Derivation of fundamental trigonometric (circular function) identities

Our starting point will be the basic addition formulas:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

and the symmetry properties of cos and sin.

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

Substituting $-\theta_2$ for θ_2 and using symmetries, we get the subtraction formulas:

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$$

Adding cosine addition and subtraction formulas, we get a cosine product formula:

$$\cos\theta_1 \cos\theta_2 = \frac{1}{2}(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2))$$

Subtracting them gives a sine product formula:

$$\sin\theta_1 \sin\theta_2 = \frac{1}{2}(\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2))$$

Adding sine addition formulas gives a sine-cosine product formula:

$$\sin\theta_1 \cos\theta_2 = \frac{1}{2}(\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2))$$

which may be obtained in reverse order by instead subtracting them, or just by interchanging θ_1 and θ_2 in the earlier formula:

$$\cos\theta_1 \sin\theta_2 = \frac{1}{2}(\sin(\theta_1 + \theta_2) - \sin(\theta_1 - \theta_2))$$

Setting $u = \theta_1 + \theta_2$ and $v = \theta_1 - \theta_2$, so $\theta_1 = \frac{u+v}{2}$ and $\theta_2 = \frac{u-v}{2}$, gives different addition and subtraction formulas for cosine and sine, where the operation is performed after the function, rather than before:

$$\cos u + \cos v = 2\cos(\frac{u+v}{2})\cos(\frac{u-v}{2})$$

$$\cos u - \cos v = -2\sin(\frac{u+v}{2})\sin(\frac{u-v}{2})$$

$$\sin u + \sin v = 2\sin(\frac{u+v}{2})\cos(\frac{u-v}{2})$$

$$\sin u - \sin v = 2\sin(\frac{u+v}{2})\sin(\frac{u-v}{2})$$

Setting $\theta_2 = -\theta_1$, and using a special value of cosine, we get the Pythagorean relationship:

$$\cos 0 = 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Setting $\theta_1 = \theta_2 = \theta$ gives the double-angle formulas:

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Beginning with the Pythagorean relationship and adding or subtracting the cosine double angle formula, gives formulas which represent the second powers of $\cos\theta$ and $\sin\theta$, and cosine of twice θ in terms of each other:

$$(\cos\theta)^2 = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$(\sin\theta)^2 = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\cos 2\theta = 2(\cos\theta)^2 - 1 = 1 - 2(\sin\theta)^2$$

Similar formulas can be obtained representing the powers of $\cos\theta$ and $\sin\theta$ up to the nth power, and cosine and sine of whole multiples of θ up to n, in terms of each other (see below.)

Taking square roots, and substituting $\frac{\theta}{2}$ for θ,

$$|\cos\frac{\theta}{2}| = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}$$

and

$$|\sin\frac{\theta}{2}| = \sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta}$$

A useful informal definition of $(\cos\theta, \sin\theta)$ is that they represent the horizontal and vertical coordinates of a point on the unit circle, with θ measuring the distance along the circle from $(1,0)$ in the clockwise direction, usually given in proportion to the full circumference, 2π. This is good enough when simple fractions of the circle are involved, but fails to be useful in generality. A good formal definition involves first defining the inverse of the sine function, whose output is the arc length θ between $(1,0)$ and a point (x,y) on the unit circle $x^2 + y^2 = 1$, calculated as an integral, so that $y = \sin\theta$.

$$(x, y) = (r\cos\theta, r\sin\theta)$$

with

$$r = \sqrt{x^2 + y^2}$$

$$x = \pm\sqrt{r^2 - y^2}$$

$$y = \pm\sqrt{r^2 - x^2}$$

$$\cos\theta = \pm\sqrt{1 - (\sin\theta)^2}$$

$$\sin\theta = \pm\sqrt{1 - (\cos\theta)^2}$$

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$\sec\theta \equiv \frac{1}{\cos\theta}$$

$$\csc\theta \equiv \frac{1}{\sin\theta}$$

so

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\cot\theta = \frac{x}{y}$$

$$\sec\theta = \frac{r}{x}$$

$$\csc\theta = \frac{r}{y}$$

Shift properties and Periodicity

$$\cos(\theta + 2\pi) = \cos(\theta)$$

$$\sin(\theta + 2\pi) = \sin(\theta)$$

$$\cos(\theta - \frac{\pi}{2}) = \sin(\theta)$$

Special values

$$(\cos 0, \sin 0) = (1, 0)$$

$$(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$$

$$(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

$$(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (0, 1)$$

By looking at the diagram, one can see that the first three of these special points determine all of the rest by various reflections. The first is straightforward. To see the second, construct an equilateral triangle by reflecting the point in the x-axis, so that twice the vertical coordinate must equal the radius, 1, and $\sin\frac{\pi}{6} = \frac{1}{2}$. Solve Pythagoras for the horizontal coordinate: $(\cos\frac{\pi}{6})^2 + (\frac{1}{2})^2 = 1$ so $\cos\frac{\pi}{6} = (\frac{3}{4})^{\frac{1}{2}} = \frac{\sqrt{3}}{2}$. Finally, when $\theta = \frac{\pi}{4}$, the horizontal and vertical coordinates of the point $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$ are equal, so solving Pythagoras we see that twice the square of either coordinate equals 1, so that the square of either coordinate equals $\frac{1}{2}$ and $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = (\frac{1}{2})^{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.