1. Estimate the area under the graph of $y = 4x^2 + 1$ between $x = 0$ and $x = 2$ by computing some Riemann sums.

   (a) Find $L_4$, the left-endpoint Riemann sum.
   
   (b) Find $R_4$, the right-endpoint Riemann sum.
   
   (c) Find $M_4$, the midpoint Riemann sum.

### Solution:

(a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

\[
L_4 = f(0)\Delta x + f\left(\frac{1}{2}\right)\Delta x + f(1)\Delta x + f\left(\frac{3}{2}\right)\Delta x \\
= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right) + (10)\left(\frac{1}{2}\right) \\
= 9.
\]

(b) $R_4 = f\left(\frac{1}{2}\right)\Delta x + f(1)\Delta x + f\left(\frac{3}{2}\right)\Delta x + f(2)\Delta x$

\[
= (2)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right) + (10)\left(\frac{1}{2}\right) + (17)\left(\frac{1}{2}\right) \\
= 17.
\]

(c) $M_4 = f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x + f\left(\frac{5}{4}\right)\Delta x + f\left(\frac{7}{4}\right)\Delta x$

\[
= \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{29}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{53}{4}\right)\left(\frac{1}{2}\right) \\
= \frac{100}{8} = \frac{25}{2}.
\]
2. The definite integral $\int_{a}^{b} f(x) \, dx$ can be interpreted as the signed area under the graph of $y = f(x)$ between $x = a$ and $x = b$. For these problems, sketch a graph of the integrand and use geometry to determine the value of the definite integral.

(a) $\int_{0}^{3} (5 + x) \, dx$

(b) $\int_{1}^{4} (4 - 2x) \, dx$

(c) $\int_{0}^{10} g(x) \, dx$, where $g(x) = \begin{cases} 3x & \text{if } x \leq 2 \\ 6 & \text{if } 2 < x \leq 5 \\ 16 - 2x & \text{if } x > 5 \end{cases}$

(d) $\int_{0}^{2} \sqrt{4 - x^2} \, dx$

**Solution:**

(a) area = 19.5

(b) signed area = $1 - 4 = -3$

(c) signed area = $6 + 18 + 9 - 4 = 29$

(d) It is one quarter of a circle of radius 2, therefore the area is $\pi$. 

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3. Use the definition of the definite integral to compute \( \int_{0}^{2} (4x^2 + 1) \, dx \).

(a) Find the right-endpoint Riemann sum \( R_n = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x \).

\[ \text{Hint:} \quad \text{Your answer should be a function of } n \text{ alone and you may need to use the formulas } \sum_{i=1}^{n} 1 = n, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \]

(b) Now take the limit as \( n \to \infty \) of your answer in part (a). This is the value of \( \int_{0}^{2} (4x^2 + 1) \, dx \).

Solution:

(a) \( \Delta x = \frac{2-0}{n} = \frac{2}{n} \).

\[
R_n = \sum_{i=1}^{n} f(0 + i\Delta x)(\frac{2}{n}) = \sum_{i=1}^{n} \left(4\left(\frac{2i}{n}\right)^2 + 1\right)\frac{2}{n} = \sum_{i=1}^{n} \left(\frac{32i^2}{n^3} + \frac{2}{n}\right) = \frac{32}{n^3} \sum_{i=1}^{n} i^2 + \frac{2}{n} \sum_{i=1}^{n} 1 = \frac{32n(n+1)(2n+1)}{6n^3} + \frac{2n}{n} = \frac{16n(n+1)(2n+1)}{3n^3} + 2.
\]

(b) Now we take the limit as \( n \to \infty \). Note that the numerator of the first term looks like \( 32n^3 \) plus some terms of degree 2 or less in \( n \). So

\[
\lim_{n \to \infty} \left(\frac{16n(n+1)(2n+1)}{3n^3} + 2\right) = \lim_{n \to \infty} \left(\frac{32n^3}{3n^3} + 2\right) = \frac{32}{3} + 2 = \frac{38}{3}.
\]
4. A function \( f(x) \) is not integrable on \([a, b]\) if the Riemann sum approximations do not converge (to a finite value) as \( n \to \infty \) or if the limiting value of the sums dependeds on the sample points chosen in each subinterval. The two function below are not integrable on \([0, 1]\). Explain why.

(a) \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \).

Hint: Show that \( R_n \geq n \).

(b) \( g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \).

Solution:

(a) Note that \( \Delta x = \frac{1}{n} \) and

\[
R_n = \sum_{i=1}^{n} f(0 + \frac{i}{n}) \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{n^2}{i^2} \right) = n \sum_{i=1}^{n} \frac{1}{i^2}.
\]

It is hard to know exactly what \( \sum_{i=1}^{n} \frac{1}{i^2} \) is, but it is clearly greater than or equal to its first term, which is 1. So \( R_n \geq n \), and so \( \lim_{n \to \infty} R_n \geq \lim_{n \to \infty} n = \infty \).

(b) In any subinterval, there are both rational and irrational points. So if we make a sequence of Riemann sums where we are always choosing rational points \( \bar{x}_i \) as our sample points, then each Riemann sum will look like

\[
\sum_{i=1}^{n} g(\bar{x}_i) \Delta x = \sum_{i=1}^{n} (1) \Delta x = 1 \text{ (since the length of the interval is 1)}.\]

Hence in the limit as \( n \to \infty \) we will get the value 1. However, if we make a sequence of Riemann sums where we are always choosing irrational points \( \bar{y}_i \) as our sample points, then each Riemann sum will look like

\[
\sum_{i=1}^{n} g(\bar{y}_i) \Delta x = \sum_{i=1}^{n} (0) \Delta x = 0.\]

Hence in the limit as \( n \to \infty \) we will get the value 0. Therefore the Riemann sums converge to different value depending on the sample point chosen.