MATH 1210 Lab 12 (11/9/2017)

Student #:_____ Instructor & Section:_____

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated. No calculators or electronics of any kind are allowed, unless stated otherwise.

1. Estimate the area under the graph of $y = 4x^2 + 1$ between $x = 0$ and $x = 2$ by computing some Riemann sums.
   (a) Find L_4, the left-endpoint Riemann sum.
   (b) Find R_4, the right-endpoint Riemann sum.
   (c) Find M_4, the midpoint Riemann sum.

Solution:

(a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

$$\begin{aligned} L_4 &= f(0)\Delta x + f(\frac{1}{2})\Delta x + f(1)\Delta x + f(\frac{3}{2})\Delta x \\ &= (1)(\frac{1}{2}) + (2)(\frac{1}{2}) + (5)(\frac{1}{2}) + (10)(\frac{1}{2}) \\ &= 9. \end{aligned}$$

(b)

$$\begin{aligned} R_4 &= f(\frac{1}{2})\Delta x + f(1)\Delta x + f(\frac{3}{2})\Delta x + f(2)\Delta x \\ &= (2)(\frac{1}{2}) + (5)(\frac{1}{2}) + (10)(\frac{1}{2}) + (17)(\frac{1}{2}) \\ &= 17. \end{aligned}$$

(c)

$$\begin{aligned} M_4 &= f(\frac{1}{4})\Delta x + f(\frac{3}{4})\Delta x + f(\frac{5}{4})\Delta x + f(\frac{7}{4})\Delta x \\ &= (\frac{5}{4})(\frac{1}{2}) + (\frac{13}{4})(\frac{1}{2}) + (\frac{29}{4})(\frac{1}{2}) + (\frac{53}{4})(\frac{1}{2}) \\ &= \frac{100}{8} = \frac{25}{2}. \end{aligned}$$

2. The definite integral $\int_a^b f(x)\,dx$ can be interpreted as the signed area under the graph of $y = f(x)$ between $x = a$ and $x = b$. For these problems, sketch a graph of the integrand and use geometry to determine the value of the definite integral.

(a) $\int_0^3 (5 + x)\,dx$

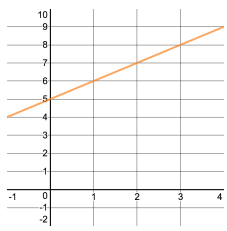
(b) $\int_1^4 (4 - 2x)\,dx$

(c) $\int_0^{10} g(x)\,dx$, where $g(x) = \begin{cases} 3x & \text{if} \quad x \leq 2 \\ 6 & \text{if} \quad 2 < x \leq 5 \\ 16 - 2x & \text{if} \quad x > 5 \end{cases}$

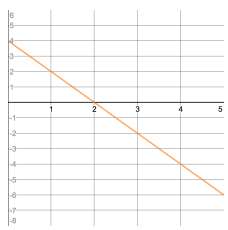
(d) $\int_0^2 \sqrt{4 - x^2}\,dx$

Solution:

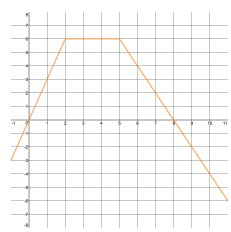
(a) area=19.5



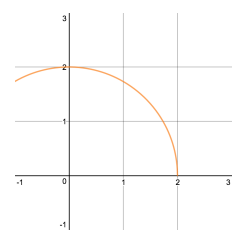
(b) signed area=$1 - 4 = -3$



(c) signed area=$6 + 18 + 9 - 4 = 29$



(d) It is one quarter of a circle of radius 2, therefore the area is π.



3. Use the definition of the definite integral to compute $\int_0^2 (4x^2+1)\,dx$.

(a) Find the right-endpiont Riemann sum $R_n = \sum_{i=1}^n f(a+i\Delta x)\Delta x$.
Hint: Your answer should be a function of n alone and you may need to use the formulas $\sum_{i=1}^n 1 = n$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

(b) Now take the limit as $n \to \infty$ of your answer in part (a). This is the value of $\int_0^2 (4x^2+1)\,dx$.

Solution:

(a) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.

$$\begin{aligned}
R_n &= \sum_{i=1}^n f(0+\frac{2i}{n})(\frac{2}{n}) \\
&= \sum_{i=1}^n (4(\frac{2i}{n})^2+1)\frac{2}{n} \\
&= \sum_{i=1}^n (\frac{32i^2}{n^3}+\frac{2}{n}) \\
&= \frac{32}{n^3}\sum_{i=1}^n i^2 + \frac{2}{n}\sum_{i=1}^n 1 \\
&= \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{2}{n}(n) \\
&= \frac{16n(n+1)(2n+1)}{3n^3} + 2.
\end{aligned}$$

(b) Now we take the limit as $n \to \infty$. Note that the numerator of the first term looks like $32n^3$ plus some terms of degree 2 or less in n. So

$$\lim_{n\to\infty}\left(\frac{16n(n+1)(2n+1)}{3n^3}+2\right) = \lim_{n\to\infty}\left(\frac{32n^3}{3n^3}+2\right) = \frac{32}{3}+2 = \frac{38}{3}.$$

4. A function $f(x)$ is not integrable on $[a, b]$ if the Riemann sum approximations do not converge (to a finite value) as $n \to \infty$ or if the limiting value of the sums dependeds on the sample points chosen in each subinterval. The two function below are not integrable on $[0, 1]$. Explain why.

   (a) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$.

   Hint: Show that $R_n \geq n$.

   (b) $g(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$

Solution:

(a) Note that $\Delta x = \frac{1}{n}$ and

$$R_n = \sum_{i=1}^{n} f(0 + \frac{i}{n})\frac{1}{n} = \sum_{i=1}^{n} (\frac{n^2}{i^2})\frac{1}{n} = n \sum_{i=1}^{n} \frac{1}{i^2}.$$

It is hard to know exactly what $\sum_{i=1}^{n} \frac{1}{i^2}$ is, but it is clearly greater than or equal to its first term, which is 1. So $R_n \geq n$, and so $\lim_{n\to\infty} R_n \geq \lim_{n\to\infty} n = \infty$.

(b) In any subinterval, there are both rational and irrational points. So if we make a sequence of Riemann sums where we are always choosing rational points $\overline{x}_i$ as our sample points, then each Riemann sum will look like $\sum_{i=1}^{n} g(\overline{x}_i)\Delta x = \sum_{i=1}^{n}(1)\Delta x = 1$ (since the length of the interval is 1). Hence in the limit as $n \to \infty$ we will get the value 1. However, if we make a sequence of Riemann sums where we are always choosing irrational points $\overline{y}_i$ as our sample points, then each Riemann sum will look like $\sum_{i=1}^{n} g(\overline{y}_i)\Delta x = \sum_{i=1}^{n}(0)\Delta x = 0$. Hence in the limit as $n \to \infty$ we will get the value 0. Therefore the Riemann sums converge to different value depending on the sample point chosen.