi} (\sin^2 x + \cos^2 x) \; dx = 2\pi$.

(b) Now argue that $\displaystyle\int_0^{2\pi} \sin^2 x \; dx = \int_0^{2\pi} \cos^2 x \; dx$.

(c) Finally, use (a) and (b) to show that $\displaystyle\int_0^{2\pi} \sin^2 x \; dx = \pi$.

(d) Lastly, compute the integral directly using the trig identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$.

Solution:

(a) Since $\sin^2 x + \cos^2 x = 1$, we have $\int_0^{2\pi} (\sin^2 x + \cos^2 x) \; dx = \int_0^{2\pi} 1 \; dx = 2\pi$.

(b) Now $\cos x = \sin(x + \frac{\pi}{2})$, and so $\cos^2 x = \sin^2(x + \frac{\pi}{2})$. It follows after setting $u = x + \frac{\pi}{2}$,

$$\int_0^{2\pi} \cos^2 x \; dx = \int_0^{2\pi} \sin^2 (x + \frac{\pi}{2}) \; dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \sin^2 u \; du = \int_0^{2\pi} \sin^2 u \; du.$$

In the last equality, I've used the fact that if $f(x)$ is periodic with period p, then for any number a and any integer k, $\int_a^{a+kp} f(x) \; dx = \int_0^{kp} f(x) \; dx$.

(c)

$$\begin{aligned} 2\pi &= \int_0^{2\pi} (\sin^2 x + \cos^2 x) \; dx \\ &= \int_0^{2\pi} \sin^2 x \; dx + \int_0^{2\pi} \cos^2 x \; dx \\ &= 2 \int_0^{2\pi} \sin^2 x \; dx \\ &\Rightarrow \int_0^{2\pi} \sin^2 x \; dx = \pi. \end{aligned}$$

(d)

$$\begin{aligned}\int_0^{2\pi} \sin^2 x \, dx &= \int_0^{2\pi} (\frac{1}{2} - \frac{1}{2}\cos(2x)) \, dx \\ &= \left(\frac{x}{2} - \frac{1}{4}\sin(2x)\right|_0^{2\pi} \\ &= (\pi - \frac{1}{4}\sin(4\pi)) - (0 - \frac{1}{4}\sin 0) \\ &= \pi.\end{aligned}$$

3. Suppose that we want to use Newton's Method to approximate a solution of

$$\int_0^x \frac{1}{1+t^2} \, dt = \frac{1}{2}$$

Set $g(x) = \int_0^x \frac{1}{1+t^2} \, dt - \frac{1}{2}$.

(a) Show that $g(0) < 0$ and $g(1) > 0$.

(b) Why must there be a solution in the interval $(0, 1)$?

(c) Set $x_1 = 0$ and use Newton's method to find x_2, a better approximation to the root.

Solution:

(a) $g(0) = \int_0^0 \frac{1}{1+t^2} \, dt - \frac{1}{2} = -\frac{1}{2} < 0$. Now $\frac{1}{1+t^2} > \frac{1}{2}$ for $0 \leq t < 1$, and so $\int_0^1 \frac{1}{1+t^2} \, dt > \int_0^1 \frac{1}{2} \, dt = \frac{1}{2}$. Therfore, $g(1) = \int_0^1 \frac{1}{1+t^2} \, dt - \frac{1}{2} > \frac{1}{2} - \frac{1}{2} = 0.$.

(b) $g(x)$ is continuous (see FTC1), and so by the Intermediate Value Theorem, there must be some point c in $(0, 1)$ where $g(c) = 0$.

(c) $x_1 = 0$. $g(0) = -\frac{1}{2}$. $g'(x) = \frac{1}{1+x^2}$ by FTC1, so $g'(0) = 1$. Therefore,

$$x_2 = 0 - \frac{-\frac{1}{2}}{1} = \frac{1}{2}.$$

4. Consider the integral $\int_1^4 (x^2+1)\ dx$.

(a) How large do we have to take n so that T_n, the nth trapezoidal approximation to the integral above, has an error of .5 or less? Recall that the error in the trapezoidal approximation to $\int_a^b f(x)\ dx$ satisfies

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2}$$

where K is the maximum of $|f''(x)|$ on $[a,b]$.

(b) Compute T_n for the value of n you found above.

(c) Compute the exact value of the integral. How close is T_n actually?

Solution:

(a) $b=4, a=1,\ f(x)=x^2+1 \Rightarrow f'(x)=2x \Rightarrow f''(x)=2$. So $K=2$. We want to find n such that

$$\frac{K(b-a)^3}{12n^2} = \frac{2(4-1)^3}{12n^2} = \frac{54}{12n^2} \leq .5 \Rightarrow 54 \leq 6n^2 \Rightarrow n^2 \geq 9$$

So we can take $n=3$.

(b) When $n=3$, $\Delta x = \frac{4-1}{3} = 1$. We have

$$T_3 = \frac{1}{2}\left(f(1)+2f(2)+2f(3)+f(4)\right) = \frac{1}{2}(2+10+20+17) = \frac{49}{2}.$$

(c)

$$\int_1^4 (x^2+1)\ dx = \left(\frac{x^3}{3}+x\right|_1^4 = (\frac{64}{3}+4)-(\frac{1}{3}+1) = 24.$$

So $|E_{T_3}| = \left|\frac{49}{2}-24\right| = \frac{1}{2} = .5$.