1. Compute the following definite integrals using symmetry and/or properties of the definite integral.

(a) \[ \int_{-\pi/4}^{\pi/4} \tan^5 x \, dx \]

(b) \[ \int_{-2}^{2} |x|^3 \, dx \]

(c) \[ \int_{0}^{4} |3x^2 - 18x + 24| \, dx \]

(d) \[ \int_{3\pi}^{6\pi} |\sin x| \, dx \]

Solution:

(a) \( \tan^5 x \) is an odd function, so \( \int_{-\pi/4}^{\pi/4} \tan^5 x \, dx = 0. \)

(b) \(|x|^3\) is even, so

\[
\int_{-2}^{2} |x|^3 \, dx = 2 \int_{0}^{2} x^3 \, dx = 2 \left( \frac{1}{4} x^4 \right) \bigg|_{0}^{2} = 8.
\]

(c) Note that \( 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4) \), and so \( 3x^2 - 18x + 24 < 0 \) when \( 2 < x < 4 \). Therefore

\[
\int_{0}^{4} |3x^2 - 18x + 24| \, dx = \int_{0}^{2} (3x^2 - 18x + 24) \, dx - \int_{2}^{4} (3x^2 - 18x + 24) \, dx \\
= \left( x^3 - 9x^2 + 24x \right) \bigg|_{0}^{2} - \left( x^3 - 9x^2 + 24x \right) \bigg|_{2}^{4} \\
= (0 - (16 - 20)) = 24.
\]

(d) Note that \(|\sin x|\) is periodic with period \( \pi \). The interval of integration includes three periods. From 0 to \( \pi \), \(|\sin x| = \sin x\) since \( \sin x \) is positive. So

\[
\int_{3\pi}^{6\pi} |\sin x| \, dx = 3 \int_{0}^{\pi} |\sin x| \, dx = 2 \int_{0}^{\pi} \sin x \, dx = 2 (-\cos x) \bigg|_{0}^{\pi} = 4.
\]
2. In this problem, we will show that \( \int_0^{2\pi} \sin^2 x \, dx = \pi \).

(a) First show that \( \int_0^{2\pi} (\sin^2 x + \cos^2 x) \, dx = 2\pi \).

(b) Now argue that \( \int_0^{2\pi} \sin^2 x \, dx = \int_0^{2\pi} \cos^2 x \, dx \).

(c) Finally, use (a) and (b) to show that \( \int_0^{2\pi} \sin^2 x \, dx = \pi \).

(d) Lastly, compute the integral directly using the trig identity \( \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos (2x) \).

Solution:

(a) Since \( \sin^2 x + \cos^2 x = 1 \), we have \( \int_0^{2\pi} (\sin^2 x + \cos^2 x) \, dx = \int_0^{2\pi} 1 \, dx = 2\pi \).

(b) Now \( \cos x = \sin (x + \frac{\pi}{2}) \), and so \( \cos^2 x = \sin^2 (x + \frac{\pi}{2}) \). It follows after setting \( u = x + \frac{\pi}{2} \),

\[
\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \sin^2 (x + \frac{\pi}{2}) \, dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \sin^2 u \, du = \int_0^{2\pi} \sin^2 u \, du.
\]

In the last equality, I’ve used the fact that if \( f(x) \) is periodic with period \( p \), then for any number \( a \) and any integer \( k \), \( \int_a^{a+kp} f(x) \, dx = \int_0^{kp} f(x) \, dx \).

(c)

\[
2\pi = \int_0^{2\pi} (\sin^2 x + \cos^2 x) \, dx \\
= \int_0^{2\pi} \sin^2 x \, dx + \int_0^{2\pi} \cos^2 x \, dx \\
= 2 \int_0^{2\pi} \sin^2 x \, dx \\
\Rightarrow \int_0^{2\pi} \sin^2 x \, dx = \pi.
\]
The integral is evaluated as follows:

\[
\int_0^{2\pi} \sin^2 x \, dx = \int_0^{2\pi} \left( \frac{1}{2} - \frac{1}{2} \cos (2x) \right) \, dx \\
= \left[ \frac{x}{2} - \frac{1}{4} \sin (2x) \right]_0^{2\pi} \\
= (\pi - \frac{1}{4} \sin (4\pi)) - (0 - \frac{1}{4} \sin 0) \\
= \pi.
\]

3. Suppose that we want to use Newton’s Method to approximate a solution of

\[
\int_0^x \frac{1}{1+t^2} \, dt = \frac{1}{2}
\]

Set \( g(x) = \int_0^x \frac{1}{1+t^2} \, dt - \frac{1}{2} \).

(a) Show that \( g(0) < 0 \) and \( g(1) > 0 \).

(b) Why must there be a solution in the interval \((0, 1)\)?

(c) Set \( x_1 = 0 \) and use Newton’s method to find \( x_2 \), a better approximation to the root.

**Solution:**

(a) \( g(0) = \int_0^0 \frac{1}{1+t^2} \, dt - \frac{1}{2} = -\frac{1}{2} < 0 \). Now \( \frac{1}{1+t^2} > \frac{1}{2} \) for \( 0 \leq t < 1 \), and so \( \int_0^1 \frac{1}{1+t^2} \, dt > \int_0^1 \frac{1}{2} \, dt = \frac{1}{2} \). Therefore, \( g(1) = \int_0^1 \frac{1}{1+t^2} \, dt - \frac{1}{2} > \frac{1}{2} - \frac{1}{2} = 0 \).

(b) \( g(x) \) is continuous (see FTC1), and so by the Intermediate Value Theorem, there must be some point \( c \) in \( (0, 1) \) where \( g(c) = 0 \).

(c) \( x_1 = 0 \). \( g(0) = -\frac{1}{2} \). \( g'(x) = \frac{1}{1+x^2} \) by FTC1, so \( g'(0) = 1 \). Therefore,

\[
x_2 = 0 - \frac{-\frac{1}{2}}{1} = \frac{1}{2}.
\]
4. Consider the integral \( \int_{1}^{4} (x^2 + 1) \, dx \).

(a) How large do we have to take \( n \) so that \( T_n \), the \( n \)th trapezoidal approximation to the integral above, has an error of .5 or less? Recall that the error in the trapezoidal approximation to \( \int_{a}^{b} f(x) \, dx \) satisfies

\[
|E_{T_n}| \leq \frac{K(b - a)^3}{12n^2}
\]

where \( K \) is the maximum of \( |f''(x)| \) on \([a, b]\).

(b) Compute \( T_n \) for the value of \( n \) you found above.

(c) Compute the exact value of the integral. How close is \( T_n \) actually?

**Solution:**

(a) \( b = 4, a = 1, f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2. \) So \( K = 2. \) We want to find \( n \) such that

\[
\frac{K(b - a)^3}{12n^2} = \frac{2(4 - 1)^3}{12n^2} = \frac{54}{12n^2} \leq .5 \Rightarrow 54 \leq 6n^2 \Rightarrow n^2 \geq 9
\]

So we can take \( n = 3. \)

(b) When \( n = 3, \Delta x = \frac{4 - 1}{3} = 1. \) We have

\[
T_3 = \frac{1}{2} (f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2} (2 + 10 + 20 + 17) = \frac{49}{2}.
\]

(c) \[
\int_{1}^{4} (x^2 + 1) \, dx = \left( \frac{x^3}{3} + x \right|_{1}^{4} = \left( \frac{64}{3} + 4 \right) - \left( \frac{1}{3} + 1 \right) = 24.
\]

So \( |E_{T_3}| = \left| \frac{49}{2} - 24 \right| = \frac{1}{2} = .5. \)